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Asymptotics of the Solutions of the Random Schrödinger Equation

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Abstract

We consider solutions of the Schrödinger equation with a weak time-dependent random potential. It is shown that when the two-point correlation function of the potential is rapidly decaying, then the Fourier transform $\hat{\zeta}_\varepsilon(t,\xi)$ of the appropriately scaled solution converges point-wise in ξ to a stochastic complex Gaussian limit. On the other hand, when the two-point correlation function decays slowly, we show that the limit of $\hat{\zeta}_\varepsilon(t,\xi)$ has the form $\hat{\zeta}_0(\xi)\exp(iB_\kappa(t,\xi))$ where $B_\kappa(t,\xi)$ is a fractional Brownian motion.

1. Introduction and the main results

We consider solutions of the Schrödinger equation

$$\begin{aligned}
&i\frac{\partial\phi}{\partial t}+\frac{1}{2}\Delta\phi-\gamma V(t,x)\phi=0,\quad x\in\mathbb{R}^d,\\
&\phi(0,x)=\phi_0(x),
\end{aligned}\tag{1.1}$$

with a random potential $V(t,x)$ in the spatial dimension $d\geqq 1$. Here $\gamma\ll 1$ is the small parameter that measures the relative strength of the (weak) random fluctuations. Throughout the paper we assume for simplicity that $\phi_0\in\mathcal{S}(\mathbb{R}^n)$. The long time behavior of the Wigner transform [13] of the solutions of (1.1), defined as

$$W(t,x,k)=\int e^{ik\cdot y}\phi\left(t,x-\frac{y}{2}\right)\phi^*\left(t,x+\frac{y}{2}\right)\frac{dy}{(2\pi)^d},$$

has been extensively studied in the past: it can be shown that when properly rescaled to allow for long distance and large time propagation, the limit of

$\mathbb{E}(W(t,x,k))$ converges as $\gamma \to 0$ to the solution of the radiative transport equation [3,4,7–9,15,19,27]

$$\bar{W}_t + k\cdot\nabla_x \bar{W} = \int \hat{R}\left(p-k, \frac{p^2-k^2}{2}\right)(\bar{W}(t,x,p) - \bar{W}(t,x,k))\mathrm{d}p. \tag{1.2}$$

This result holds under the assumption that $V(t,x)$ is a spatially and temporally homogeneous mean-zero random field with the two-point correlation function

$$R(t,x) = \mathbb{E}[V(s,y)V(t+s,x+y)],$$

whose power spectrum

$$\hat{R}(\omega,k) = \int R(t,x)\mathrm{e}^{-ik\cdot x - i\omega t}\,\mathrm{d}x\,\mathrm{d}t$$

appears in (1.2). The harder case, when the random potential is time-independent, was studied in [7–9,19,27]. In addition, it has been shown that the limit is often self-averaging, that is, given any test function $\eta(x,k) \in \mathcal{S}(\mathbb{R}^{2d})$, $\langle W,\eta\rangle \to \langle \bar{W},\eta\rangle$ in probability [1,2,4–6,21,22]. However, this result does not hold strongly, that is, point-wise in x and k. Here we denote

$$\langle W,\eta\rangle = \int W(x,k)\eta(x,k)\,\mathrm{d}x\,\mathrm{d}k.$$

On the other hand, surprisingly, the solution $\phi(t,x)$ of (1.1) itself seems to be much less studied – an obvious reason for this is that $\phi(t,x)$ becomes highly oscillatory after propagation over long distances, while the Wigner transform is a macroscopic quantity. The goal of the present paper is to understand the behavior of $\phi(t,x)$ after propagation over long distances and also to study the effect of the slow spatial and temporal decay of the correlation function $R(t,x)$ on the behavior of solutions, long time limit and self-averaging properties.

As we are interested in the long time, large propagation distances effect of the random inhomogeneities, we consider temporal and spatial scales of the order $t \sim O(\varepsilon^{-1})$ and $x \sim O(\varepsilon^{-1})$ with $\varepsilon = \varepsilon(\gamma) \ll 1$, a small parameter depending on γ, to be determined later. Finding an appropriate length and time scale $O(\varepsilon^{-1})$, on which one observes a non-trivial behavior, as a functional of $\gamma \ll 1$ is part of the problem. Let us recast (1.1) as an equation for the rescaled function $\phi_\varepsilon(t,x) = \phi(t/\varepsilon, x/\varepsilon)$:

$$\begin{aligned} &i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi_\varepsilon - \gamma V\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\phi_\varepsilon = 0, \\ &\phi_\varepsilon(0,x) = \phi_0(x/\varepsilon). \end{aligned} \tag{1.3}$$

In particular, we have $\hat{\phi}_\varepsilon(0,\xi) = \varepsilon^d\hat{\phi}_0(\varepsilon\xi)$. We assume that the spatial power spectrum has the form

$$\tilde{R}(t,k) = \mathrm{e}^{-\mathfrak{g}(k)|t|}\hat{R}(k), \tag{1.4}$$

where $\hat{R}(k) \in L^1(\mathbb{R}^d)$, and

$$\tilde{R}(t,k) = \int e^{-ik\cdot x} R(t,x)\,dx.$$

The space–time power energy spectrum is then

$$\hat{R}(\omega,k) = \frac{2\mathfrak{g}(k)\hat{R}(k)}{\omega^2 + \mathfrak{g}^2(k)}. \tag{1.5}$$

Rapidly decaying correlations

We first address the case in which the correlation function $R(t,x)$ decays sufficiently rapidly. In order to formulate our main result in this situation, let

$$D(p,\xi) = \frac{2\hat{R}(p)}{(2\pi)^d[\mathfrak{g}(p) - i(\xi\cdot p - |p|^2/2)]} \tag{1.6}$$

and

$$D(\xi) = \int D(p,\xi)dp = 2\int \frac{\hat{R}(p)}{\mathfrak{g}(p) - i(\xi\cdot p - |p|^2/2)}\frac{dp}{(2\pi)^d}. \tag{1.7}$$

Let also

$$\mathcal{L}F(\xi) := \int \mathrm{Re}D(p,\xi)[F(p) - F(\xi)]dp, \tag{1.8}$$

for any F bounded and measurable, and let $\widehat{W}(t,\xi)$ be the Duhamel solution of the equation

$$\begin{cases} \partial_t \widehat{W}(t,\xi) = \mathcal{L}\widehat{W}(t,\xi), \\ \widehat{W}(0,\xi) = |\hat{\phi}_0(\xi)|^2. \end{cases} \tag{1.9}$$

Note that (1.9) is simply the integrated in x form of the kinetic equation (1.2), that is, if $W(t,x,\xi)$ solves (1.2) then $\hat{W}(t,\xi) = \int W(t,x,\xi)\,dx$ solves (1.9).

The first result of this paper is the following theorem concerning the commonly considered situation when the function $R(t,x)$ decays rapidly.

Theorem 1.1. *Assume that $V(t,x)$ is a spatially homogeneous mean-zero Gaussian and Markovian in a time random field with the two-point correlation function $R(t,x)$ and a spatial power spectrum $\tilde{R}(t,k)$ of the form* (1.4) *with*

$$\int \frac{\hat{R}(p)dp}{\mathfrak{g}(p)} < +\infty. \tag{1.10}$$

Let $\varepsilon = \gamma^2$, and define

$$\hat{\zeta}_\varepsilon(t,\xi) = \frac{1}{\varepsilon^d}\hat{\phi}_\varepsilon(t,\xi/\varepsilon)e^{i|\xi|^2 t/(2\varepsilon)}, \tag{1.11}$$

where $\phi_\varepsilon(t,x)$ is the solution of (1.3). Then, for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in law, as $\varepsilon \to 0$, to

$$\hat{\zeta}(t,\xi) = \mathrm{e}^{-tD_\xi/2}\hat{\phi}_0(\xi) + Z(t,\xi) \tag{1.12}$$

Here $Z(t,\xi)$ is a centered, complex valued Gaussian random variable, whose variance equals

$$\mathbb{E}|Z(t,\xi)|^2 = \widehat{W}(t,\xi) - \mathrm{e}^{-t\mathrm{Re}D_\xi}|\hat{\phi}_0(\xi)|^2.$$

Let us recall [14] that a random variable $Z = X + iY$ is a centered complex Gaussian if X and Y are mean-zero Guassian independent random variables with $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$.

Note that Theorem 1.1 implies, in particular, that $\mathbb{E}\{|\hat{\zeta}(t,\xi)|^2\} = \widehat{W}(t,\xi)$ is the solution of (1.9), as would be expected from the usual kinetic theory for waves. However, this theorem gives much more precise information on the limit of the whole random field $\hat{\zeta}_\varepsilon(t,\xi)$, not just its second absolute moment.

Slowly decaying correlations

Suppose now that the spatial power spectrum has the form

$$\hat{R}(p) = \frac{a(p)}{|p|^{2\alpha+d-2}} \tag{1.13}$$

and the spectral gap is

$$\mathfrak{g}(p) = \mu|p|^{2\beta} \tag{1.14}$$

for some $\alpha < 1$, $0 \leqq \beta \leqq 1/2$, $\mu > 0$, and a compactly supported, non-negative, bounded measurable function $a(p)$. We assume that $a(p)$ is continuous at $p = 0$ and $a(0) > 0$. Observe that in order for (1.10) to hold we need to assume that $\alpha + \beta < 1$. Our second result concerns the case when the correlation function decays slowly, so that $\alpha + \beta > 1$. This implies that $\alpha \in (\frac{1}{2}, 1)$. Let us first define the constants

$$K_1(\alpha,\beta,\mu) = \Omega_d \int_0^{+\infty} \mathrm{e}^{-\mu\rho^{2\beta}} \frac{\mathrm{d}\rho}{\rho^{2\alpha-1}}, \tag{1.15}$$

where Ω_d is the surface area of the unit sphere in $\mathbb{R}^d$, and

$$K_2(\xi;\alpha,\mu) = \int_0^{+\infty} \mathrm{e}^{-\mu\rho} \frac{\mathrm{d}\rho}{\rho^{2\alpha-1}} \int_{\mathbb{S}^{d-1}} \mathrm{e}^{i|\xi|\rho\omega\cdot e_1} S(\mathrm{d}\omega). \tag{1.16}$$

Theorem 1.2. *Assume that the two-point space–time correlation function $R(t,x)$ has the form (1.4) with $\hat{R}(p)$ and $\mathfrak{g}(p)$ as in (1.13) and (1.14), and that $\alpha + \beta > 1$, $1/2 < \alpha < 1$ and $\beta \leqq 1/2$. Let $\varepsilon = \gamma^{1/\kappa}$, with*

$$\kappa = \frac{\alpha + 2\beta - 1}{2\beta} = 1 - \frac{1-\alpha}{2\beta}. \tag{1.17}$$

Then, for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$, defined by (1.11), converges in law, as $\varepsilon \to 0$, to the random variable

$$\bar{\zeta}_0(t,\xi) = \hat{\phi}_0(\xi)\mathrm{e}^{i\sqrt{D(\xi)}B_\kappa(t)}, \tag{1.18}$$

where $B_\kappa(t;\xi)$ is a standard scalar fractional Brownian motion, its variance D given by

$$D = \frac{a(0)K_1(\alpha,\beta,\mu)}{\kappa(2\kappa-1)(2\pi)^d}. \tag{1.19}$$

when $\beta < 1/2$, and

$$D(\xi) = \frac{a(0)K_2(\xi;\alpha,\mu)}{\alpha(2\alpha-1)(2\pi)^d}, \tag{1.20}$$

when $\beta = 1/2$.

We note that there are several important differences between the rapidly decorrelating case considered in Theorem 1.1 and the slowly decorrelating case in Theorem 1.2. First of all, the time scale of $\hat{\zeta}_\varepsilon(t,\xi)$ now is not γ^{-2} but rather $\gamma^{-1/\kappa}$. In particular, it is no longer universal but depends, rather, on the parameters α and β when $\alpha+\beta > 1$. On the other hand, if we fix the ratio ε of the overall propagation distance and the correlation length of the medium, then the strength of the heterogeneities $\gamma = \varepsilon^\kappa = \varepsilon^{1-(1-\alpha)/2\beta}$ that produces a non-trivial effect also decreases when α and β increase. This shows that weaker fluctuations generate a macroscopic effect in the presence of long range (in space and time) correlations.

To illustrate why the correct time scale in the setting of Theorem 1.2 is $O(\varepsilon^{-1/\kappa})$, one may consider a simplified version of our equation (with $\mu = 1$), obtained by dropping the Laplacian,

$$i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t} = -\gamma V\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\phi_\varepsilon, \quad \phi_\varepsilon(0,x) = \phi_0\left(\frac{x}{\varepsilon}\right). \tag{1.21}$$

Then $\phi_\varepsilon(t,x) = \phi_0(x/\varepsilon)\exp(iW_\varepsilon(t,x))$, with

$$W_\varepsilon(t,x) = -\frac{\gamma}{\varepsilon}\int_0^t V\left(\frac{s}{\varepsilon},\frac{x}{\varepsilon}\right)\mathrm{d}s,$$

and

$$\begin{aligned}\mathbb{E}[W_\varepsilon^2(t,x)] &= \frac{2\gamma^2}{\varepsilon^2}\int_0^t\int_0^s R\left(\frac{s-s'}{\varepsilon},0\right)\mathrm{d}s\,\mathrm{d}s'\\ &= \frac{2\gamma^2}{\varepsilon^2}\int_0^t\int_0^{s_1}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{-|p|^{2\beta}s_2/\varepsilon}a(p)\mathrm{d}p}{(2\pi)^d|p|^{2\alpha+d-2}}\mathrm{d}s_2\,\mathrm{d}s_1.\end{aligned}$$

Upon a change of variables $p = \varepsilon^{1/(2\beta)}q/(s_2)^{1/(2\beta)}$ we get

$$\begin{aligned}\mathbb{E}[W_\varepsilon^2(t,x)] &= \frac{2\gamma^2}{\varepsilon^{2-(1-\alpha)/\beta}}\frac{a(0)K_1(\alpha,\beta,1)}{(2\pi)^d}\int_0^t\mathrm{d}s_1\int_0^{s_1}\frac{\mathrm{d}s_2}{s_2^{(1-\alpha)/\beta}}\\ &= \frac{2\gamma^2}{\varepsilon^{2-(1-\alpha)/\beta}}\frac{a(0)K_1(\alpha,\beta,1)}{(2\pi)^d}\frac{\beta^2t^{(\alpha+2\beta-1)/\beta}}{(\alpha+\beta-1)(\alpha+2\beta-1)} = Dt^{2\kappa},\end{aligned}$$

with the choice of γ as in Theorem 1.2, and D as in (1.19). Note that this not only correctly identifies the time scale, but also the fractional diffusion coefficient for $\beta < 1/2$. At $\beta = 1/2$ the Laplacian plays a non-trivial role and the formula for the diffusion coefficient is different.

Let us also point out a difference between the evolution of the energy and the phase of the wave: while Theorems 1.1 and 1.2 show that the time scale on which the phase evolves depends very much on the nature of the correlations of the random medium, this does not seem to be the case for the wave energy density. Indeed, the amplitude of the limit process $\bar{\zeta}_0(t,\xi)$ in (1.18) is trivial: $|\bar{\zeta}_0(t,\xi)| = |\hat{\phi}_0(\xi)|$ hence, unlike the phase, the wave energy has not yet been affected by the random fluctuations on the time scale $O(\gamma^{-1/\kappa})$. Moreover, while the total scattering cross-section

$$\Sigma = \int \hat{R}\left(p-k, \frac{p^2-k^2}{2}\right) \mathrm{d}p$$

is infinite in the regime of slowly decaying correlations, with the parameters α and β as in Theorem 1.2, the transport equation (1.2) still makes sense because of the regularizing effect of the difference $W(t,x,p) - W(t,x,k)$ that appears in the right side of (1.2). Hence, we believe that even in this range of parameters, wave energy evolves on the time scale $O(\gamma^{-2})$, as in the rapidly decorrelating case. Thus, the slow decay of correlations leads to time-separation of the energy and phase evolutions, a phenomenon we plan to address in detail elsewhere.

Let us mention that to the best of our knowledge the first study of wave propagation in random media with slowly decaying correlations was done in the one-dimensional case [12,20], where it was shown that a pulse going through a random medium with long range correlations performs a fractional Brownian motion around its mean position, as opposed to the regular Brownian motion in the rapidly decorrelating case [11]. On the other hand, motion of particles in such random media leading to fractional Brownian limits was considered in [10,17,18]. The main contributions of the present paper are that, first, the full limit process of the wave field is identified (we are not aware of any such results for waves in any regime in dimensions higher than one), and, second, it is shown that slow decay of correlations induces a different time scale for the field and energy.

The paper is organized as follows: in Section 2 we consider the Duhamel expansion for (1.1) that is the basis for our considerations. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In both proofs we first identify the limit of $\mathbb{E}(\hat{\zeta}_\varepsilon(t,\xi))$ using first the Duhamel expansion for the expression under the expectation and then applying the rules for computing the expectation of a product of Gaussian random variables. These involve summing over all possible partitions of the set indexing the random variables into pairs. We will refer to such a partition as a *pairing*. This is the same strategy as the one used in [7–9] to obtain the kinetic limit in the rapidly decorrelating case. Here, however, the estimation is simpler since the potential is time-dependent. Summation over pairings relies on the assumption that the random potential has Gaussian statistics, as is common in this approach. On the other hand, the new aspect in the case of slowly decaying correlations is that all pairings contribute to the limit, not just the time-ordered pairings as in the rapidly decorrelating case. The next step in the proofs of Theorems 1.1 and 1.2 is to identify the limit of the higher moments of $\hat{\zeta}_\varepsilon(t,\xi)$ including the mixed moments of the form $\mathbb{E}[\hat{\zeta}_\varepsilon^N(t,\xi)\hat{\zeta}_\varepsilon^{*M}(t,\xi)]$.

2. The Duhamel expansion

We may rewrite (1.3) as an integral in time equation

$$\hat{\phi}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi)\mathrm{e}^{-i\varepsilon|\xi|^2 t/2} + \frac{\gamma}{i\varepsilon}\int_0^t \int \frac{\hat{V}(s_1/\varepsilon, \mathrm{d}p_1)}{(2\pi)^d} \times \hat{\phi}_\varepsilon\left(s_1, \xi - \frac{p_1}{\varepsilon}\right)\mathrm{e}^{-i\varepsilon|\xi|^2(t-s_1)/2}\,\mathrm{d}s_1.$$

Hence, the function $\hat{\zeta}_\varepsilon(t,\xi)$ given by (1.11) solves

$$\hat{\zeta}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi) + \frac{\gamma}{i\varepsilon}\int_0^t\int \frac{\hat{V}(s_1/\varepsilon,\mathrm{d}p_1)}{(2\pi)^d}\hat{\zeta}_\varepsilon(s_1,\xi-p_1)\mathrm{e}^{i(|\xi|^2-|\xi-p_1|^2)s_1/(2\varepsilon)}\,\mathrm{d}s_1, \tag{2.1}$$

as $\hat{\zeta}(0,\xi) = \hat{\phi}_0(\xi)$. Iterating (2.1) leads to an infinite series expansion for $\hat{\zeta}_\varepsilon(t,\xi)$:

$$\hat{\zeta}_\varepsilon(t,\xi) = \sum_{n=0}^\infty \hat{\zeta}_n^\varepsilon(t,\xi), \tag{2.2}$$

with the individual terms of the form

$$\hat{\zeta}_n^\varepsilon(t,\xi) = \left[\frac{\gamma}{i\varepsilon(2\pi)^d}\right]^n \int_{\Delta_n(t)} \mathrm{d}\mathbf{s}^{(n)} \int \hat{V}\left(\frac{s_1}{\varepsilon},\mathrm{d}p_1\right)\ldots\hat{V}\left(\frac{s_n}{\varepsilon},\mathrm{d}p_n\right) \times\hat{\phi}_0(\xi - p_1 - \cdots - p_n)\mathrm{e}^{iG_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)})/\varepsilon}, \tag{2.3}$$

with the phase

$$\begin{aligned} G_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) &= \sum_{k=1}^n (|\xi - p_1 - \cdots - p_{k-1}|^2 - |\xi - p_1 - \cdots - p_k|^2)\frac{s_k}{2} \\ &= A_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) - B_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}). \end{aligned} \tag{2.4}$$

Here we use the notation $p_0 = 0$, $\mathbf{s}^{(n)} = (s_1,\ldots,s_n) \in \mathbb{R}^n$, $\mathbf{p}^{(n)} = (p_1,\ldots,p_n) \in \mathbb{R}^{nd}$, so that $\mathrm{d}\mathbf{s}^{(n)} = \mathrm{d}s_1\mathrm{d}s_2\ldots\mathrm{d}s_n$. We have also split the phase into

$$A_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^n (\xi\cdot p_m)s_m, \tag{2.5}$$

$$B_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^n s_m p_m\cdot\left(\sum_{j=1}^{m-1}p_j\right) + \frac{1}{2}\sum_{m=1}^n s_m|p_m|^2.$$

Finally, $\Delta_n(t)$ denotes the time simplex

$$\Delta_n(t) = \{(s_1, s_2, \dots, s_n) : \ 0 \leqq s_n \leqq s_{n-1} \leqq \dots \leqq s_1 \leqq t\}.$$

The next proposition shows that the series (2.2) converges almost surely and, moreover, one can take the expectation term-wise for $\varepsilon > 0$ and $\gamma > 0$ fixed. This will allow us to work with term-wise estimates for each $\mathbb{E}(\hat{\zeta}_\varepsilon^n)$ separately in the proof of Theorems 1.1 and 1.2.

Proposition 2.1. (i) *The series* (2.2) *for the function* $\hat{\zeta}_\varepsilon(t, \xi)$ *converges almost surely for all values of* $\gamma, \varepsilon \in (0, 1]$ *and* $\phi_0 \in C_c^\infty(\mathbb{R}^d)$.

(ii) *Moreover, for each* $(t, \xi) \in \mathbb{R}^{1+d}$ *fixed, we have*

$$\mathbb{E}\hat{\zeta}_\varepsilon(t, \xi) = \sum_{n=0}^{\infty} \mathbb{E}\hat{\zeta}_n^\varepsilon(t, \xi). \tag{2.6}$$

Proof. We may assume without loss of generality that $\gamma = \varepsilon = 1$. Using an elementary result from analysis, see for example Theorem 1.38 of [24], the conclusion of the lemma follows, provided we can show that $\sum_{n=0}^{\infty} [\mathbb{E}|\hat{\zeta}_n^\varepsilon(t, \xi)|^2]^{1/2} < +\infty$.

Note that

$$\begin{aligned}
\mathbb{E}|\hat{\zeta}_n^\varepsilon(t, \xi)|^2 &= \frac{1}{(2\pi)^{2nd}} \int_{\Delta_n(t)} \mathrm{d}\mathbf{s}^{(n)} \int_{\Delta_n(t)} \mathrm{d}\tilde{\mathbf{s}}^{(n)} \int \mathbb{E}\left[\prod_{k=1}^{n} \hat{V}(s_k, \mathrm{d}p_k) \prod_{k=1}^{n} \hat{V}^*(\tilde{s}_k, \mathrm{d}\tilde{p}_k)\right] \\
&\quad \times \hat{\phi}_0(\xi - \sum_{j=1}^{n} p_j) \hat{\phi}_0^*(\xi - \sum_{j=1}^{n} \tilde{p}_j) \mathrm{e}^{iG_n(\mathbf{s}^{(n)}, \mathbf{p}^{(n)})} \mathrm{e}^{-iG_n(\tilde{\mathbf{s}}^{(n)}, \tilde{\mathbf{p}}^{(n)})} \\
&\leqq \frac{t^n (2n-1)!!}{n!^2 (2\pi)^{2nd}} \left[\int \hat{R}(\mathrm{d}p)\right]^n \|\hat{\phi}_0\|_\infty^2 \leqq \frac{C^n}{n!}
\end{aligned} \tag{2.7}$$

for some constant $C > 0$ independent of n, and the conclusion of the proposition follows. □

3. Proof of Theorem 1.1

We now prove Theorem 1.1, that is, we consider the case in which the two-point correlation function decays sufficiently rapidly so that the phase obeys a deterministic limit. We shall assume in the course of the proof of Theorem 1.1 that $\gamma = \varepsilon^{1/2}$.

Outline of the proof

The proof is based on working with the Duhamel expansion (2.2) and, in particular, with the series for the moments $\mathbb{E}\{[\hat{\zeta}_\varepsilon(t, \xi)]^n [[\hat{\zeta}_\varepsilon(t, \xi)]^m]^*\}$ with arbitrary non-negative integers n, m. To clarify the exposition, we first consider the moments of $\hat{\zeta}_\varepsilon(t, \xi)$, that is, set $m = 0$ above, starting with the case $n = 1$, $m = 0$. Next, we compute the limit of the moments of the form $\mathbb{E}\{[\hat{\zeta}_\varepsilon(t, \xi)]^n\}$, using the expansion into pairings. The key observation is that the products of the form $\mathbb{E}\{\hat{\zeta}^{\varepsilon}_{j_1}(t,\xi)\hat{\zeta}^{\varepsilon}_{j_2}(t,\xi)\ldots\hat{\zeta}^{\varepsilon}_{j_n}(t,\xi)\}$ are approximately equal to $\mathbb{E}\{\hat{\zeta}^{\varepsilon}_{j_1}(t,\xi)\}$ $E\{\hat{\zeta}^{\varepsilon}_{j_2}(t,\xi)\}\ldots\mathbb{E}\{\hat{\zeta}^{\varepsilon}_{j_n}(t,\xi)\}$ due to an oscillatory phase. The next step is to pass to the mixed moments with $m \neq 0$. There, we first consider the second absolute moment $\mathbb{E}\{|\hat{\zeta}_{\varepsilon}(t,\xi)|^2\}$ and prove that in the limit it converges to the solution of the spatially homogeneous radiative transport equation (1.9). In the general case, we prove that $\mathbb{E}([\hat{\zeta}_{\varepsilon}(t,\xi)]^n[\hat{\zeta}_{\varepsilon}(t,\xi)]^m)$ behaves as $\varepsilon \to 0$ as $\mathbb{E}(Z(t,\xi)^n Z(t,\xi)^{*m})$, where $Z(t,\xi)$ is a complex Gaussian with mean and variance as described in Theorem 1.1. Later, after we introduce the appropriate terminology, we will describe more precisely the exact role of the oscillatory phase in the last step of computing the general moment.

3.1. Convergence of the expectation $\mathbb{E}(\hat{\zeta}_{\varepsilon}(t,\xi))$

The main result of this section is the following

Proposition 3.1. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}_{\varepsilon}(t,\xi) = \hat{\phi}_0(\xi)\mathrm{e}^{-tD_{\xi}/2}, \tag{3.1}$$

for all $t \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^d\backslash\{0\}$.

The initial step in the proof is the following uniform bound for the individual terms of (2.6).

Proposition 3.2. *For all* $T > 0$, $n \geqq 0$ *and all* $\xi \in \mathbb{R}^d\backslash\{0\}$, *there exists a constant* $C(T;\xi)$ *such that*

$$\sup_{t\in[0,T]}|\mathbb{E}\hat{\zeta}^{\varepsilon}_n(t,\xi)| \leqq \frac{C^n(T;\xi)}{n!} \tag{3.2}$$

for all $\varepsilon \in (0,1]$.

As a consequence, we may interchange the limit $\varepsilon \downarrow 0$ and the summation in n.

Corollary 3.3. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}_{\varepsilon}(t,\xi) = \sum_{n=0}^{\infty}\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}^{\varepsilon}_n(t,\xi), \tag{3.3}$$

for all $t \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^d\backslash\{0\}$.

Next, we identify the limit of the individual terms in the right side of (3.3).

Proposition 3.4. *We have* $\mathbb{E}\hat{\zeta}^{\varepsilon}_n(t,\xi) = 0$ *when n is odd and*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}^{\varepsilon}_{2n}(t,\xi) = \frac{1}{n!}\left(\frac{-tD(\xi)}{2}\right)^n\hat{\phi}_0(\xi) \tag{3.4}$$

for all $t \in \mathbb{R}$, $n \in \mathbb{N}$ *and* $\xi \in \mathbb{R}^d\backslash\{0\}$.

This together with (3.3) implies convergence of the expectation in (3.1).

Proof of Proposition 3.2

Of course, only the case of even n-s requires a proof, as the expectation vanishes for n odd. Note that

$$
\begin{aligned}
|\mathbb{E}\hat{\zeta}^{\varepsilon}_{2n}(t,\xi)| &= \left[\frac{1}{\varepsilon^{1/2}(2\pi)^d}\right]^{2n} \Bigg| \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \\
&\quad \times \int \mathbb{E}\left[\hat{V}\left(\frac{s_1}{\varepsilon}, \mathrm{d}p_1\right)\dots\hat{V}\left(\frac{s_{2n}}{\varepsilon}, \mathrm{d}p_{2n}\right)\right]\hat{\phi}_0(\xi - p_1 - \cdots - p_{2n}) \\
&\quad \times \mathrm{e}^{iG_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon}\Bigg| \\
&\leqq \frac{C^n\|\hat{\phi}_0\|_\infty}{\varepsilon^n} \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \int \left|\mathbb{E}\left[\hat{V}\left(\frac{s_1}{\varepsilon}, \mathrm{d}p_1\right)\dots\hat{V}\left(\frac{s_{2n}}{\varepsilon}, \mathrm{d}p_{2n}\right)\right]\right| \\
&= \frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!\varepsilon^n} \int_0^t \dots \int_0^t \mathrm{d}\mathbf{s}^{(2n)} \int \left|\mathbb{E}\left[\hat{V}\left(\frac{s_1}{\varepsilon}, \mathrm{d}p_1\right)\dots\hat{V}\left(\frac{s_{2n}}{\varepsilon}, \mathrm{d}p_{2n}\right)\right]\right|.
\end{aligned}
\tag{3.5}
$$

The last step above uses the symmetry of the integrand in $s_1, \dots, s_{2n}$ which brings about the factorial in the dominator. Using the relation

$$
\mathbb{E}\left[\hat{V}(t,\mathrm{d}p)\hat{V}(s,\mathrm{d}q)\right] = (2\pi)^d \mathrm{e}^{-\mathfrak{g}(p)|t-s|}\delta(p+q)\hat{R}(p)\mathrm{d}p\mathrm{d}q, \tag{3.6}
$$

and the rules of computing the $2n$-th joint moment of mean zero Gaussian random variables, we conclude that the utmost right-hand side of (3.5) can be estimated by

$$
\frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!\varepsilon^n} \sum_{\mathcal{F}} \int_0^t \dots \int_0^t \mathrm{d}\mathbf{s}^{(2n)} \int \mathrm{d}\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} \mathrm{e}^{-\mathfrak{g}(p_k)|s_k - s_l|/\varepsilon}\delta(p_k + p_l)\hat{R}(p_k), \tag{3.7}
$$

where the summation extends over all pairings formed over vertices $1, \dots, 2n$. We recall that a pairing for the set $S = \{1, 2, \dots, 2n\}$ is a partition of S into n pairs of numbers (lr), such that each element of S appears in exactly one of the pairs. If a pair (lr) is present in a given pairing $\mathcal{F}$ and $l < r$, we say that l is a left vertex and r is a right vertex.

Changing variables $s_k' := s_k/\varepsilon$ we obtain that expression (3.7) equals

$$
\begin{aligned}
&\frac{C^n\|\phi_0\|_\infty}{(2n)!} \sum_{\mathcal{F}} \int \mathrm{d}\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} \left[\varepsilon\int_0^{t/\varepsilon}\int_0^{t/\varepsilon} \mathrm{e}^{-\mathfrak{g}(p_k)|s_k - s_l|}\mathrm{d}s_k\mathrm{d}s_l\right]\delta(p_k+p_l)\hat{R}(p_k) \\
&\leqq \frac{C^n t^n\|\phi_0\|_\infty}{(2n)!} \sum_{\mathcal{F}} \int \prod_{(k,l)\in\mathcal{F}} \delta(p_k + p_l)\frac{\hat{R}(p_k)}{\mathfrak{g}(p_k)}\mathrm{d}\mathbf{p}^{(2n)} \\
&= \frac{C^n t^n\|\hat{\phi}_0\|_\infty}{2^n n!}\left[\int \frac{\hat{R}(p)}{\mathfrak{g}(p)}\mathrm{d}p\right]^n.
\end{aligned}
\tag{3.8}
$$

In the last step above we used the fact that the total number of possible pairings for a set of $2n$ elements is $(2n-1)!!$. Now, the conclusion of Proposition 3.2 follows. □

The above argument actually shows the following.

Proposition 3.5. *There exists a constant C such that for all $n \geqq 1$, $t > 0$, $\varepsilon \in (0, 1]$*

$$\sum_{\mathcal{F}} \int \cdots \int_{\Delta_{2n(t)}} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k - s_l|/\varepsilon} \delta(p_k + p_l) \hat{R}(p_k) \leqq \frac{(Ct\varepsilon)^n}{n!}, \tag{3.9}$$

where the summation extends over all pairings formed over $\{1, \ldots, 2n\}$.

Proof of Proposition 3.4

Let us introduce some terminology: the pairing $(1, 2), \ldots, (2n-1, 2n)$ shall be called a *time-ordered pairing*. For a given pairing $\mathcal{F}$ we let

$$\mathcal{I}_\varepsilon(t; \mathcal{F}) := \int_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k - s_l|/\varepsilon} \delta(p_k + p_l) \hat{R}(p_k). \tag{3.10}$$

As a conclusion of Proposition 3.5, we obtain, in particular, that

$$\mathcal{I}(\mathcal{F}) = \limsup_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-n} \mathcal{I}_\varepsilon(t; \mathcal{F}) < +\infty, \tag{3.11}$$

for any pairing $\mathcal{F}$. We will now show that $\mathcal{I}(\mathcal{F}) = 0$ if $\mathcal{F}$ is not a time-ordered pairing, and then identify the actual limit of $\varepsilon^{-n}\mathcal{I}_\varepsilon(\mathcal{F})$ for the time-ordered pairings, completing the proof of Proposition 3.4. We start with non-time-ordered pairings.

Lemma 3.6. *Suppose that $\mathcal{F}$ is not a time-ordered pairing. Then,*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-n} \mathcal{I}_\varepsilon(t; \mathcal{F}) = 0, \tag{3.12}$$

for any $T > 0$.

Proof. This lemma shall be proved by induction on n – the number of edges of a pairing. First, we verify it for $n = 2$. Then, we have to consider two pairings, $\mathcal{F}_1 = \{(1, 3), (2, 4)\}$ and $\mathcal{F}_2 = \{(1, 4), (2, 3)\}$. Start with the first one. Suppose that $\kappa \in (0, 1)$ and consider the sets of the following times: $A_1 = [|s_1 - s_3| \geqq \varepsilon^\kappa]$ and $A_2 = [|s_2 - s_4| \geqq \varepsilon^\kappa]$, as well as $A_3 = A_1^c \cup A_2^c$. Consider the expressions

$$I_i(\varepsilon) = \int_{\Delta_4(t)\cap A_i} ds_1 \ldots ds_4 \int dp_1 dp_2$$
$$\times \exp\{-[\mathfrak{g}(p_1)(s_1 - s_3) + \mathfrak{g}(p_2)(s_2 - s_4)]/\varepsilon\} \hat{R}(p_1)\hat{R}(p_2),$$

for $i = 1, 2, 3$, then

$$\mathcal{I}_\varepsilon(t; \mathcal{F}_1) \leqq \sum_{i=1}^{3} I_i(\varepsilon).$$

We will see that $I_1(\varepsilon)$ and $I_2(\varepsilon)$ are small because the integrand is exponentially small in ε, while $I_3(\varepsilon)$ because the domain of integration is small. Indeed, observe that

$$\begin{aligned} I_1(\varepsilon) &\leqq \int_0^t \int_0^t \mathrm{d}s_1 \mathrm{d}s_3 \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{d}s_2 \mathrm{d}s_4 \\ &\quad \times \int \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{e}^{-\varepsilon^{\kappa-1}\mathfrak{g}(p_1)/2} \mathrm{e}^{-[\mathfrak{g}(p_1)|s_1-s_3|+\mathfrak{g}(p_2)|s_2-s_4|]/(2\varepsilon)} \hat{R}(p_1)\hat{R}(p_2) \\ &= (2t\varepsilon)^2 \int \mathrm{e}^{-\varepsilon^{\kappa-1}\mathfrak{g}(p_1)/2} \frac{\hat{R}(p_1)\mathrm{d}p_1}{\mathfrak{g}(p_1)} \int \frac{\hat{R}(p_2)\mathrm{d}p_2}{\mathfrak{g}(p_2)} \end{aligned}$$

and it follows from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-2} I_1(\varepsilon) = 0. \tag{3.13}$$

Similarly, one can prove that (3.13) holds for $I_2(\varepsilon)$. On the other hand, we note that if $0 \leqq s_1 - s_3 \leqq \varepsilon^\kappa$ and $0 \leqq s_2 - s_4 \leqq \varepsilon^\kappa$ (so that $(s_1, s_2, s_3, s_4) \in A_3$) then (since $0 \leqq s_3 \leqq s_2$), we have $0 \leqq s_1 - s_4 \leqq 2\varepsilon^\kappa$ as well. Hence,

$$I_3(\varepsilon) \leqq Ct\varepsilon^{3\kappa}$$

and (3.13) follows for $I_3(\varepsilon)$, provided that $\kappa > 2/3$. We have shown in this way that

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-2} \mathcal{I}(t;\mathcal{F}_1) = 0.$$

A similar argument also yields an analogous statement for $\mathcal{I}(t;\mathcal{F}_2)$.

Assume, now, for the sake of the induction argument that (3.12) holds for some $n \geqq 2$ and for all non-time-ordered pairings with $2k$ vertices with $k \leqq n$. Let $\mathcal{F}$ be a non-time-ordered pairing consisting of $n+1$ edges. As before, we choose $\kappa \in (0,1)$ which shall be specified later. For a given edge $e = (k_0, l_0)$ of a pairing $\mathcal{F}$ set

$$A(e) = [|s_{k_0} - s_{l_0}| \geqq \varepsilon^\kappa] \subseteq \Delta_{2n+2}(t),$$

and $A(\mathfrak{e}) = \bigcup_{e\in\mathcal{F}} A^c(e)$. Define also, again for $e \in \mathcal{F}$,

$$I_e(\varepsilon) = \int_{\Delta_{2n}(t)\cap A(e)} \mathrm{d}\mathbf{s}^{(2n+2)} \int \mathrm{d}\mathbf{p}^{(2n+2)} \prod_{(k,l)\in\mathcal{F}} \mathrm{e}^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon} \delta(p_k+p_l)\hat{R}(p_k),$$

as well as

$$I_{\mathfrak{e}}(\varepsilon) = \int_{\Delta_{2n}(t)\cap A(\mathfrak{e})} \mathrm{d}\mathbf{s}^{(2n+2)} \int \mathrm{d}\mathbf{p}^{(2n+2)} \prod_{(k,l)\in\mathcal{F}} \mathrm{e}^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon} \delta(p_k+p_l)\hat{R}(p_k).$$

Note that for the first term, as in the computation for the pairing $\mathcal{F}_1$ with $n = 2$, we have

$$
\begin{aligned}
I_e(\varepsilon) &\leqq \int_0^t \dots \int_0^t \mathrm{d}\mathbf{s}^{(2n+2)} \int \mathrm{d}\mathbf{p}^{(2n+2)} \mathrm{e}^{-\mathfrak{g}(p_{k_0})\varepsilon^{\kappa-1}/2} \\
&\quad \times \prod_{(k,l)\in\mathcal{F}} \mathrm{e}^{-\mathfrak{g}(p_k)|s_k-s_l|/(2\varepsilon)} \delta(p_k+p_l)\hat{R}(p_k) \\
&\leqq \int \mathrm{d}\mathbf{p}^{(2n+2)} \mathrm{e}^{-\mathfrak{g}(p_{k_0})\varepsilon^{\kappa-1}/2} \\
&\quad \times \prod_{(k,l)\in\mathcal{F}} \delta(p_k+p_l)\hat{R}(p_k) \prod_{(k,l)\in\mathcal{F}} \int_0^t \mathrm{d}s_k \int_{\mathbb{R}} \mathrm{d}s_l \mathrm{e}^{-\mathfrak{g}(p_k)|s_k-s_l|/(2\varepsilon)} \\
&= (2t\varepsilon)^{n+1} \left[\int \frac{\hat{R}(p)\mathrm{d}p}{\mathfrak{g}(p)} \right]^n \int \mathrm{e}^{-\mathfrak{g}(p)\varepsilon^{\kappa-1}/2} \frac{\hat{R}(p)\mathrm{d}p}{\mathfrak{g}(p)},
\end{aligned}
$$

thus the term in the exponent is very large and negative, whence

$$\lim_{\varepsilon\to 0} \varepsilon^{-(n+1)} I_e(\varepsilon) = 0.$$

On the other hand, for $I_{\mathfrak{e}}(\varepsilon)$ we have two possibilities: either it splits into a union of two sub-pairings or not. More precisely, either (1) there exists m_0 such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_i$, $i = 1, 2$ are pairings formed over $\{1, \dots, 2m_0\}$ and $\{2m_0+1, \dots, 2n+2\}$ respectively, or (2) there exists a sequence of edges $e_i = (k_i, l_i)$, $i = 1, \dots, m$ such that $k_1 = 1$, $k_{i+1} < l_i < l_{i+1}$, for $i = 1, \dots, m-1$, and $l_m = 2n+2$. In the first case we have

$$
\begin{aligned}
I_{\mathfrak{e}}(\varepsilon) &\leqq \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{2m_0-1}} \mathrm{d}s_{2m_0} \int \mathrm{d}p_1 \dots \mathrm{d}p_{2m_0} \\
&\quad \times \prod_{(k,l)\in\mathcal{F}_1} \mathrm{e}^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon} \delta(p_k+p_l)\hat{R}(p_k) \\
&\quad \times \left\{ \int_{\Delta_{2(n+1-m_0)}(s_{2m_0})} \mathrm{d}s_{2m_0+1} \dots \mathrm{d}s_{2n+2} \int \mathrm{d}p_{2m_0+1} \dots \mathrm{d}p_{2n+2} \right. \\
&\quad \left. \times \prod_{(k,l)\in\mathcal{F}_2} \mathrm{e}^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon} \delta(p_k+p_l)\hat{R}(p_k). \right\}.
\end{aligned}
$$

Hence,

$$I_{\mathfrak{e}}(\varepsilon) \leqq \mathcal{I}_\varepsilon(t;\mathcal{F}_1)\mathcal{I}_\varepsilon(t;\mathcal{F}_2),$$

and thus (3.12) holds in light of the induction hypothesis. In the second case, for $\mathbf{s}^{(2n+2)} \in A(\mathfrak{e})$ we have $0 \leqq s_1 - s_{2n+2} \leqq m\varepsilon^\kappa$, therefore

$$\mathcal{I}_\varepsilon(t;\mathcal{F}) \leqq Ct\varepsilon^{(2n+1)\kappa}$$

and (3.12) holds (with n replaced by $n+1$), provided that $\kappa > (n+1)/(2n+1)$. □

The contribution of the time-ordered pairings

The last step in the proof of Proposition 3.4 is to consider the contribution of the time-ordered pairings. We have shown so far that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}^{\varepsilon}_{2n}(t,\xi)=J_n(t,\xi), \tag{3.14}$$

where

$$J_n(t,\xi)=\hat{\phi}_0(\xi)\lim_{\varepsilon\downarrow 0}\frac{(-1)^n}{[\varepsilon(2\pi)^d]^n}\int_{\Delta_{2n}(t)}\mathrm{d}\mathbf{s}^{(2n)}\int\mathrm{d}\mathbf{p}^{(2n)}\prod_{k=1}^{n}\hat{R}(p_{2k-1})\delta(p_{2k-1}+p_{2k})$$
$$\times\mathrm{e}^{-\mathfrak{g}(p_{2k-1})(s_{2k-1}-s_{2k})/\varepsilon}\exp\left\{iG_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon\right\} \tag{3.15}$$

where $G_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})$ is given by (2.4). For a time-ordered pairing, taking into account the delta-functions, we have

$$G_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})=\sum_{m=1}^{n}\left[\xi\cdot p_{2m-1}-\frac{1}{2}|p_{2m-1}|^2\right](s_{2m-1}-s_{2m}).$$

Hence, (3.15) can be written as

$$J_n(t,\xi)=\hat{\phi}_0(\xi)\lim_{\varepsilon\downarrow 0}\frac{(-1)^n}{[\varepsilon(2\pi)^d]^n}\int_{\Delta_{2n}(t)}\mathrm{d}\mathbf{s}^{(2n)}\int\mathrm{d}\mathbf{p}^{(2n)}$$
$$\times\prod_{k=1}^{n}\hat{R}(p_{2k-1})\delta(p_{2k-1}+p_{2k})\mathrm{e}^{-Q(p_{2k-1})(s_{2k-1}-s_{2k})/\varepsilon}, \tag{3.16}$$

with

$$Q(p)=\mathfrak{g}(p)-i\left(\xi\cdot p-\frac{1}{2}|p|^2\right).$$

Changing variables $s'_{2m}=(s_{2m-1}-s_{2m})/\varepsilon$ we obtain, after dropping the primes:

$$J_n(t,\xi)=\hat{\phi}_0(\xi)\lim_{\varepsilon\downarrow 0}\frac{(-1)^n}{(2\pi)^{nd}}\int_0^t\mathrm{d}s_1\int_0^{s_1/\varepsilon}\mathrm{d}s_2\int_0^{s_1-\varepsilon s_2}\mathrm{d}s_3\dots$$
$$\times\int_0^{s_{2n-3}-\varepsilon s_{2n-2}}\mathrm{d}s_{2n-1}\int_0^{s_{2n-1}/\varepsilon}\mathrm{d}s_{2n}$$
$$\times\int\dots\int\prod_{k=1}^{n}\hat{R}(p_{2k-1})\mathrm{d}p_{2k-1}\prod_{k=1}^{n}\mathrm{e}^{-Q(p_{2k-1})s_{2k}}. \tag{3.17}$$

One can now compute the limit in (3.17):

$$J_n(t,\xi)=\hat{\phi}_0(\xi)\frac{(-1)^n}{(2\pi)^{nd}}\int_0^t\mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_3\dots$$
$$\times\int_0^{s_{2n-3}}\mathrm{d}s_{2n-1}\int\dots\int\prod_{k=1}^{n}\frac{\hat{R}(p_{2k-1})}{Q(p_{2k-1})}\mathrm{d}p_{2k-1}$$
$$=\hat{\phi}_0(\xi)\frac{(-1)^n t^n}{(2\pi)^{nd}n!}\left(\int\frac{\hat{R}(p)}{Q(p)}\mathrm{d}p\right)^n=\hat{\phi}_0(\xi)\frac{(-tD(\xi))^n}{2^n n!}, \tag{3.18}$$

where $D(\xi)$ is given by (1.7).

This completes the proof of Proposition 3.4. □

3.2. The limit of an arbitrary moment

As the next step we compute the limit of moments of $\hat{\zeta}_\varepsilon(t,\xi)$, without the powers of $\hat{\zeta}^*_\varepsilon(t,\xi)$ present. We denote the right-hand side of (3.1) by $\bar{\zeta}(t,\xi)$.

Proposition 3.7. *We have, for all $t \geqq 0$, a positive integer N and $\xi \neq 0$:*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left\{[\hat{\zeta}_\varepsilon(t,\xi)]^N\right\} = [\bar{\zeta}(t,\xi)]^N. \tag{3.19}$$

Proof. Consider the expansion

$$\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N = \sum_{n_1,\ldots,n_N=0}^{\infty} \hat{\zeta}^\varepsilon_{n_1}(t,\xi)\ldots\hat{\zeta}^\varepsilon_{n_N}(t,\xi), \tag{3.20}$$

where each term $\hat{\zeta}^\varepsilon_n(t,\xi)$ is given by (2.3). Evaluating the expectation in (3.20) and using an argument as in the proof of part (ii) of Proposition 2.1 gives

$$\mathbb{E}\left\{\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N\right\} = \sum_{n_1,\ldots,n_N=0}^{\infty} J^\varepsilon_{n_1,\ldots,n_N}(t,\xi), \tag{3.21}$$

where

$$J^\varepsilon_{n_1,\ldots,n_N}(t,\xi) = \mathbb{E}\left[\hat{\zeta}^\varepsilon_{n_1}(t,\xi)\ldots\hat{\zeta}^\varepsilon_{n_N}(t,\xi)\right], \tag{3.22}$$

or, equivalently,

$$\begin{aligned}
J^\varepsilon_{n_1\ldots n_N}(t,\xi) &= \left[\frac{i}{\varepsilon^{1/2}(2\pi)^d}\right]^{2n} \int\!\!\int_{D^t_{n_1\ldots n_N}} \mathrm{d}\mathbf{s}_1\ldots\mathrm{d}\mathbf{s}_N \\
&\quad\times \int \prod_{j=1}^{N}\left[\hat{\phi}_0(\xi - p_{j1} - \cdots - p_{jn_j})\mathrm{e}^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}\right] \\
&\quad\times\mathbb{E}\left[\hat{V}\left(\frac{s_{11}}{\varepsilon},\mathrm{d}p_{11}\right)\ldots\hat{V}\left(\frac{s_{1n_1}}{\varepsilon},\mathrm{d}p_{1n_1}\right)\right. \\
&\quad\left.\cdots\hat{V}\left(\frac{s_{N1}}{\varepsilon},\mathrm{d}p_{2N1}\right)\ldots\hat{V}\left(\frac{s_{Nn_N}}{\varepsilon},\mathrm{d}p_{Nn_N}\right)\right],
\end{aligned}$$

where $n_1 + \cdots + n_N = 2n$, $\mathbf{s}_j = (s_{j1},\ldots,s_{jn_j})$, $\mathbf{p}_j = (p_{j1},\ldots,p_{jn_j})$ and $D^t_{n_1\ldots n_N} := \Delta_{n_1}(t)\times\ldots\times\Delta_{n_N}(t)$. We evaluate the expectation using the pairings, as in (3.5), and get

$$J^\varepsilon_{n_1,n_2}(t,\xi) = \sum_{\mathcal{F}} J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F}). \tag{3.23}$$

Here the summation extends over all pairings formed over pairs of integers (jk), with $j = 1,\ldots,N$, and $k = 1,\ldots,n_j$. We introduce a lexicographical ordering between pairs, that is, we say that $(jk) \prec (j'k')$ if $j < j'$, or if $j = j'$, then $k \leqq k'$. If (e,f) is an edge of a pairing we say that e is a left vertex if $e \prec f$. Also, given a vertex $e = (jk)$ we will use the notation $s(e) = s_{jk}$, $p(e) = p_{jk}$. The following analog of Proposition 3.2 holds.

Proposition 3.8. *For each $T > 0$ there exist constants $J_{n_1\dots n_N}(t,\xi)$ such that*

$$\sup_{t\in[0,T]} |J^{\varepsilon}_{n_1\dots n_N}(t,\xi)| \leqq J_{n_1\dots n_N}(T,\xi), \quad \forall\varepsilon\in(0,1] \tag{3.24}$$

and

$$\sum_{n_1\dots n_N=0}^{+\infty} J_{n_1\dots n_N}(T,\xi) < +\infty.$$

Proof. Suppose that $n_1+\dots+n_N = 2n$. Estimates following (3.5) essentially hold without changes, that is, we start with

$$\begin{aligned}
|J^{\varepsilon}_{n_1\dots n_N}(t,\xi)| &\leqq \frac{C^n\|\hat\phi_0\|_\infty^N}{\varepsilon^n}\iint\limits_{D^t_{n_1\dots n_N}} \mathrm{d}\mathbf{s}_1\dots\mathrm{d}\mathbf{s}_N \\
&\quad\times\int\left|\mathbb{E}\left[\hat V\left(\frac{s_{11}}{\varepsilon},\mathrm{d}p_{11}\right)\dots\right.\right.\\
&\quad\left.\left.\times\hat V\left(\frac{s_{1n_1}}{\varepsilon},\mathrm{d}p_{1n_1}\right)\dots\hat V\left(\frac{s_{N1}}{\varepsilon},\mathrm{d}p_{N1}\right)\dots\hat V\left(\frac{s_{Nn_N}}{\varepsilon},\mathrm{d}p_{Nn_N}\right)\right]\right|\\
&\leqq \frac{C^n\|\hat\phi_0\|_\infty^N}{n_1!\dots n_N!\varepsilon^n}\int_0^t\dots\int_0^t \mathrm{d}\mathbf{s}_1\dots\mathrm{d}\mathbf{s}_N\\
&\quad\times\int\left|\mathbb{E}\left[\hat V\left(\frac{s_{11}}{\varepsilon},\mathrm{d}p_{11}\right)\dots\hat V\left(\frac{s_{1n_1}}{\varepsilon},\mathrm{d}p_{1n_1}\right)\cdots\right.\right.\\
&\quad\left.\left.\times\hat V\left(\frac{s_{N1}}{\varepsilon},\mathrm{d}p_{N1}\right)\dots\hat V\left(\frac{s_{Nn_N}}{\varepsilon},\mathrm{d}p_{Nn_N}\right)\right]\right|.
\end{aligned}$$

This can be estimated, as in (3.8), and we obtain

$$|J^{\varepsilon}_{n_1\dots n_N}(t,\xi)| \leqq \frac{C_T^n\|\hat\phi_0\|_\infty^N}{n_1!\dots n_N!}\#(\mathcal{F}),$$

where $\#(\mathcal{F})$ is the total number of the pairings, and is equal to

$$\#(\mathcal{F}) = (n_1+\dots+n_N-1)!! = (2n-1)!!.$$

We conclude that

$$|J^{\varepsilon}_{n_1\dots n_N}(t,\xi)| \leqq \frac{C_T^n(2n-1)!!}{n_1!\dots n_N!}\|\hat\phi_0\|_\infty^N. \tag{3.25}$$

On the other hand, we have

$$\sum_{n=0}^{\infty}\sum_{n_1+\dots+n_N=2n}\frac{C_T^n(2n-1)!!}{n_1!\dots n_N!} = \sum_{n=0}^{\infty}\frac{C_T^nN^{2n}(2n-1)!!}{(2n)!} = \sum_{n=0}^{\infty}\frac{(NC_T)^n}{n!} < +\infty,$$

and the conclusion of Proposition 3.8 follows. □

The non-trivial pairings

We now return to the proof of Proposition 3.7. As a consequence of Proposition 3.8, we may pass to the limit $\varepsilon \downarrow 0$ term-wise in the series (3.21):

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left\{\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N\right\} = \sum_{n_1\ldots n_N=0}^{\infty} \sum_{\mathcal{F}} \lim_{\varepsilon\downarrow 0} J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F}), \tag{3.26}$$

where $J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F})$ is given by

$$\begin{aligned} J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F}) = \frac{(-1)^n}{[(2\pi)^d\varepsilon]^n} &\iint_{D^t_{n_1\ldots n_N}} \mathrm{d}\mathbf{s}_1\ldots\mathrm{d}\mathbf{s}_N \int \mathrm{d}\mathbf{p}_1\ldots\mathrm{d}\mathbf{p}_N \\ &\times \prod_{(jk,j'm)\in\mathcal{F}} \left[\mathrm{e}^{-\mathfrak{g}(p_{jk})|s_{jk}-s_{j'm}|/\varepsilon}\hat{R}(p_{jk})\delta(p_{jk}+p_{j'm})\right] \\ &\times \prod_{j=1}^{N}\left[\mathrm{e}^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}\hat{\phi}_0(\xi-p_{j1}-\cdots-p_{jn_j})\right], \end{aligned} \tag{3.27}$$

and we only need to study the limit of $J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F})$ for a fixed pairing $\mathcal{F}$. Recall that in the case of the first moment of $\hat{\zeta}_\varepsilon$, that is, for $N=1$, this limit did not vanish only for the time-ordered pairings. We will show, in Lemma 3.9 below, for a general $N>1$ that only the following pairings $\mathcal{F}$ contribute to the limit: first, for any bond $(jk)(j'k')$ we must have $j=j'$ – that is, there are no bonds crossings between various $\hat{\zeta}^\varepsilon_j$. This imposes, in particular, the condition that all n_j have to be even, and also splits naturally $\mathcal{F}$ as a product $\mathcal{F}_1\otimes\cdots\otimes\mathcal{F}_N$. The second requirement is that each $\mathcal{F}_j$ is a time-ordered pairing on n_j vertices.

We now formalize this claim. Let Π be the set of all permutations of the vertices

$$\{(1;1),\ldots(1;n_1),\ldots,(N;1),\ldots,(N,n_N)\}.$$

We divide the domain of integration $D^t_{n_1\ldots n_N} = \Delta_{n_1}(t)\times\Delta_{n_N}(t)$ into the sets $\Delta(\sigma)$, $\sigma\in\Pi$ as follows: a point $\mathbf{s}:=(s_{11},\ldots,s_{1n_1},\ldots,s_{N1},\ldots,s_{Nn_N})\in\Delta(\sigma)$, if $s_{\sigma(1;1)}\geqq s_{\sigma(1;2)}\ldots\geqq s_{\sigma(N;n_N)}$ and $\mathbf{s}\in D^t_{n_1\ldots n_N}$. This gives rise to a decomposition $D^t_{n_1\ldots n_N}=\bigcup_{\sigma\in\Pi}\Delta(\sigma)$. Note that the set $\Delta(\sigma)$ may be empty for some permutations σ because $s_{jk}\leqq s_{jk'}$ for all $j=1,\ldots,N$ and $k\leqq k'$ if $\mathbf{s}\in D^t_{n_1\ldots n_N}$, hence, for instance, $s_{12}>s_{11}$ is impossible. Then we can write

$$J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F}) = \sum_{\sigma\in\Pi} J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F},\sigma),$$

where $J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F},\sigma)$ corresponds to the integration over $\Delta(\sigma)$. By the same argument as in the proof of Lemma 3.6, we can prove that that

$$\lim_{\varepsilon\downarrow 0} J^\varepsilon_{n_1\ldots n_N}(t,\xi;\mathcal{F},\sigma)=0,$$

unless $\mathcal{F}=\mathcal{F}_\sigma := (\sigma(1;1),\sigma(1;2))(\sigma(1;3),\sigma(1;4))\ldots(\sigma(N;n_N-1),\sigma(N;n_N))$, that is, for each domain $\Delta(\sigma)$ there is only one pairing that potentially may contribute to the limit, and such pairings are the analogs of the time-ordered pairings introduced before. It follows that

$$\bar{J}_{n_1\dots n_N}(t,\xi;\sigma)=\sum_{\mathcal{F}}\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1\dots n_N}(t,\xi;\mathcal{F},\sigma)=\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1\dots n_N}(t,\xi;\mathcal{F}_\sigma,\sigma).$$

Let $(\bar{e}_{2k-1},\bar{e}_{2k})$, $k=1,\dots,n$, be the edges of a time-ordered pairing $\mathcal{F}_\sigma$ as above, that is, $\bar{e}_1=\sigma(1;1)$, $\bar{e}_2=\sigma(1;2)$, and so on. We we will now show that, as we have claimed above, in order for the pairing to contribute to the limit all its edges must be of the form $(\bar{e}_{2k-1},\bar{e}_{2k})=((j;2l-1),(j;2l))$ for some $j=1,\dots,N$ and $l=1,\dots,[n_j/2]$, that is, no vertices corresponding to two different simplices should be paired. This, as we have mentioned, forces all n_j, $j=1,\dots,N$ to be even.

Lemma 3.9. *All pairings $\mathcal{F}_\sigma$ containing an edge of the form $(\bar{e}_{2k-1},\bar{e}_{2k})=((j;i_1),(j';i_2))$, where $j\neq j'$, satisfy*

$$\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1\dots n_N}(t,\xi;\mathcal{F}_\sigma,\sigma)=0. \tag{3.28}$$

It follows from Lemma 3.9 that

$$\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1\dots n_N}(t,\xi;\mathcal{F})=\prod_{j=1}^{N}J_{n_j}(t,\xi), \tag{3.29}$$

where $\mathcal{F}$ is a pairing that is the union of N time-ordered pairings formed over the sets

$$\{(1;1),\dots,(1;n_1)\},\dots,\{(N;1),\dots,(N;n_N)\}.$$

In all other cases

$$\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1\dots n_N}(t,\xi;\mathcal{F})=0.$$

Now, the conclusion of Proposition 3.7 follows. It remains only to prove Lemma 3.9. □

Proof of Lemma 3.9 Let us illustrate what happens in the simplest example when $N=2$, and $n_1=n_2=1$. The corresponding term is

$$\hat{\zeta}^{\varepsilon}_1(t,\xi)=\frac{1}{i\sqrt{\varepsilon}}\int_0^t\int\frac{\hat{V}(s_1/\varepsilon,\mathrm{d}p_1)}{(2\pi)^d}\mathrm{e}^{i(|\xi|^2-|\xi-p_1|^2)s_1/(2\varepsilon)}\hat{\phi}_0(\xi-p_1)\mathrm{d}s_1.$$

The only possible pairing in $\mathbb{E}\{(\hat{\zeta}^{\varepsilon}_1(t,\xi))^2\}$ is $((1;1)(2;1))$, which has $j=1\neq j'=2$, and we have

$$\begin{aligned}\mathbb{E}\left\{(\hat{\zeta}^{\varepsilon}_1(t,\xi))^2\right\}&=-\frac{1}{\varepsilon}\int_0^t\int_0^t\int\frac{\mathrm{e}^{-\mathfrak{g}(p_1)|s_1-s_2|/\varepsilon}\hat{R}(p_1)}{(2\pi)^d}\\&\quad\times\mathrm{e}^{i(|\xi|^2-|\xi-p_1|^2)s_1/(2\varepsilon)}\mathrm{e}^{i(|\xi|^2-|\xi+p_1|^2)s_2/(2\varepsilon)}\\&\quad\times\hat{\phi}_0(\xi-p_1)\hat{\phi}_0(\xi+p_1)\mathrm{d}p_1\mathrm{d}s_1\mathrm{d}s_2\\&=\int\mathcal{K}_\varepsilon(p_1,t)\hat{\phi}_0(\xi-p_1)\hat{\phi}_0(\xi+p_1)\frac{\hat{R}(p_1)}{(2\pi)^d}\mathrm{d}p_1,\end{aligned}$$

where

$$K_\varepsilon(p_1,t) := \int_0^t \mathrm{d}s_2 \mathrm{e}^{-i|p_1|^2 s_2/\varepsilon} \left\{ \int_0^{t/\varepsilon} \mathrm{d}s_1 \int \exp\{-\mathfrak{g}(p_1)|s_1 - s_2/\varepsilon| \right.$$
$$\left. +i(\xi \cdot p_1 - |p_1|^2/2)(s_1 - s_2/\varepsilon)\} \right\}.$$

It is clear that $|K_\varepsilon(p,t)| \leqq t\mathfrak{g}^{-1}(p_1)$ for all $\varepsilon \in (0,1]$, $t \in \mathbb{R}$ hence by virtue of Lebesgue dominated convergence theorem and (1.10), we can write that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left\{(\hat{\zeta}_1^\varepsilon(t,\xi))^2\right\} = \int \left[\lim_{\varepsilon\downarrow 0} K_\varepsilon(p_1,t)\right] \hat{\phi}_0(\xi - p_1)\hat{\phi}_0(\xi + p_1)\frac{\hat{R}(p_1)}{(2\pi)^d}\mathrm{d}p_1, \tag{3.30}$$

provided we can argue that the limit $\lim_{\varepsilon\downarrow 0} \mathcal{K}_\varepsilon(p_1,t)$ exists almost everywhere in p_1. On the other hand, since

$$\left| \int_{x_1}^{x_2} \mathrm{e}^{iax}\, \mathrm{d}x \right| \leqq \frac{4}{a},$$

for all $x_1, x_2 \in \mathbb{R}$, $a \neq 0$, it follows that

$$\begin{aligned} K_\varepsilon(p_1,t) &= \int_0^t \mathrm{d}s_2 \int_{-s_2/\varepsilon}^{(t-s_2)/\varepsilon} \mathrm{d}s_1 \int \exp\{-\mathfrak{g}(p_1)|s_1| \\ &\quad +i(\xi\cdot p_1 - |p_1|^2/2)s_1 - i|p_1|^2 s_2/\varepsilon\} \\ &= \int_{-t/\varepsilon}^{t/\varepsilon} \exp\left\{-\mathfrak{g}(p_1)|s_1| + i(\xi\cdot p_1 - |p_1|^2/2)s_1\right\} \\ &\quad \times \left(\int_{-\varepsilon s_1}^{t-\varepsilon s_1} \exp\{-i|p_1|^2 s_2/\varepsilon\}\mathrm{d}s_2\right)\mathrm{d}s_1 \end{aligned}$$

satisfies

$$|K_\varepsilon(p_1,t)| \leqq \frac{C\varepsilon}{|p_1|^2}\int_{-\infty}^{\infty} \exp\{-\mathfrak{g}(p_1)|s_1|\}\mathrm{d}s_1 \leqq \frac{C\varepsilon}{|p_1|^2\mathfrak{g}(p_1)} \to 0$$

as $\varepsilon \downarrow 0$ for each $p_1 \neq 0$. Hence we conclude from (3.30) that $\lim_{\varepsilon\downarrow 0}\mathbb{E}\{(\hat{\zeta}_1^\varepsilon(t,\xi))^2\} = 0$. The reason this limit vanishes is the oscillatory phase in the expression for $K_\varepsilon(p_1,t)$.

The general case in (3.28) proceeds along the lines of the above computation for $\mathbb{E}[(\hat{\zeta}_1^\varepsilon(t,\xi))^2]$, using the oscillatory phase. Consider the edge $(\bar{e}_{2k-1}, \bar{e}_{2k})$ corresponding to the smallest values of such "mixed" s. That is, all smaller times are "non-mixed" (they are paired to the next time in their simplex). To avoid even lengthier notation we will assume that all of those smaller times come from the same simplex: $s(\bar{e}_{2k-1}) \geqq s(\bar{e}_{2k}) \geqq s_{j,r} \geqq \cdots \geqq s_{j,n_j}$, and to fix our attention we let $j = N$, $\bar{e}_{2k-1} = (1; n_1)$, $\bar{e}_{2k} = (N; r-1)$ and $s_{1n_1} \geqq s_{N,r-1}$. The other cases can be argued in the same way. Note that then $n_N - r + 1 = 2n - 2k$ (recall that $n_1 + \cdots + n_N = 2n$) has to be even, and we should also have $j = 1$, $j' = N$, $i_2 = r - 1$ and $i_1 = n_1$. Let us denote

$\mathrm{d}\mathbf{s}'_{j,m} = ds_{j,1}\ldots \mathrm{d}s_{j,m}$, $\mathrm{d}\mathbf{s}''_{j,m} = ds_{j,m}\ldots \mathrm{d}s_{j,n_j}$, with $j = 1,\ldots,N$, and

$$\Delta'_m(t;\sigma) = [t \geqq s(\bar{e}_1) \geqq \cdots s(\bar{e}_{2m}) \geqq 0],$$
$$\Delta''_m(t;\sigma) = [t \geqq s(\bar{e}_{2m}) \geqq \cdots s(\bar{e}_{2n}) \geqq 0].$$

Denote also by $G_m(\mathbf{s},\mathbf{p})$ the expression (2.4), where the range of summation has been restricted to $k = 1,\ldots,m$ and by $G'_{nm}(\mathbf{s},\mathbf{p}) := G_n(\mathbf{s},\mathbf{p}) - G_m(\mathbf{s},\mathbf{p})$. Using (3.27) we can write

$$\begin{aligned} J^{\varepsilon}_{n_1\ldots n_N}(t,\xi;\mathcal{F}_\sigma) &= \frac{(-1)^n}{[(2\pi)^d\varepsilon]^n}\int\prod_{m=1}^{n}\Big[\hat{R}(p_{\bar{e}_{2m-1}})\delta(p_{\bar{e}_{2m-1}}+p_{\bar{e}_{2m}})\Big]\mathrm{d}\mathbf{p}_1\ldots\mathrm{d}\mathbf{p}_N\\ &\times\prod_{j=1}^{N}\hat{\phi}_0(\xi - p_{j1}-\cdots-p_{jn_j})\int_{\Delta'_{k-1}(t;\sigma)}\mathrm{d}\mathbf{s}'_{1,n_1-1}\mathrm{d}\mathbf{s}_2\ldots\mathrm{d}\mathbf{s}_{N-1}\mathrm{d}\mathbf{s}'_{N,r-2}\\ &\times\prod_{m=1}^{k-1}\Big[\mathrm{e}^{-\mathfrak{g}(p_{\bar{e}_{2m-1}})|s_{\bar{e}_{2m-1}}-s_{\bar{e}_{2m}}|/\varepsilon}\Big]\exp\left\{i\sum_{j=2}^{N-1}G_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon\right\}\\ &\times\mathrm{e}^{iG_{n_1-1}(\mathbf{s}_1,\mathbf{p}_1)/\varepsilon}\mathrm{e}^{iG_{r-2}(\mathbf{s}_N,\mathbf{p}_N)/\varepsilon}\\ &\times\int_0^{s(\bar{e}_{2k-2})}\mathrm{d}s_{1,n_1}\int_0^{s_{1,n_1}}\mathrm{d}s_{2,r-1}\mathrm{e}^{-\mathfrak{g}(p_{1,n_1})(s_{1,n_1}-s_{N,r-1})/\varepsilon}\\ &\times\exp\left\{i\left[\xi\cdot p_{1,n_1}+\frac{1}{2}|p_{1,n_1}|^2 - p_{1,n_1}\cdot\left(\sum_{m=1}^{n_1}p_{1,m}\right)\right](s_{1,n_1}-s_{N,r-1})/\varepsilon\right\}\\ &\times\exp\left\{ip_{1,n_1}\cdot\left(-\sum_{m=1}^{n_1-1}p_{1,m}+\sum_{m=1}^{r-1}p_{N,m}\right)s_{N,r-1}/\varepsilon\right\}\mathcal{I}_\varepsilon(s_{N,r-1},\mathbf{p}_N), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned}\mathcal{I}_\varepsilon(s_{N,r-1},\mathbf{p}_N) := \int_{\Delta''_{k-1}(s_{N,r-1};\sigma)} & \mathrm{e}^{iG'_{n_N,r}(\mathbf{s}''_{N,r},\mathbf{p}_N)/\varepsilon}\\ &\times\prod_{m=0}^{(n_N-r-1)/2}\mathrm{e}^{-\mathfrak{g}(p_{N,r+2m})(s_{N,r+2m}-s_{N,r+2m+1})/\varepsilon}\mathrm{d}\mathbf{s}''_{N,r}.\end{aligned}$$

Observe that

$$G'_{n_N,r}(\mathbf{s}''_{N,r},\mathbf{p}_N) = \sum_{m=0}^{(n_N-r-1)/2} C_{r,m}(\mathbf{p}_N)(s_{N,r+2m}-s_{N,r+2m+1}),$$

where

$$C_{r,m}(\mathbf{p}_N) := (\xi\cdot p_{N,r+2m}) - \left(\sum_{j=1}^{r-1}p_{N,j}\right)\cdot p_{N,r+2m} - \frac{1}{2}|p_{N,r+2m}|^2.$$

Performing the change of variables $s'_{N,r+l} = s_{N,r+l}/\varepsilon$, and then subsequently $s''_{N,r+2m+1} := s'_{N,r+2m} - s'_{N,r+2m+1}$, $s''_{N,r+2m} := s'_{N,r+2m}$ for the variables "following" the edge $(\bar{e}_{2k-1}, \bar{e}_{2k})$, that is, for $l = r, \dots, n_N$, we conclude that

$$\begin{aligned}
\mathcal{I}_\varepsilon(s_{N,r-1}, \mathbf{p}_N) &= \varepsilon^{n_N-r+1} \int_{\Delta_{n_N-r+1}(s_{N,r-1}/\varepsilon)} \\
&\quad \times \prod_{m=0}^{(n_N-r-1)/2} \mathrm{e}^{[-\mathfrak{g}(p_{N,r+2m})+iC_{r,m}(\mathbf{p}_N)]s_{N,r+2m}} \mathrm{d}\mathbf{s}''_{N,r} \\
&= \varepsilon^{2n-2k} |\Delta_{(n_N-r+1)/2}(s_{N,r-1}/\varepsilon)| \left\{ \prod_{m=0}^{(n_N-r-1)/2} [\mathfrak{g}(p_{N,r+2m}) - iC_{r,m}(\mathbf{p}_N)] \right\}^{-1} \\
&\quad + \varepsilon^{2n-2k} o(1),
\end{aligned}$$

as $\varepsilon \ll 1$ and $n_N - r + 1 = 2n - 2k$. Hence,

$$J^\varepsilon_{n_1\dots n_N}(t, \xi; \mathcal{F}_\sigma, \sigma) = \tilde{J}^\varepsilon_{n_1\dots n_N}(t, \xi; \sigma) + o(1),$$

where

$$\begin{aligned}
\tilde{J}^\varepsilon_{n_1\dots n_N}(t, \xi; \sigma) = \frac{(-1)^n}{(2\pi)^{nd}(n-k)!} &\int \mathrm{d}\mathbf{p}_1 \dots \mathrm{d}\mathbf{p}_N \\
&\times \prod_{m=1}^{n} \delta(p_{\bar{e}_{2m-1}} + p_{\bar{e}_{2m}}) \hat{R}(p_{\bar{e}_{2m-1}}) \\
&\times \left\{ \prod_{m=0}^{(n_N-r-1)/2} [\mathfrak{g}(p_{N,r+2m}) - iC_{r,m}(\mathbf{p}_N)] \right\}^{-1} \\
&\times \prod_{j=1}^{N} \hat{\phi}_0(\xi - p_{j1} - \dots - p_{jn_j}) K_\varepsilon(t, \mathbf{p}_1, \dots, \mathbf{p}_N)
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
K_\varepsilon(t, \mathbf{p}_1, \dots, \mathbf{p}_N) &= \varepsilon^{-k} \int_{\Delta'_{k-1}(t;\sigma)} \mathrm{d}\mathbf{s}'_{1,n_1-1} \mathrm{d}\mathbf{s}_2 \dots \mathrm{d}\mathbf{s}_{N-1} \mathrm{d}\mathbf{s}'_{N,r-2} \\
&\quad \times \prod_{j=1}^{k-1} \mathrm{e}^{-\mathfrak{g}(p_{\bar{e}_{2j-1}})|s_{\bar{e}_{2j-1}} - s_{\bar{e}_{2j}}|/\varepsilon} \mathrm{e}^{iG_{n_1-1}(\mathbf{s}_1, \mathbf{p}_1)/\varepsilon} \mathrm{e}^{iG_{r-2}(\mathbf{s}_N, \mathbf{p}_N)/\varepsilon} \\
&\quad \times \exp\left\{ i \sum_{j=2}^{N-1} G_{n_j}(\mathbf{s}_j, \mathbf{p}_j)/\varepsilon \right\} \\
&\quad \times \int_0^{s(\bar{e}_{2k-2})} \mathrm{d}s_{1,n_1} \int_0^{s_{1,n_1}} \mathrm{d}s_{N,r-1} s^{n-k}_{N,r-1} \mathrm{e}^{-\mathfrak{g}(p_{1,n_1})(s_{1,n_1} - s_{N,r-1})/\varepsilon}
\end{aligned}$$

$$\times \exp\left\{ i\left[\xi \cdot p_{1,n_1} + 1/2|p_{1,n_1}|^2 - p_{1,n_1} \cdot \left(\sum_{m=1}^{n_1} p_{1,m}\right)\right](s_{1,n_1} - s_{2,r-1})/\varepsilon\right\}$$
$$\times \exp\left\{ -ip_{1,n_1} \cdot \left(\sum_{m=1}^{n_1-1} p_{1,m} - \sum_{m=1}^{r-1} p_{N,m}\right) s_{N,r-1}/\varepsilon\right\}.$$

Note that the expression in parentheses appearing in the last exponent equals

$$S(\mathbf{p}_1, \mathbf{p}_N) := 2\sum{}' p_{1,m} + \sum \pm p_{jm},$$

where the first sum extends over all indices that correspond to the vertices $(1; m)$ that appear in the edges of the form $((1; m), (N; l))$ and the other sum extends over those $p_{j,m}$, $j = 1$, or N that are not paired with the $p_{1,m}$ appearing in the first sum.

We conclude from the Lebesgue dominated convergence theorem, as in the example in the case $N = 2$ we considered previously, that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \tilde{J}^{\varepsilon}_{n_1\dots n_N}(t, \xi; \sigma) &:= \frac{(-1)^n}{(2\pi)^{2nd}(n-k)!} \int d\mathbf{p}_1 \dots d\mathbf{p}_N \\
&\times \prod_{m=1}^{n} \delta(p_{\bar{e}_{2m-1}} + p_{\bar{e}_{2m}}) \hat{R}(p_{\bar{e}_{2m-1}}) \\
&\times \left\{ \prod_{m=0}^{(n_N-r-1)/2} [\mathfrak{g}(p_{N,r+2m}) - iC_{r,m}(\mathbf{p}_1, \dots, \mathbf{p}_N)] \right\}^{-1} \\
&\times \prod_{j=1}^{N} \hat{\phi}_0(\xi - p_{j1} - \dots - p_{jn_j}) \times \lim_{\varepsilon\downarrow 0} K_\varepsilon(t, \mathbf{p}_1, \dots, \mathbf{p}_N). \quad (3.33)
\end{aligned}$$

To compute $\lim_{\varepsilon\to 0} K_\varepsilon(t, \mathbf{p}_1, \dots, \mathbf{p}_N)$ we change the s variables according to $s'_{jm} := s_{jm}/\varepsilon$ and then let again $s'_{1,r-1} := s_{1,n_1}$, $s'_{N,r-1} := s_{1,n_1} - s_{N,r-1}$. We obtain that

$$\begin{aligned}
K_\varepsilon(t, \mathbf{p}_1, \dots, \mathbf{p}_N) &= \varepsilon^n \int_{\Delta'_{t/\varepsilon}(\sigma)} d\mathbf{s}'_{1,n_1-1} d\mathbf{s}_2 \dots d\mathbf{s}_{N-1} d\mathbf{s}'_{N,r-2} \\
&\times \prod_{j=1}^{k-1} e^{-\mathfrak{g}(p_{\bar{e}_{2j-1}})|s_{\bar{e}_{2j-1}} - s_{\bar{e}_{2j}}|} e^{iG_{n_1}(\mathbf{s}_1, \mathbf{p}_1)} e^{iG_r(\mathbf{s}_N, \mathbf{p}_N)} \int_0^{s(\bar{e}_{2k-2})} ds_{N,r-1} \\
&\times e^{-\mathfrak{g}(p_{1,n_1})s_{N,r-1}} \exp\left\{ i\left[\xi \cdot p_{1,n_1} + 1/2|p_{1,n_1}|^2 - p_{1,n_1} \cdot \left(\sum_{m=1}^{n_1} p_{1,m}\right)\right] s_{N,r-1}\right\} \\
&\times \int_{s_{N,r-1}}^{s(\bar{e}_{2k-2})} (s_{1,n_1} - s_{N,r-1})^{n-k} \exp\left\{2ip_{1,n_1} \cdot S(\mathbf{p}_1, \mathbf{p}_N)(s_{1,n_1} - s_{N,r-1})\right\} ds_{1,n_1}.
\end{aligned}$$
(3.34)

Since for any $a \neq 0$, $T > 0$ and an integer $m \geqq 0$ we have an estimate

$$\sup_{0<A<B<T/\varepsilon} \left| \int_A^B s^m e^{ias} ds \right| \leqq C_{T,a} \varepsilon^{-m}, \tag{3.35}$$

where $C_{T,a} < +\infty$, the integral in the last line of (3.34) can be estimated by $C(\mathbf{p}_1, \ldots, \mathbf{p}_N)\varepsilon^{k-n}$, with $C(\mathbf{p}_1, \ldots, \mathbf{p}_N) < +\infty$, except possibly for a set of measure zero where $p_{1,n_1} \cdot S(\mathbf{p}_1, \mathbf{p}_N) = 0$. As a result, we obtain that

$$K_\varepsilon(t, \mathbf{p}_1, \ldots, \mathbf{p}_N) \leqq \varepsilon^k \int_{\Delta'_{k-1}(t/\varepsilon;\sigma)} \prod_{m=1}^{2k-2} ds(\bar{e}_m)$$
$$\times \prod_{m=1}^{k-1} e^{-\mathfrak{g}(p_{\bar{e}_{2m-1}}) s_{\bar{e}_{2m-1}}} \int_0^{s(\bar{e}_{2k-2})} e^{-\mathfrak{g}(p_{\bar{e}_{2k-1}}) s_{1,n_1}} ds_{1,n_1}.$$

Here $\bar{e}_{2k-1} := (1, n_1)$. Thus $K_\varepsilon(t, \mathbf{p}_1, \ldots, \mathbf{p}_N) \leqq C'(\mathbf{p}_1, \ldots, \mathbf{p}_N)\varepsilon$, where $C'(\mathbf{p}_1, \ldots, \mathbf{p}_N) < +\infty$, except possibly for a set of zero measure, and (3.28) follows. This concludes the proof of Lemma 3.9. □

3.3. Convergence of the mixed moments

We now turn to the convergence of mixed moments of the form $\mathbb{E}[[\hat{\zeta}_\varepsilon(t,\xi)]^{M+N} [\hat{\zeta}_\varepsilon(t,\xi)]^{*M}]$. Suppose that $N, M \geqq 0$ are integers and consider the expansion

$$\mathbb{E}\left[[\hat{\zeta}_\varepsilon(t,\xi)]^{M+N} [\hat{\zeta}_\varepsilon(t,\xi)]^{*M} \right] = \sum_{\mathbf{n}}^{\infty} H_{\mathbf{n}}^\varepsilon(t,\xi) \tag{3.36}$$

where the summation extends over all multi-indices $\mathbf{n} := (n_{jl})_{l=1,\ldots,N_j, j=1,2}$, $N_1 := N + M$, $N_2 := M$ and

$$H_{\mathbf{n}}^\varepsilon(t,\xi) := \mathbb{E}\left[\prod_{l=1}^{N+M} \hat{\zeta}_{n_{1l}}^\varepsilon(t,\xi) \prod_{l=1}^{M} (\hat{\zeta}_{n_{2l}}^\varepsilon(t,\xi))^* \right]. \tag{3.37}$$

With the notation $2|\mathbf{n}| := \sum_{j,l} n_{jl}$, $|\mathbf{n}|^* := 2|\mathbf{n}| - \sum_l n_{2l}$ we obtain

$$H_{\mathbf{n}}^\varepsilon(t,\xi) = \frac{(-1)^{|\mathbf{n}|^*}}{(2\pi)^{2|\mathbf{n}|d} \varepsilon^{|\mathbf{n}|}} \int \cdots \int_{D_{\mathbf{n}}^t} \prod d\mathbf{s}_{\mathbf{n}}$$
$$\times \int \prod_{j=1}^{2} \prod_{l=1}^{N_j} \left[\hat{\phi}_{0j}\left(\xi - \sum_{k=1}^{n_{jl}} p_{lk}^{(j)} \right) e^{i G_{n_{jl}}^{(j)}(\mathbf{s}_l^{(j)}, \mathbf{p}_l^{(j)})/\varepsilon} \right]$$
$$\times \mathbb{E}\left[\prod_{l=1}^{N_1} \prod_{k=1}^{n_{1l}} \hat{V}\left(\frac{s_{lk}^{(1)}}{\varepsilon}, dp_{lk}^{(1)} \right) \prod_{l=1}^{N_2} \prod_{k=1}^{n_{2l}} \hat{V}^*\left(\frac{s_{lk}^{(2)}}{\varepsilon}, dp_{lk}^{(2)} \right) \right],$$

where $\mathbf{s}_l^{(j)} = (s_{l1}^{(j)}, \ldots, s_{ln_l}^{(j)})$, $\mathbf{p}_l^{(j)} = (p_{l1}^{(j)}, \ldots, p_{ln_j}^{(j)})$ and $D_{\mathbf{n}}^t := \prod_{j=1}^{2} \prod_{l=1}^{N_j} \Delta_{n_{jl}}(t)$. The following result can be shown as a generalization of Proposition 3.8

Proposition 3.10. *For each $T > 0$ there exist constants $H_{\mathbf{n}}(T,\xi)$ such that*

$$\sup_{t\in[0,T]} |H^{\varepsilon}_{\mathbf{n}}(t,\xi)| \leqq H_{\mathbf{n}}(T,\xi), \quad \forall \varepsilon \in (0,1] \tag{3.38}$$

and $\sum_{\mathbf{n}} H_{\mathbf{n}}(T,\xi) < +\infty$.

A direct corollary from the above proposition is

Corollary 3.11. *For any non-negative integers N, M we have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left[[\hat{\zeta}_\varepsilon(t,\xi)]^{M+N}[\hat{\zeta}_\varepsilon(t,\xi)]^{*M}\right] = \sum_{\mathbf{n}}^{\infty} \lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{\mathbf{n}}(t,\xi). \tag{3.39}$$

Therefore, as in the case of "pure" moments $\mathbb{E}[(\hat{\zeta}(t,\xi))^N]$, we may consider the term-wise limits of $H^{\varepsilon}_{\mathbf{n}}$.

The second absolute moment

We start by identifying the limit of $\mathbb{E}|\hat{\zeta}_\varepsilon(t,\xi)|^2$, as this provides one of the two main building blocks for other non-pure moments, the other being $|\bar{\zeta}(t,\xi)|^2$.

Proposition 3.12. *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left|\hat{\zeta}_\varepsilon(t,\xi)\right|^2 = \widehat{W}(t,\xi), \tag{3.40}$$

with $\widehat{W}(t,\xi)$ given by (1.9).

Note that

$$\left|\hat{\zeta}_\varepsilon(t,\xi)\right|^2 = \sum_{n_1,n_2=0}^{\infty} \hat{\zeta}^{\varepsilon}_{n_1}(t,\xi)[\hat{\zeta}^{\varepsilon}_{n_2}(t,\xi)]^*, \tag{3.41}$$

where each term $\hat{\zeta}^{\varepsilon}_{n}(t,\xi)$ is given by (2.3). Evaluating the expectation in (3.41) and using an argument as in the proof of part (ii) of Proposition 2.1 gives

$$\mathbb{E}\left|\hat{\zeta}_\varepsilon(t,\xi)\right|^2 = \sum_{n_1,n_2=0}^{\infty} H^{\varepsilon}_{n_1,n_2}(t,\xi), \tag{3.42}$$

where $2n = n_1 + n_2$ is even, and

$$H^{\varepsilon}_{n_1,n_2}(t,\xi) = \mathbb{E}\left[\hat{\zeta}^{\varepsilon}_{n_1}(t,\xi)\hat{\zeta}^{\varepsilon *}_{n_2}(t,\xi)\right], \tag{3.43}$$

or, equivalently,

$$H^{\varepsilon}_{n_1,n_2}(t,\xi) = (-1)^{n-n_2}\left[\frac{1}{\varepsilon^{1/2}(2\pi)^d}\right]^{2n} \iint_{D^t_{n_1,n_2}} \mathrm{d}\mathbf{s}_1 \mathrm{d}\mathbf{s}_2$$
$$\times \int \prod_{j=1}^{2} \left[\hat{\phi}_{0j}(\xi - p_{j1} - \cdots - p_{jn_j}) \mathrm{e}^{iG^{(j)}_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}\right]$$
$$\times \mathbb{E}\left[\hat{V}\left(\frac{s_{11}}{\varepsilon}, \mathrm{d}p_{11}\right) \ldots \hat{V}\left(\frac{s_{1n_1}}{\varepsilon}, \mathrm{d}p_{1n_1}\right) \hat{V}^*\left(\frac{s_{21}}{\varepsilon}, \mathrm{d}p_{21}\right) \ldots\right.$$
$$\left.\times \hat{V}^*\left(\frac{s_{2n_2}}{\varepsilon}, \mathrm{d}p_{2n_2}\right)\right],$$

where

$$\hat{\phi}_{0j} := \begin{cases} \hat{\phi}_0, & \text{if } j = 1 \\ \hat{\phi}_0^*, & \text{if } j = 2, \end{cases}$$
$$G^{(j)}_{n_j} := \begin{cases} G_{n_j}, & \text{if } j = 1 \\ -G_{n_j}, & \text{if } j = 2, \end{cases}$$

$\mathbf{s}_j = (s_{j1}, \ldots, s_{jn_j})$, $\mathbf{p}_j = (p_{j1}, \ldots, p_{jn_j})$ and $D^t_{n_1,n_2} := \Delta_{n_1}(t) \times \Delta_{n_2}(t)$. We recall that $\hat{V}^*(s, \mathrm{d}p) = \hat{V}(s, -\mathrm{d}p)$.

As a consequence of Corollary 3.11 we may pass to the limit $\varepsilon \downarrow 0$ term-wise:

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left|\hat{\zeta}_\varepsilon(t,\xi)\right|^2 = \sum_{n_1,n_2=0}^{\infty} \sum_{\mathcal{F}} \lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}), \tag{3.44}$$

where $H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F})$ is given by

$$H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}) := \frac{(-1)^{n-n_2}}{(2\pi)^{nd}\varepsilon^n} \iint_{D^t_{n_1,n_2}} \mathrm{d}\mathbf{s}_1 \mathrm{d}\mathbf{s}_2 \int \mathrm{d}\mathbf{p}_1 \mathrm{d}\mathbf{p}_2$$
$$\times \prod_{j=1}^{2} \left[\mathrm{e}^{iG^{(j)}_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon} \hat{\phi}_{0j}(\xi - p_{j1} - \cdots - p_{jn_j})\right]$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \left[\mathrm{e}^{-\mathfrak{g}(p_{jk})|s_{jk}-s_{j'm}|/\varepsilon} \hat{R}(p_{jk}) \delta(p_{jk} + (-1)^{j'-1} p_{j'm})\right] \tag{3.45}$$

and we only need to study $\lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F})$ for a fixed pairing $\mathcal{F}$. Here the pairings are of the form (j,k) with $j = 1, 2$ and $k = 1, \ldots, n_j$.

We divide the temporal domain of integration $D^t_{n_1,n_2}$ as follows. Consider the set of all permutations σ of (j,k). We let $\Delta(\sigma)$ be a subset of $D^t_{n_1,n_2}$ such that $t \geqq s_{\sigma(11)} \geqq \cdots \geqq s_{\sigma(2,n_2)} \geqq 0$. We say that a permutation σ is admissible if $\Delta(\sigma)$ is not empty. For an admissible permutation, define $H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F},\sigma)$ as in

(3.45), replacing the domain $D^t_{n_1,n_2}$ by $\Delta(\sigma)$. We call a permutation and a pairing $\mathcal{F}$ *significant* if

$$\lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F},\sigma)\neq 0.$$

Let $\mathcal{F}_\sigma$ be the pairing

$$\mathcal{F}_\sigma=(\sigma(1;1),\sigma(1;2))(\sigma(1;3),\sigma(1;4))\dots(\sigma(2;n_2-1),\sigma(2;n_2)).$$

From what we have already shown in the analysis of non-absolute moments, it follows that for a given permutation σ, any pairing $\mathcal{F}\neq\mathcal{F}_\sigma$ is not significant.

Notational interlude Let us introduce some notation. For a given significant permutation σ, each bond $((j;k),(j';k'))\in\mathcal{F}_\sigma$ may be only of the following two types: (I) $j=j'$ and $|k-k'|=1$, or (II) $j=1$, $j'=2$ (and k,k' could be arbitrary). We denote by $\mathcal{B}_1(\mathcal{F}_\sigma)$, $\mathcal{B}_2(\mathcal{F}_\sigma)$ the sets of the bonds of types (I) and (II), respectively, and shall omit writing $\mathcal{F}_\sigma$ in the notation for these types of the bonds when it is obvious from the context. The bonds of $\mathcal{F}_\sigma$ are ordered as follows: $b\prec b'$ if $b=(\sigma(j_1;i_1),\sigma(j_2;i_2))$, $b'=(\sigma(j_1';i_1'),\sigma(j_2';i_2'))$ and $(j_1;i_1)$ precedes $(j_1';i_1')$ in the lexicographical ordering. For any $b\in\mathcal{F}_\sigma$ we define $\iota_1(b)=\sigma(j_1;i_1)$ and $\iota_2(b)=\sigma(j_2;i_2)$, where $(j_1;i_1)$ precedes $(j_2;i_2)$ in the lexicographical ordering (note that the word "preceding" and the symbol $\prec$ are taken here in the "less than or equal" sense). Assume that $b\in\mathcal{F}_\sigma$ and $\iota_1(b)=(j;i)$, then we set $\mathrm{j}(b):=j$. Suppose that $b\in\mathcal{B}_2$. Let b_{prec} be a bond belonging to $\mathcal{B}_2$ and immediately preceding b among the bonds in $\mathcal{B}_2$, let $m_j(b)$ be the number of bonds from $\mathcal{B}_1$ of the type $((j;i)(j;i+1))$ occurring between b_{prec} and b, where $j=1,2$, and let $m(b)=m_1(b)+m_2(b)$. We denote $\iota(b)=(1;j)$, $l_1(\mathcal{F}_\sigma)=\#\mathcal{B}_1$, $l_2(\mathcal{F}_\sigma)=\#\mathcal{B}_2$ and by b_{fin} we designate the last element from $\mathcal{B}_2$, and set $\mathbf{P}_b:=\sum_{b'\prec b,b'\in\mathcal{B}_2}p_{\iota(b')}$.

The following result holds.

Lemma 3.13. *Let us identify the variables $p_{j,i}=(-1)^{j+j'}p_{j',i'}$ for all bonds $b=((j;i)(j',i'))$. For any admissible permutation σ we have*

$$\begin{aligned}H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma)&=\frac{(-1)^{n-n_2}}{(2\pi)^{nd}\varepsilon^n}\int\int_{\Delta(\sigma)}d\mathbf{s}_1d\mathbf{s}_2\int\prod_{b\in\mathcal{F}}dp_{\iota_1(b)}\left|\hat{\phi}\left(\xi-\mathbf{P}_{b_{fin}}\right)\right|^2\\&\times\prod_{b\in\mathcal{F}}\Big[\exp\left\{-\mathfrak{g}(p_{\iota_1(b)})(s_{\iota_1(b)}-s_{\iota_2(b)})/\varepsilon\right\}\hat{R}(p_{\iota_1(b)})\Big]\\&\times\prod_{b\in\mathcal{B}_1}\exp\left\{(-1)^{\mathrm{j}(b)-1}i(|\xi-\mathbf{P}_b|^2-|\xi-\mathbf{P}_b-p_{\iota_1(b)}|^2)(s_{\iota_1(b)}-s_{\iota_2(b)})/\varepsilon\right\}\\&\times\prod_{b\in\mathcal{B}_2}\exp\left\{(-1)^{\mathrm{j}(b)-1}i(|\xi-\mathbf{P}_{b_{\text{prec}}}|^2-|\xi-\mathbf{P}_b|^2)(s_{\iota_1(b)}-s_{\iota_2(b)})/\varepsilon\right\}.\end{aligned}\tag{3.46}$$

Proof. For a given bond b define

$$G(b) := \sum (-1)^{j-1}(|\xi - p_{j,1} - \cdots - p_{j,k-1}|^2 - |\xi - p_{j,1} - \cdots - p_{j,k}|^2)\frac{s_{j,k}}{2},$$

where the summation extends over all $(j;k)$ that are vertices of the bonds preceding b. We claim that for a terminal bond b_{fin}, with the identification of variables $p_{\iota_1(b)} = -p_{\iota_2(b)}$ if $b \in \mathcal{B}_1$ and $p_{\iota_1(b)} = p_{\iota_2(b)}$ if $b \in \mathcal{B}_2$, we can write

$$\sum_{j=1}^{2} G_{n_j}^{(j)}(\mathbf{s}_j, \mathbf{p}_j) = \sum_{b\in\mathcal{B}_1}(-1)^{\mathrm{j}(b)-1}(|\xi - \mathbf{P}_b|^2 - |\xi - \mathbf{P}_b - p_{\iota_1(b)}|^2)\frac{(s_{\iota_1(b)} - s_{\iota_2(b)})}{2}$$
$$+ \sum_{b\in\mathcal{B}_2}(-1)^{\mathrm{j}(b)-1}(|\xi - \mathbf{P}_{b_{\mathrm{prec}}}|^2 - |\xi - \mathbf{P}_b|^2)\frac{(s_{\iota_1(b)} - s_{\iota_2(b)})}{2}. \tag{3.47}$$

Note that

$$\sum_{j=1}^{2} G_{n_j}^{(j)}(\mathbf{s}_j, \mathbf{p}_j) = G(b_{last}),$$

where b_{last} is the last bond in the ordering described above. Thus, to prove (3.47) it suffices to show that

$$G(b) = \sum_{\substack{b' \prec b \\ b' \in \mathcal{B}_1}} (-1)^{\mathrm{j}(b')-1}(|\xi - \mathbf{P}_{b'}|^2 - |\xi - \mathbf{P}_{b'} - \mathbf{p}_{\iota_1(b')}|^2)\frac{(s_{\iota_1(b')} - s_{\iota_2(b')})}{2}$$
$$+ \sum_{\substack{b' \prec b \\ b' \in \mathcal{B}_2}} (-1)^{\mathrm{j}(b')-1}(|\xi - \mathbf{P}_{b'_{\mathrm{prec}}}|^2 - |\xi - \mathbf{P}_{b'}|^2)\frac{(s_{\iota_1(b')} - s_{\iota_2(b')})}{2}. \tag{3.48}$$

Formula (3.48) will be shown by induction on b with respect to the bond ordering. Note that the initial bond b_{ini} can be only one of $(1;1)(1;2)$, $(1;1)(2;1)$, $(2;1)(2;2)$, or $(2;1)(1;1)$. Suppose first that the initial bond b_{ini} belongs to $\mathcal{B}_1$ and is of the form $(1;1), (1;2)$. Then,

$$G(b_{ini}) = (|\xi|^2 - |\xi - p_{1,1}|^2)\frac{s_{1,1}}{2} + (|\xi - p_{1,1}|^2 - |\xi - p_{1,1} - p_{1,2}|^2)\frac{s_{1,2}}{2}$$
$$= (|\xi|^2 - |\xi - p_{1,1}|^2)\frac{(s_{1,1} - s_{1,2})}{2}, \tag{3.49}$$

due to the identification $p_{1,1} = -p_{1,2}$. The case when $b_{ini} = (2;1), (2;2)$ is analogous, and leads to

$$G(b_{ini}) = (|\xi|^2 - |\xi - p_{2,1}|^2)\frac{(s_{2,2} - s_{2,1})}{2},$$

hence the factor $(-1)^{\mathrm{j}(b)-1}$ in front.

Assume that the initial bond is of the form (1; 1), (2; 1). Then, using the identification $p_{1,1} = p_{1,2}$ we obtain

$$G(b_{ini}) = (|\xi|^2 - |\xi - p_{1,1}|^2)\frac{s_{1,1}}{2} - (|\xi|^2 - |\xi - p_{2,1}|^2)\frac{s_{2,1}}{2}$$
$$= (|\xi|^2 - |\xi - p_{1,1}|^2)\frac{(s_{1,1} - s_{2,1})}{2}. \tag{3.50}$$

Suppose that (3.48) holds for a certain b. Consider first the case when $b = ((1; i), (1; i+1))$. Since σ is an admissible permutation, the following bond b_{fol} may be either $((1; i+2), (1; i+3))$, $((2; j), (2; j+1))$, or $((1; i+2), (2; j))$. In the first two cases the verification that (3.49) holds for b_{fol} is straightforward. We check only the case when $b_{fol} = ((1; i+2), (2; j))$. Then,

$$G(b_{fol}) = G(b) + (|\xi - \mathbf{P}_b|^2 - |\xi - \mathbf{P}_b - p_{1,i}|^2)\frac{s_{1,i}}{2}$$
$$-(|\xi - \mathbf{P}_b|^2 - |\xi - \mathbf{P}_b - p_{2,j}|^2)\frac{s_{2,j}}{2}$$

and (3.48) follows due to the identity $p_{1,i} = p_{2,j}$, thus (3.47) holds as well. Note that the last two terms combined give the factor $(-1)^{\mathrm{j}(b_{fol})-1}(s_{\iota_1(b)} - s_{\iota_2(b)})$. This finishes the proof of Lemma 3.13. □

In order to write compactly the formula arising after integrating out the s and p-variables corresponding to $b \in \mathcal{B}_1$, we introduce an extra bond $\mathfrak{b}$, with the convention that $m_j(\mathfrak{b})$, $j = 1, 2$, is the number of bonds in $\mathcal{B}_1$ of the type $((j; i)(j; i+1))$ following b_{fin}, and also set $s_{\iota_1}(\mathfrak{b}) = 0$ and $\mathfrak{b}_{\text{prec}} = b_{fin}$. Then, changing variables $p_b \to \mathbf{P}_b$ for $b \in \mathcal{B}_2$, we obtain that

$$\lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t, \xi; \mathcal{F}_\sigma, \sigma) = \lim_{\varepsilon\downarrow 0} \tilde{H}^{\varepsilon}_{n_1,n_2}(t, \xi; \mathcal{F}_\sigma, \sigma),$$

where

$$\tilde{H}^{\varepsilon}_{n_1,n_2}(t, \xi; \mathcal{F}_\sigma, \sigma) := \frac{(-1)^{n-n_2}}{(2\pi)^{l_2(\mathcal{F}_\sigma)d}\varepsilon^{l_2(\mathcal{F}_\sigma)}}\left\{\prod_{b\in\tilde{\mathcal{B}}_2} m(b)!\right\}^{-1}$$
$$\times \int_{\Delta_{2l_2(\mathcal{F}_\sigma)}(t)} \prod_{b\in\mathcal{B}_2} ds_b \int \prod_{b\in\mathcal{B}_2} \mathrm{d}\mathbf{P}_b$$
$$\times \prod_{b\in\tilde{\mathcal{B}}_2} (s_{\iota_2(b_{\text{prec}})} - s_{\iota_1(b)})^{m(b)} (D(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{m_1(b)} (D^*(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{m_2(b)}$$
$$\times \prod_{b\in\mathcal{B}_2}\Bigg[\mathrm{e}^{-\mathfrak{g}\left(\mathbf{P}_b - \mathbf{P}_{b_{\text{prec}}}\right)(s_{\iota_1(b)} - s_{\iota_2(b)})/\varepsilon}\,\hat{R}\left(\mathbf{P}_b - \mathbf{P}_{b_{\text{prec}}}\right)$$
$$\times \exp\left\{(-1)^{\mathrm{j}(b)-1} i(|\xi - \mathbf{P}_{b_{\text{prec}}}|^2 - |\xi - \mathbf{P}_b|^2)(s_{\iota_1(b)} - s_{\iota_2(b)})/\varepsilon\right\}\Bigg]$$
$$\times \left|\hat{\phi}\left(\xi - \mathbf{P}_{b_{fin}}\right)\right|^2, \tag{3.51}$$

where $D(\xi)$ is given by (1.7), and we adopt the convention that $s_{\iota_2(b_{\text{prec}})} = t$ and $\mathbf{P}_{b_{\text{prec}}} = 0$ when $b = b_{ini}$. We let $\varepsilon \downarrow 0$ and obtain

$$\lim_{\varepsilon \downarrow 0} \tilde{H}^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma) = H_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma),$$

where

$$\begin{aligned} H_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma) &:= (-1)^{n-n_2}\left\{\prod_{b\in\tilde{\mathcal{B}}_2} m(b)!\right\}^{-1} \int_{\Delta_{l_2(\mathcal{F}_\sigma)}(t)} \prod_{b\in\mathcal{B}_2} \mathrm{d}s_b \\ &\times \int \prod_{b\in\mathcal{B}_2} \mathrm{d}\mathbf{P}_b \left|\hat{\phi}\left(\xi - \mathbf{P}_{b_{fin}}\right)\right|^2 \\ &\times \left[\prod_{b\in\tilde{\mathcal{B}}_2} (s_{\iota_2(b_{\text{prec}})} - s_{\iota_1(b)})^{m(b)} (D(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{m_1(b)} (D^*(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{m_2(b)}\right] \\ &\times \prod_{b\in\mathcal{B}_2} (D^{(\mathrm{j}(b))}(\mathbf{P}_b - \mathbf{P}_{b_{\text{prec}}}, \xi - \mathbf{P}_{b_{\text{prec}}})/2), \end{aligned} \tag{3.52}$$

with

$$D^{(j)}(p,\xi) := \begin{cases} D(p,\xi), & \text{if } j = 1 \\ D^*(p,\xi), & \text{if } j = 2, \end{cases}$$

with $D(p,\xi)$ given by (1.6).

Note that, modulo the value of $D^{(\mathrm{j}(b))}$ (it may equal either to D or D^* depending on $\mathrm{j}(b)$), the right side of (3.52) is the same for all admissible permutations σ, for which $n_1 + n_2 = 2n$ and $\mathcal{B}_2$ are fixed, and that for all such σ and any $b \in \mathcal{B}_2$, the values of $m_1(b)$ and $m_2(b)$ do not change. The total number of such pairings is

$$2^{l_2(\mathcal{B}_2)} \prod_{b\in\tilde{\mathcal{B}}_2} \binom{m_1(b) + m_2(b)}{m_1(b)} = 2^{l_2(\mathcal{B}_2)} \prod_{b\in\tilde{\mathcal{B}}_2} \binom{m(b)}{m_1(b)}.$$

Here $l_2(\mathcal{B}_2)$ equals $l_2(\mathcal{F}_\sigma)$ for an arbitrary pairing $\mathcal{F}_\sigma$ with a given $\mathcal{B}_2$, n_1 and n_2. Now we sum up the terms $H_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma)$ over such permutations σ. Then, as $n_2 - l_2(\mathcal{B}_2)$ is even (it is equal to twice the number of vertices in the second simplex connected by bonds in $\mathcal{B}_1$), we get

$$\begin{aligned} H_{n_1,n_2}(t,\xi;\mathcal{B}_2) &:= \sum_\sigma H_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma) \\ &= (-1)^{n-l_2(\mathcal{B}_2)} 2^{l_2(\mathcal{B}_2)} \\ &\times \left\{\prod_{b\in\tilde{\mathcal{B}}_2} m_1(b)! m_2(b)!\right\}^{-1} \int_{\Delta_{l_2(\mathcal{B}_2)}(t)} \prod_{b\in\mathcal{B}_2} \mathrm{d}s_b \int \prod_{b\in\mathcal{B}_2} \mathrm{d}\mathbf{P}_b \left|\hat{\phi}\left(\xi - \mathbf{P}_{b_{fin}}\right)\right|^2 \\ &\times \prod_{b\in\tilde{\mathcal{B}}_2} \left[(s_{\iota_2(b_{\text{prec}})} - s_{\iota_1(b)})^{m(b)} (D(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{m_1(b)} (D^*(\xi - \mathbf{P}_{b_{\text{prec}}})/2)^{*m_2(b)}\right] \\ &\times \prod_{b\in\mathcal{B}_2} \mathrm{Re}\,(D(\mathbf{P}_b - \mathbf{P}_{b_{\text{prec}}}, \xi - \mathbf{P}_{b_{\text{prec}}})/2). \end{aligned} \tag{3.53}$$

Next, we sum $H_{n_1,n_2}(t,\xi;\mathcal{B}_2)$ over all $\mathcal{B}_2$ whose cardinality equals a fixed number p, and over n_1 and n_2, corresponding to such $\mathcal{B}_2$, and obtain that

$$
\begin{aligned}
\bar{H}_p(t,\xi) &:= \sum_{n_1,n_2,|\mathcal{B}_2|=p} H_{n_1,n_2}(t,\xi;\mathcal{B}_2) \\
&= \sum_{m_1^{(1)},\dots,m_p^{(1)}=0}^{+\infty} \sum_{m_1^{(2)},\dots,m_p^{(2)}=0}^{+\infty} \left\{\prod_{i=1}^p m_i^{(1)}! \prod_{i=1}^p m_i^{(2)}!\right\}^{-1} \\
&\quad \times \int_{\Delta_p(t)} \mathrm{d}\mathbf{s} \int \mathrm{d}\mathbf{P} \left|\hat{\phi}\left(\xi-\mathbf{P}_p\right)\right|^2 \prod_{i=1}^p \mathrm{Re}D(\mathbf{P}_i-\mathbf{P}_{i-1},\xi-\mathbf{P}_{i-1}) \\
&\quad \times \prod_{i=1}^{p+1}\Big[(-1)^{m_i^{(1)}+m_i^{(2)}}(s_{i-1}-s_i)^{m_i^{(1)}+m_i^{(2)}} \\
&\quad \times (D/2)^{m_i^{(1)}}(\xi-\mathbf{P}_{i-1})(D^*/2)^{m_i^{(2)}}(\xi-\mathbf{P}_{i-1})\Big]. \qquad (3.54)
\end{aligned}
$$

Finally, we change variables $\mathbf{P}_i := \xi - \mathbf{P}_i$ and sum $H_{n_1,n_2}(t,\xi;\mathcal{B}_2)$ over all $p = |\mathcal{B}_2|$. We conclude that

$$
\begin{aligned}
\sum_{p=0}^{+\infty} \bar{H}_p(t,\xi) &= \sum_{p=0}^{+\infty} \int_{\Delta_p(t)} \mathrm{d}\mathbf{s} \int \mathrm{d}\mathbf{P} \left|\hat{\phi}\left(\mathbf{P}_p\right)\right|^2 \prod_{i=1}^p \mathrm{Re}D(\mathbf{P}_{i-1}-\mathbf{P}_i,\mathbf{P}_{i-1}) \\
&\quad \times \prod_{i=1}^{p+1} \exp\left\{-(s_{i-1}-s_i)[D(\mathbf{P}_{i-1})+D^*(\mathbf{P}_{i-1})]/2\right\}. \qquad (3.55)
\end{aligned}
$$

Here $\mathbf{P}_0 := \xi$.

The last observation is that the right side is the Duhamel expansion for the solution $\widehat{W}(t,\xi)$ of (1.9), finishing the proof of Proposition 3.12. □

The result proved in the foregoing can be also stated in the following way (which will be useful for us in the sequel).

Proposition 3.14. *We have*

$$
\sum_{n_1,n_2=0}^{\infty} \sum_{\mathcal{F}:\mathcal{B}_2(\mathcal{F})\neq\emptyset} \lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}) = \widehat{W}(t,\xi) - |\bar{\zeta}(t,\xi)|^2 \qquad (3.56)
$$

and

$$
\sum_{n_1,n_2=0}^{\infty} \sum_{\mathcal{F}:\mathcal{B}_2(\mathcal{F})=\emptyset} \lim_{\varepsilon\downarrow 0} H^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}) = |\bar{\zeta}(t,\xi)|^2. \qquad (3.57)
$$

This identifies separately the pairings with no bonds in $\mathcal{B}_2$ that lead to the "ballistic" part $|\bar{\zeta}(t,\xi)|^2$, and those with some bonds in $\mathcal{B}_2$ that lead to the scattering contribution to the solution of the transport equation.

Higher mixed moments

In this section we identify the limit of the general moments of the form $\mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)^{N+M}\hat{\zeta}_\varepsilon(t,\xi)^{*M}]$ finishing the proof of Theorem 1.1.

Proposition 3.15. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\hat{\zeta}_\varepsilon(t,\xi)^{N+M}\hat{\zeta}_\varepsilon(t,\xi)^{*M}\right]=\mathbb{E}\left[\hat{\zeta}(t,\xi)^{N+M}\hat{\zeta}(t,\xi)^{*M}\right] \tag{3.58}$$

for all $N, M \geqq 0$*. Here* $\hat{\zeta}(t,\xi)$ *is given by* (1.12).

We proceed as in the case of the second absolute moment: using the rules of computing expectations of moments of Gaussian variables and Proposition 3.10, we arrive at the formula

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[[\hat{\zeta}_\varepsilon(t,\xi)]^{M+N}[\hat{\zeta}_\varepsilon(t,\xi)]^{*M}\right]=\sum_{\mathbf{n}}\sum_{\mathcal{F}}\lim_{\varepsilon\downarrow 0}H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F}), \tag{3.59}$$

where $H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F})$ is given by

$$\begin{aligned}H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F})=&\frac{1}{\varepsilon^{|\mathbf{n}|/2}}\int\cdots\int_{D^t_{\mathbf{n}}}\prod ds_e\\&\times\int\prod\frac{dp_e}{(2\pi)^d}\prod_{(e,e')\in\mathcal{F}}\left[e^{-\mathfrak{g}(p_e)|s_e-s_{e'}|/\varepsilon}\hat{R}(p_e)\delta(p_e+(-1)^{\mathfrak{j}(e')-1}p_{e'})\right]\\&\times\prod_{l=1}^{2}\prod_{j=1}^{2}\left[\hat{\phi}_{0j}\left(\xi-\sum_{e:\mathfrak{j}(e)=j,\mathfrak{l}(e)=l}p_e\right)e^{iG^{(j)}_{n_{jl}}(\mathbf{s}_{l,j},\mathbf{p}_{l,j})/\varepsilon}\right].\end{aligned} \tag{3.60}$$

Here $\mathcal{F}$ is a pairing formed between vertices (e,e') that are triples of the form (j,l,k), where $j=1,2$ (terms with $j=1$ come from non-conjugated $\hat{\zeta}_\varepsilon$ and those with $j=2$ come from $\hat{\zeta}^*_\varepsilon$), $l=1,\ldots,N_j$ (N_1 is the number of non-conjugated $\hat{\zeta}$ and N_2 is the number of conjugated ones), and $k=1,\ldots,n_{j,l}$, and $|\mathbf{n}|=\sum n_{j,l}$ is the total number of $\hat{V}$ appearing. In addition, for any $e=(j,l,k)$ we let $\mathfrak{j}(e):=j$, $\mathfrak{l}(e):=l$ and $\mathbf{s}_{j,l}:=(s^{(j,l)}_1,\ldots,s^{(j,l)}_{n_{j,l}})$, $\mathbf{p}_{j,l}:=(p^{(j,l)}_1,\ldots,p^{(j,l)}_{n_{j,l}})$. Likewise, we use this notation for any variable indexed by e.

Thanks to Proposition 3.10 we only need to study $H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F})$ for a fixed pairing $\mathcal{F}$. Let σ be a permutation of the triples (j,l,k). We can write $D_{\mathbf{n}}(t)=\bigcup_\sigma\Delta_{|\mathbf{n}|}(\sigma)$, where $\Delta_{|\mathbf{n}|}(\sigma):=[t\geqq s_{\sigma(1,1,1)}\geqq\cdots\geqq s_{\sigma(2,N_2,n_{2,N_2})}\geqq 0]$ and the union extends over all such permutations. The expression in (3.60) can then be rewritten as a sum of $H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F},\sigma)$ corresponding to the integrals over all $\Delta(\sigma)$. In analogy with the previously considered cases, we call σ significant if there exists $\mathcal{F}$ such that $\lim_{\varepsilon\downarrow 0}H^\varepsilon_{\mathbf{n}}(t,\xi;\mathcal{F},\sigma)\neq 0$. A pairing $\mathcal{F}$ for which the above holds is also called significant. It can be argued, as before, that for each significant σ there exists at most one significant pairing

$$\mathcal{F}_\sigma:=(\sigma(1,1,1),\sigma(1,1,2)),\ldots(\sigma(2,N_2,n_{2,N_2-1}),\sigma(2,N_2,n_{2,N_2})), \tag{3.61}$$

such that σ and $\mathcal{F}_\sigma$ are significant. For a given significant permutation σ we order vertices according to (3.61). This order induces a corresponding ordering of bonds of $\mathcal{F}_\sigma$. For a given bond $b \in \mathcal{F}_\sigma$ we can define now $\iota_1(b), \iota_2(b)$ in analogy with what has been done in the computation of the second absolute moment.

Let b_{prec} be a bond immediately preceding b in $\mathcal{B}_2$. In addition, if $b \in \mathcal{B}_{2,k}^{(j)}$ we denote by $\hat{b}_{\text{prec}}$ the bond immediately preceding b in this class. Let also

$$G(b) := \frac{1}{2}\widehat{\sum}(-1)^{j-1}\left(\left|\xi - p_1^{(j,l)} - \cdots - p_{k-1}^{(j,l)}\right|^2 - \left|\xi - p_{l,1}^{(j)} - \cdots - p_k^{(j,l)}\right|^2\right) s_k^{(j,l)},$$

where the summation extends over all $(j; l, k)$ that are the vertices of the bonds preceding b (including b itself).

The computation for the non-absolute moments in the proof of Proposition 3.7 already shows that a pairing corresponding to a significant permutation may not contain a bond connecting a vertex (j, l, k) to a vertex (j', l', k') with $j = j'$ but $l \neq l'$, as this leads to a large oscillatory phase. Therefore, a permutation may be significant only when the pairing $\mathcal{F}_\sigma$ consists of bonds of the following types: (I) (j, l, k), (j', l', k') when $j = j'$, $l = l'$ and $|k - k'| = 1$, or (II) $j = 1$, $j' = 2$ and (l, k), (l', k') could be arbitrary. We denote the sets of bonds corresponding to conditions (I) and (II) by $\mathcal{B}_1(\mathcal{F}_\sigma)$ and $\mathcal{B}_2(\mathcal{F}_\sigma)$ respectively, and call those from the latter set mixed bonds.

Consider a collection P of pairs (l_1, l_1'), (l_2, l_2'), $\ldots$, (l_m, l_m'), with $\{l_1, \ldots, l_m\} \subseteq \{1, \ldots, N + M\}$ and $\{l_1', \ldots l_m'\} \subseteq \{1, \ldots, N\}$. Such a collection is admissible if every l_j and l_j' appears only once in P.

If P is admissible, we denote by $\mathcal{S}(P)$ the family of all pairings $\mathcal{F}$ such that

$$\mathcal{B}_2(\mathcal{F}) = \{((1, l_1, k_1), (2, l_1', k_1')), \ldots ((1, l_m, k_m), (2, l_m', k_m'))\}$$

for some $k_1, k_1', \ldots, k_m, k_m'$. A simple generalization of Proposition 3.14 is the following.

Proposition 3.16. *For an admissible collection of pairs P we have*

$$\mathcal{R}(P) := \sum_{\mathbf{n}} \sum_{\mathcal{F}\in\mathcal{S}(P)} \lim_{\varepsilon\downarrow 0} H_{\mathbf{n}}^{\varepsilon}(t, \xi; \mathcal{F}) = \left[\widehat{W}(t, \xi) - |\bar{\zeta}(t, \xi)|^2\right]^m |\bar{\zeta}(t, \xi)|^{2(M-m)}[\bar{\zeta}(t, \xi)]^N. \tag{3.62}$$

We say that a set of mixed bonds $\mathcal{B}_2(\mathcal{F})$ is *proper* if it has the following property: if a bond $((1; l, k), (2; l', k')) \in \mathcal{B}_2(\mathcal{F})$ then for all other bonds $((1; l, j), (2; l'', j')) \in \mathcal{B}_2(\mathcal{F})$ we have $l'' = l'$. The main observation of this section is the following.

Proposition 3.17. *Suppose that for a given σ the pairing $\mathcal{F}_\sigma$ is significant. Then $\mathcal{B}_2(\mathcal{F}_\sigma)$ is proper.*

Proof. To simplify the notation we consider only the case $N = 0$, $M = 2$. The proof in the general case follows the same idea. Let $\mathcal{B}_{21} := \mathcal{B}_{21}^{(1)} \cup \mathcal{B}_{21}^{(2)}$ be the subset of $\mathcal{B}_2$ consisting of bonds $((1; 1, k), (2; 1, k'))$ that form $\mathcal{B}_{21}^{(1)}$ and $((1; 2, k), (2; 2, k'))$ that form $\mathcal{B}_{21}^{(2)}$. Likewise $\mathcal{B}_{22} := \mathcal{B}_{22}^{(1)} \cup \mathcal{B}_{22}^{(2)}$, where the bonds forming $\mathcal{B}_{22}^{(1)}$, $\mathcal{B}_{22}^{(2)}$ are $((1; 1, k), (2; 2, k'))$ and $((1; 2, k), (2; 1, k'))$, respectively. Note that obviously $\mathcal{B}_{21} := \mathcal{B}_2 \backslash \mathcal{B}_{22}$. Suppose that contrary to the statement of the proposition we have both $\mathcal{B}_{21} \neq \emptyset$ and $\mathcal{B}_{22} \neq \emptyset$. Without loss of generality, assume that there exists a bond $b_{ini} := ((1, 1, k), (2, 2, k'))$, which is in $\mathcal{B}_{22}^{(1)}$, such that all $b' \neq b_{ini}$ from $\mathcal{B}_2$ preceding that bond are from $\mathcal{B}_{21}$. Let $\mathbf{P}_b := \sum_{\substack{b' \prec b \\ b' \in \mathcal{B}_{21}^{(k)}}} p_{\iota_1(b')}$, if $b = ((j; k, l), (j'; k, l'))$ for some j, j', l, l'. If the summation extends over the empty set, we let $\mathbf{P}_b := 0$. Let b_k be the bonds from $\mathcal{B}_{21}^{(k)}$, $k = 1, 2$, immediately preceding b_{ini}. We have

$$
\begin{aligned}
G(b_{ini}) = & \sum_{\substack{b' \prec b_{ini} \\ b' \in \mathcal{B}_1}} (-1)^{\mathrm{j}(b')-1} (|\xi - \mathbf{P}_{b'}|^2 - |\xi - \mathbf{P}_{b'} - \mathbf{p}_{\iota_1(b')}|^2) \frac{(s_{\iota_1(b')} - s_{\iota_2(b')})}{2} \\
& + \sum_{\substack{b' \prec b_{ini}, b' \neq b_{ini} \\ b' \in \mathcal{B}_2}} (-1)^{\mathrm{j}(b')-1} (|\xi - \mathbf{P}_{\hat{b}'_{\mathrm{prec}}}|^2 - |\xi - \mathbf{P}_{b'}|^2) \frac{(s_{\iota_1(b')} - s_{\iota_2(b')})}{2} \\
& + \sum_{j=1}^{2} (-1)^{j-1} (|\xi - \mathbf{P}_{b_j}|^2 - |\xi - \mathbf{P}_{b_j} - p_{\iota_1(b)}|^2) \frac{s_{\iota_j(b_{ini})}}{2}.
\end{aligned} \tag{3.63}
$$

The third line above may be rewritten as

$$
\begin{aligned}
& (|\xi - \mathbf{P}_{b_1}|^2 - |\xi - \mathbf{P}_{b_1} - p_{\iota_1(b)}|^2) \frac{s_{\iota_1(b)} - s_{\iota_2(b)}}{2} \\
& + \Big[(|\xi - \mathbf{P}_{b_1}|^2 - |\xi - \mathbf{P}_{b_1} - p_{\iota_1(b)}|^2) \\
& - (|\xi - \mathbf{P}_{b_2}|^2 - |\xi - \mathbf{P}_{b_2} - p_{\iota_1(b)}|^2) \Big] \frac{s_{\iota_2(b)}}{2}.
\end{aligned} \tag{3.64}
$$

Since the coefficient by $s_{\iota_2(b)}$ in (3.64) vanishes only on a set of measure zero we can use, again, estimate (3.35) to conclude that $|H_{\mathbf{n}}^{\varepsilon}(t, \xi; \mathcal{F}_\sigma, \sigma)| \leqq C\varepsilon$ for some constant $C > 0$ and the conclusion of the proposition follows. □

We now finish the proof of Proposition 3.15. As a consequence of Propositions 3.16 and 3.17 we conclude that

$$
\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[[\zeta_\varepsilon(t, \xi)]^{N+M} \left[\zeta_\varepsilon^*(t, \xi)\right]^M \right] \\
& = \sum_{m=0}^{+\infty} m! \binom{N+M}{m} \binom{M}{m} \left[\widehat{W}(t, \xi) - |\bar{\zeta}(t, \xi)|^2 \right]^m |\bar{\zeta}(t, \xi)|^{2(M-m)} [\bar{\zeta}(t, \xi)]^N.
\end{aligned} \tag{3.65}
$$

This coincides with the moments of a complex valued Gaussian $Z(t, \xi)$ appearing in the statement of Theorem 1.1. The proofs of Proposition 3.15 and Theorem 1.1 are now complete. □

4. Proof of Theorem 1.2

The overall steps in the proof of Theorem 1.2 are similar to that of Theorem 1.1: we expand $\hat{\zeta}_\varepsilon(t, \xi)$ into the Duhamel expansion series (2.2) and then, first, use Proposition 2.1 to establish convergence of $\mathbb{E}(\hat{\zeta}_\varepsilon(t, \xi))$, and, second, address convergence of the higher moments of $\hat{\zeta}_\varepsilon(t, \xi)$. The main difference with the proof of Theorem 1.1 is that now not only the time-ordered pairings contribute in the limit $\varepsilon \to 0$ but, rather, all pairings have a non-trivial contribution. This leads to a non-Markovian limit. Furthermore, it turns out that the limit of $|\hat{\zeta}_\varepsilon(t, \xi)|$ is trivial: it converges in probability to its initial value $|\hat{\phi}_0(\xi)|$ for all $t \geqq 0$. That makes the identification of the limit of $\hat{\zeta}_\varepsilon(t, \xi)$ simpler than in the rapidly decorrelating case.

4.1. Convergence of the expectation

We first establish the analog of Corollary 3.1.

Proposition 4.1. *We have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\hat{\zeta}_\varepsilon(t, \xi) = \hat{\phi}_0(\xi)\mathbb{E}\left[e^{iB_\kappa(t;\xi)}\right], \tag{4.1}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \backslash \{0\}$. Here $B_\kappa(t)$ is the fractional Brownian motion with κ given by (1.17) and the diffusion coefficient given by (1.19) for $\beta < 1/2$ and by (1.20) for $\beta = 1/2$.

Outline of the proof The strategy of the proof of Proposition 4.1 is similar to what we have done in the rapidly decorrelating case. First, we will establish the following uniform bound:

Proposition 4.2. *For all $T > 0$, $n \geqq 0$ and all $\xi \in \mathbb{R}^d \backslash \{0\}$ there exists a constant $C(T; \xi)$ such that*

$$\sup_{t\in[0,T]} |\mathbb{E}\hat{\zeta}_n^\varepsilon(t, \xi)| \leqq \frac{C^n(T; \xi)}{n!} \tag{4.2}$$

for all $\varepsilon \in (0, 1]$.

As before, this allows us to interchange the limit $\varepsilon \to 0$ and the summation in n.

Corollary 4.3. *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_\varepsilon(t, \xi) = \sum_{n=0}^{\infty} \lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_n^\varepsilon(t, \xi), \tag{4.3}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \backslash \{0\}$.

This corollary is an immediate consequence of the estimate (4.2). The last step in the proof of Proposition 4.1 is to identify the limit of the individual terms in the right side of (4.3).

Proposition 4.4. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}_n^{\varepsilon}(t,\xi)=\hat{\phi}_0(\xi)\mathbb{E}\left[\frac{(iB_\kappa(t;\xi))^n}{n!}\right], \tag{4.4}$$

for all $t\in\mathbb{R}$ *and* $\xi\in\mathbb{R}^d\backslash\{0\}$.

This implies the conclusion of Proposition 4.1.

The proof of Proposition 4.2 We suppose that $\mathfrak{g}(p)=\mu|p|^{2\beta}$ and $\hat{R}(p)=a(p)/|p|^{2\alpha+d-2}$ for parameters μ,β,α and a function $a(p)$ as in the statement of Theorem 1.2. As in (3.5), we have the estimate

$$\begin{aligned}|\mathbb{E}\hat{\zeta}_{2n}^{\varepsilon}(t,\xi)|&\leqq\frac{\|\hat{\phi}_0\|_\infty}{(2n)!}\left[\frac{\gamma}{\varepsilon(2\pi)^d}\right]^{2n}\int_0^t\cdots\int_0^t\mathrm{d}\mathbf{s}^{(2n)}\\&\times\int\left|\mathbb{E}\left[\hat{V}\left(\frac{s_1}{\varepsilon},\mathrm{d}p_1\right)\ldots\hat{V}\left(\frac{s_{2n}}{\varepsilon},\mathrm{d}p_{2n}\right)\right]\right|\\&\leqq\frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!}\frac{\gamma^{2n}}{\varepsilon^{2n}}\sum_{\mathcal{F}}\int_0^t\cdots\int_0^t\mathrm{d}\mathbf{s}^{(2n)}\int\mathrm{d}\mathbf{p}^{(2n)}\\&\times\prod_{(k,l)\in\mathcal{F}}\mathrm{e}^{-\mu|p_k|^{2\beta}|s_k-s_l|/\varepsilon}\delta(p_k+p_l)\frac{a(p_k)}{|p_k|^{2\alpha+d-2}},\end{aligned}\tag{4.5}$$

where the summation extends over all pairings formed over vertices $\{1,\ldots,2n\}$. Changing variables $p_k':=p_k/\varepsilon^{1/(2\beta)}$ and using expression (1.17) for κ we rewrite (4.5) as

$$\begin{aligned}|\mathbb{E}\hat{\zeta}_{2n}^{\varepsilon}(t,\xi)|&\leqq\frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!}\left(\frac{\gamma}{\varepsilon^\kappa}\right)^{2n}\sum_{\mathcal{F}}\int_0^t\cdots\int_0^t\mathrm{d}\mathbf{s}^{(2n)}\int\mathrm{d}\mathbf{p}^{(2n)}\\&\times\prod_{(k,l)\in\mathcal{F}}\mathrm{e}^{-\mu|p_k|^{2\beta}|s_k-s_l|}\\&\times\delta(p_k+p_l)\frac{a(\varepsilon^{1/(2\beta)}p_k)}{|p_k|^{2\alpha+d-2}}=\frac{C^n\|\hat{\phi}_0\|_\infty}{2^n n!}\left(\frac{\gamma}{\varepsilon^\kappa}\right)^{2n}\\&\times\left[\int_0^t\int_0^t\int\mathrm{e}^{-\mu|p|^{2\beta}|s_1-s_2|}\frac{a(\varepsilon^{1/(2\beta)}p)}{|p|^{2\alpha+d-2}}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}p\right]^n.\end{aligned}\tag{4.6}$$

We used the fact that the total number of the pairings is $(2n-1)!!$ in the last step above. As $\varepsilon=\gamma^{1/\kappa}$, we may recast (4.6) as

$$
\begin{aligned}
|\mathbb{E}\hat{\zeta}^{\varepsilon}_{2n}(t,\xi)| &\leqq \frac{C^n\|\hat{\phi}_0\|_\infty}{n!}\left[\int_0^t\int_0^t\int \mathrm{e}^{-\mu|p|^{2\beta}|s_1-s_2|}\frac{a(\varepsilon^{1/(2\beta)}p)}{|p|^{2\alpha+d-2}}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}p\right]^n \\
&\leqq \frac{C^n\|\hat{\phi}_0\|_\infty}{n!}\left[\int_0^t \mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\int\frac{\mathrm{e}^{-\mu|p|^{2\beta}s_2}}{|p|^{2\alpha+d-2}}\mathrm{d}p\right]^n \\
&= \frac{C^n(M_*(t))^n\|\hat{\phi}_0\|_\infty}{n!}, \qquad (4.7)
\end{aligned}
$$

where

$$
M_*(t)=\int\frac{\mathrm{e}^{-\mu|p|^{2\beta}t}-1+\mu|p|^{2\beta}t}{\mu^2|p|^{2\alpha+4\beta+d-2}}\mathrm{d}p<+\infty
$$

for $\alpha<1$ and $\alpha+\beta>1$. Estimate (4.2) now follows. □

The proof of Proposition 4.4 We proceed now with the limit identification. As we have mentioned, the fundamental difference with the rapidly de-correlating case considered in Section 3.1 lies in the fact that the terms corresponding to an arbitrary pairing may have a non-vanishing limit, as $\varepsilon\downarrow 0$ – recall that in the previous case only those corresponding to the time-ordered pairings have non-vanishing limits. As before, starting with (2.3) we have

$$
\mathbb{E}\zeta^{\varepsilon}_{2n}(t,\xi)=\hat{\phi}_0(\xi)\sum_{\mathcal{F}}\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}),
$$

where

$$
\begin{aligned}
\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}) &= \left[\frac{\gamma}{i\varepsilon(2\pi)^{d/2}}\right]^{2n}\int_{\Delta_{2n}(t)}\mathrm{d}\mathbf{s}^{(2n)}\int\mathrm{d}\mathbf{p}^{(2n)} \\
&\quad\times\prod_{(k,l)\in\mathcal{F}}\mathrm{e}^{-\mu|p_k|^{2\beta}(s_k-s_l)/\varepsilon}\frac{a(p_k)\delta(p_k+p_l)}{|p_k|^{2\alpha+d-2}}\mathrm{e}^{iG_{2n}(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon}. \qquad (4.8)
\end{aligned}
$$

Thanks to estimate (4.3), what remains is to identify the limits

$$
\mathcal{I}_{2n}(t;\mathcal{F})=\lim_{\varepsilon\downarrow 0}\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}).
$$

An upper bound for the integrand We now proceed to rewrite $\mathcal{I}^{\varepsilon}_{2n}(t;\mathcal{F})$ in such a form that the Lebesgue dominated convergence theorem could be applied to the integrand in the limit $\varepsilon\downarrow 0$. To begin, we make a change of variables $s_i=\sum_{j=i}^{2n}\tau_j$. Consider the phase $G_{2n}(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})$ and the decomposition (2.4)–(2.5). Note that the terms corresponding to A_{2n} and B_{2n}, after the change of variables, equal, respectively,

$$
\tilde{A}(\tau^{(n)},\mathbf{p}^{(n)})=\sum_{m=1}^{n}\left(\xi\cdot\sum_{j=1}^{m}p_j\right)\tau_m, \qquad (4.9)
$$

with $\tau^{(2n)} = (\tau_1, \ldots, \tau_{2n}) \in \mathbb{R}^{2n}$ and

$$\tilde{B}(\tau^{(n)}, \mathbf{p}^{(n)}) = \sum_{m=1}^{n} \tau_m Q_m(\mathbf{p}^{(n)}), \tag{4.10}$$

where

$$Q_m(\mathbf{p}^{(n)}) = \frac{1}{2}\left|\sum_{j=1}^{m} p_j\right|^2. \tag{4.11}$$

Using the new variables, and introducing an additional variable τ_0, we can rewrite (4.8) in the following way

$$\begin{aligned}
\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \mathrm{e}^t \left[\frac{\gamma}{i\varepsilon(2\pi)^{d/2}}\right]^{2n} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathrm{d}\tau^{(2n+1)} \\
\times \int \mathrm{d}\mathbf{p}^{(2n)} \delta(t - \tau_0 - \cdots - \tau_{2n}) \\
\times \prod_{(km)\in\mathcal{F}} \frac{a(p_k)}{|p_k|^{2\alpha+d-2}} \delta(p_k + p_m) \exp\left\{-\mu|p_k|^{2\beta} \sum_{j=k}^{m-1} \tau_j/\varepsilon\right\} \\
\times \exp\left\{-\sum_{j=0}^{2n} \tau_j + i\tilde{G}_{2n}(\tau^{(2n)}, \mathbf{p}^{(2n)})/\varepsilon\right\},
\end{aligned} \tag{4.12}$$

where $\tau^{(2n+1)} = (\tau_0, \ldots, \tau_{2n})$. Next, using the fact that

$$\delta(t) = \int \mathrm{e}^{-itz} \frac{\mathrm{d}z}{2\pi},$$

we obtain

$$\begin{aligned}
\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) &= \frac{(-1)^n \mathrm{e}^t}{2\pi} \left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathrm{d}\tau^{(2n+1)} \int \mathrm{d}\mathbf{p}^{(2n)} \int \mathrm{d}z \mathrm{e}^{-izt} \\
&\times \prod_{(km)\in\mathcal{F}} \left(\frac{a(p_k)}{|p_k|^{2\alpha+d-2}} \delta(p_k + p_m)\right. \\
&\times \left.\exp\left[-\mu|p_k|^{2\beta} \sum_{j=k}^{m-1} \tau_j/\varepsilon - (1 - iz) \sum_{j=0}^{2n} \tau_j + i\tilde{G}_{2n}(\tau^{(2n)}, \mathbf{p}^{(2n)})/\varepsilon\right]\right) \\
&= \frac{(-1)^n \mathrm{e}^t}{2\pi} \left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n} \int_0^{+\infty} \cdots \int_0^{+\infty} \mathrm{d}\tau^{(2n+1)} \int \mathrm{d}\mathbf{p}^{(2n)} \\
&\times \int \mathrm{d}z \mathrm{e}^{-izt} \left(\prod_{(km)\in\mathcal{F}} \frac{a(p_k)\delta(p_k + p_m)}{|p_k|^{2\alpha+d-2}}\right)
\end{aligned}$$

$$\times \exp\Bigg\{-(1-iz)\tau_0 - \sum_{j=1}^{2n}\Bigg[\sum_{(km)\in\mathcal{F}} 1_{[k,m)}(j)\mu|p_k|^{2\beta}/\varepsilon + 1 - iz\Bigg]\tau_j$$

$$+ i\tilde{G}_{2n}(\tau^{(2n)}, \mathbf{p}^{(2n)})/\varepsilon\Bigg\}. \tag{4.13}$$

Integrating out the τ-variables gives

$$\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n e^t}{2\pi}\left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n}\int d\mathbf{p}^{(2n)}$$
$$\times \int \frac{e^{-izt}dz}{1-iz}\left(\prod_{(km)\in\mathcal{F}}\frac{a(p_k)\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}}\right)$$
$$\times\Bigg\{\prod_{j=1}^{2n}\Bigg[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}/\varepsilon + 1$$
$$+i\left(\tilde{Q}_j(\mathbf{p}^{(2n)})/\varepsilon - \sum_{k=1}^{j}\xi\cdot p_k/\varepsilon - z\right)\Bigg]\Bigg\}^{-1}. \tag{4.14}$$

Substituting $p_k' := p_k/\varepsilon^{1/(2\beta)}$, as in the passage from (4.5) to (4.6), and using the relation $\gamma = \varepsilon^{\kappa}$ leads to

$$\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n e^t}{(2\pi)^{nd+1}}\int d\mathbf{p}^{(2n)}\int\frac{e^{-izt}dz}{1-iz}$$
$$\times\prod_{(km)\in\mathcal{F}}\frac{a(\varepsilon^{1/(2\beta)}p_k)}{|p_k|^{2\alpha+d-2}}\delta(p_k+p_m)$$
$$\times\Bigg\{\prod_{j=1}^{2n}\Bigg[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta} + 1$$
$$+i\left(\tilde{Q}_j(\mathbf{p}^{(2n)})\varepsilon^{1/\beta-1} - \sum_{k=1}^{j}\xi\cdot p_k\varepsilon^{1/(2\beta)-1} - z\right)\Bigg]\Bigg\}^{-1}. \tag{4.15}$$

Let us denote by $\mathcal{L}(\mathcal{F})$ the set of all left vertices of $\mathcal{F}$, and for an edge $e = (km) \in \mathcal{F}$ set $\ell(m) = k$. The expression under the multiple integral on the right side can be majorized by

$$\frac{\|a\|_\infty^n}{(1+|z|)|p_{\ell(2n)}|^{2\alpha+d-2}\left(\mu|p_{\ell(2n)}|^{2\beta}+1+|z|\right)}$$
$$\times\prod_{\substack{j\in\mathcal{L}(\mathcal{F})\\ j\neq\ell(2n)}}\left\{\frac{1}{|p_j|^{2\alpha+d-2}}\left[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}+1\right]^{-1}\right\}$$

$$\leqq \frac{\|a\|_\infty^n}{(1+|z|)|p_{\ell(2n)}|^{2\alpha+d-2}\left(\mu|p_{\ell(2n)}|^{2\beta}+1+|z|\right)} \times \prod_{\substack{j\in\mathcal{L}(\mathcal{F})\\ j\neq\ell(2n)}} \left\{\frac{1}{|p_j|^{2\alpha+d-2}}\left[\mu|p_j|^{2\beta}+1\right]^{-1}\right\}. \tag{4.16}$$

We used the simple fact that for a vertex $j \in \mathcal{L}(\mathcal{F})$ we have $1_{[k,m)}(j) = 1$ if we take the edge with $k = j$ in the summation over the edges of $\mathcal{F}$, above. Now, the expression in the right side of (4.16) is integrable with respect to the measure $\mathrm{d}\mu = \mathrm{d}z \prod_{j\in\mathcal{L}(\mathcal{F})} \mathrm{d}p_j$, since $\alpha+\beta > 1$ and $\alpha \in (1/2, 1)$.

Computation of the limit of $\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F})$. The integrability of expression (4.16) allows us to apply the dominated convergence theorem in the expression (4.15) for $\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F})$ and pass to the limit under the integral sign, concluding that for $\beta < 1/2$ we have, as both $1/\beta > 1$ and $1/(2\beta) > 1$:

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \lim_{\varepsilon\downarrow 0}\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n\mathrm{e}^t}{2\pi}\left[\frac{a(0)}{(2\pi)^d}\right]^n \hat{\phi}_0(\xi) \times \int \mathrm{d}\mathbf{p}^{(2n)} \int \frac{\mathrm{e}^{-izt}\mathrm{d}z}{1-iz} \prod_{(km)\in\mathcal{F}} \frac{\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}} \times \left\{\prod_{j=1}^{2n}\left(\mu\sum_{(km)} 1_{[k,m)\in\mathcal{F}}(j)|p_k|^{2\beta}+1-iz\right)\right\}^{-1}, \tag{4.17}$$

while for $\beta = 1/2$ we get

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \lim_{\varepsilon\downarrow 0}\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n\mathrm{e}^t}{2\pi}\left[\frac{a(0)}{(2\pi)^d}\right]^n \hat{\phi}_0(\xi)\int \mathrm{d}\mathbf{p}^{(2n)} \times \int \frac{\mathrm{e}^{-izt}\mathrm{d}z}{1-iz} \prod_{(km)\in\mathcal{F}} \frac{\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}} \times \left\{\prod_{j=1}^{2n}\left[\mu\sum_{(km)\in\mathcal{F}} 1_{[k,m)}(j)|p_k|^{2\beta}+1-i\left(z+\sum_{k=1}^{j}\xi\cdot p_k\right)\right]\right\}^{-1}. \tag{4.18}$$

To unify the notation we introduce $\zeta(\beta) := 0$ for $\beta < 1/2$ and $\zeta(1/2) := 1$. Then, retracing our steps above, we may rewrite both (4.17) and (4.18) as (compare to (4.8))

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \left[\frac{-a(0)}{(2\pi)^d}\right]^n \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \int \mathrm{d}\mathbf{p}^{(2n)} \times \prod_{(k,l)\in\mathcal{F}} \mathrm{e}^{-\mu|p_k|^{2\beta}(s_k-s_l)} \frac{\delta(p_k+p_l)}{|p_k|^{2\alpha+d-2}} \mathrm{e}^{i\zeta(\beta)\sum_{j=1}^{2n} s_j\xi\cdot p_j}. \tag{4.19}$$

The case $\beta < 1/2$. Now, we relate expression (4.19) to the fractional Brownian motion. Consider, first, the case $\beta < 1/2$. Then, after integrating out the p-variables (4.19) becomes

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \left[\frac{-a(0)K_1(\alpha,\beta,\mu)}{(2\pi)^d}\right]^n \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \prod_{(k,l)\in\mathcal{F}} |s_k - s_l|^{(\alpha-1)/\beta}, \quad (4.20)$$

with $K_1(\alpha,\beta,\mu)$ as in (1.15). A direct computation shows that

$$\sum_{\mathcal{F}} \prod_{(pq)\in\mathcal{F}} \left|s_p - s_q\right|^{2\kappa-2} = c_\kappa^{2n}\mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(\mathrm{d}k_p)\right], \quad (4.21)$$

where $w(\mathrm{d}k)$ is a complex Gaussian noise with $\mathbb{E}[w(\mathrm{d}k)w(dk')] = \delta(k+k')dkdk'$, and $c_\kappa > 0$ is given by

$$c_\kappa = \left(\frac{\Gamma(2\kappa-1)\sin(\pi\kappa)}{\pi}\right)^{1/2}. \quad (4.22)$$

Both in (4.21) and below, the expectation of the improper multiple stochastic integral is understood as the limit of the expectation of the integrals over the intervals $k \in (-A, A)$ as $A \to +\infty$. Then, (4.20) with κ defined by (1.17) can be restated as

$$\begin{aligned}\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) &= \left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \\ &\quad\times \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \lim_{A\to+\infty} \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-A}^{A} \frac{\mathrm{e}^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(\mathrm{d}k_p)\right] \\ &= \left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \\ &\quad\times \lim_{A\to+\infty} \int_{\Delta_{2n}(t)} \mathrm{d}\mathbf{s}^{(2n)} \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-A}^{A} \frac{\mathrm{e}^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(\mathrm{d}k_p)\right], \quad (4.23)\end{aligned}$$

with the last equality justified by the Lebesgue dominated convergence theorem. Taking into account the symmetry in the s_j-variables of the expression in the right-hand side of (4.23) we obtain that

$$\begin{aligned}\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) = \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \lim_{A\to+\infty} \int_0^t \cdots \int_0^t \mathrm{d}\mathbf{s}^{(2n)} \\ \times \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-A}^{A} \frac{\mathrm{e}^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(\mathrm{d}k_p)\right]\end{aligned}$$

$$= \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n$$
$$\times \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{+\infty}\frac{e^{ik_p t}-1}{ik_p|k_p|^{\kappa-1/2}}w(dk_p)\right]. \qquad (4.24)$$

Note that the improper multiple stochastic integral in the last line makes sense. Using the harmonizable representation of the standard fractional Brownian motion, see Proposition 7.2.8, p. 328 of [25], we deduce that (4.24) can be reformulated as

$$\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) = \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2 d_\kappa^2}{(2\pi)^d}\right]^n \mathbb{E}B_\kappa^{2n}(t).$$

Here $B_\kappa(t)$ is the standard (that is, of zero mean and variance one) fractional Brownian motion with the Hurst exponent κ and

$$d_\kappa = \left(\frac{\pi}{\kappa\Gamma(2\kappa)\sin(\kappa\pi)}\right)^{1/2} = \left(\frac{\pi}{\kappa(2\kappa-1)\Gamma(2\kappa-1)\sin(\kappa\pi)}\right)^{1/2}.$$

Observe that, fortunately, $c_\kappa d_\kappa = 1/(\sqrt{\kappa(2\kappa-1)})$. To summarize, we we have shown that for $\beta < 1/2$

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi)\mathbb{E}e^{i\sqrt{D}B_\kappa(t)}, \qquad (4.25)$$

where D is given by (1.19).

The case $\beta = 1/2$. For $\beta = 1/2$ the calculation is very similar. Then, the Hurst exponent κ given by (1.17) is equal to α, and the right side of (4.24) equals

$$\sum_{\mathcal{F}}\lim_{\varepsilon\downarrow 0}\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{1}{(2n)!}\mathbb{E}[i\sqrt{D(\xi)}B_\alpha(t)]^{2n}$$

where $D(\xi)$ is as in (1.20). This finishes the proof of Proposition 4.4. □

4.2. The limit of higher moments

The next step in the proof of Theorem 1.2 is to show the following.

Proposition 4.5. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)]^N = [\hat{\phi}_0(\xi)]^N\mathbb{E}e^{iN\sqrt{D(\xi)}B_\kappa(t)} \qquad (4.26)$$

for all integers $N \geqq 1$.

Proof. Consider the expansion

$$\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N = \sum_{n_1,\ldots,n_N=0}^{\infty} \hat{\zeta}^{\varepsilon}_{n_1}(t,\xi)\ldots\hat{\zeta}^{\varepsilon}_{n_N}(t,\xi), \tag{4.27}$$

where each term $\hat{\zeta}^{\varepsilon}_n(t,\xi)$ is given by (2.3). Evaluating the expectation in (4.27) and using an argument as in the proof of part (ii) of Proposition 2.1 gives

$$\mathbb{E}\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N = \sum_{n_1,\ldots,n_N=0}^{\infty} J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi), \tag{4.28}$$

where

$$J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi) = \mathbb{E}\left[\hat{\zeta}^{\varepsilon}_{n_1}(t,\xi)\ldots\hat{\zeta}^{\varepsilon}_{n_N}(t,\xi)\right], \tag{4.29}$$

or, equivalently,

$$\begin{aligned} J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi) &= (-1)^n\left[\frac{\gamma}{\varepsilon(2\pi)^{d/2}}\right]^{2n}\int_{\Delta_{n_1}(t)}\cdots\int_{\Delta_{n_N}(t)}\prod_{j=1}^{N}\mathrm{d}\mathbf{s}_1\ldots\mathrm{d}\mathbf{s}_N \qquad (4.30)\\ &\times\int\mathbb{E}\left[\hat{V}\left(\frac{s_{11}}{\varepsilon},\mathrm{d}p_{11}\right)\ldots\hat{V}\left(\frac{s_{1n_1}}{\varepsilon},\mathrm{d}p_{1n_1}\right)\cdots\hat{V}\left(\frac{s_{N1}}{\varepsilon},\mathrm{d}p_{N1}\right)\ldots\right.\\ &\left.\times\hat{V}\left(\frac{s_{Nn_N}}{\varepsilon},\mathrm{d}p_{Nn_N}\right)\right]\\ &\times\hat{\phi}_0(\xi-p_{11}-\cdots-p_{1n_1})\ldots\hat{\phi}_0(\xi-p_{N1}-\cdots-p_{Nn_N})\prod_{j=1}^{N}\mathrm{e}^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}, \end{aligned}$$

where $\mathbf{s}_j = (s_{j1},\ldots,s_{jn_j})$ and $\mathbf{p}_j = (p_{j1},\ldots,p_{jn_j})$. We evaluate the expectation using pairings and get

$$J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi) = \sum_{\mathcal{F}} J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi;\mathcal{F}). \tag{4.31}$$

Here, the summation extends over all pairings formed over pairs of integers (jk), with $j = 1,\ldots,N$, and $k = 1,\ldots,n_j$. We introduce a lexicographical ordering between pairs, that is, we say that $(jk) \prec (j'k')$ if $j < j'$, or if $j = j'$ then $k \leqq k'$. If (e,f) is an edge of a pairing we say that e is a left vertex if $e \prec f$. Also, given a vertex $e = (jk)$ we will use the notation $s(e) = s_{jk}$, $p(e) = p_{jk}$. The following bound holds.

Proposition 4.6. *There exist constants $J_{n_1,\ldots,n_N}(t,\xi)$ such that*

$$\sup_{t\in[0,t]}|J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)| \leqq J_{n_1,\ldots,n_N}(T,\xi), \quad \forall\varepsilon\in(0,1] \tag{4.32}$$

and

$$\sum_{n_1,\ldots,n_N=0}^{+\infty} J_{n_1,\ldots,n_N}(T,\xi) < +\infty.$$

Proof. Using the relation $\varepsilon = \gamma^{1/\kappa}$ in (4.30), dropping the phases and symmetrizing gives

$$|J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)| \leqq \frac{C^n\|\hat{\phi}_0\|^N_\infty}{n_1!\ldots n_N!}\frac{1}{\varepsilon^{2n(1-\kappa)}}\int_0^t\cdots\int_0^t\prod_{j=1}^N \mathrm{d}\mathbf{s}_1\ldots\mathrm{d}\mathbf{s}_N$$
$$\times\left|\int\mathbb{E}\left[\hat{V}\left(\frac{s_{11}}{\varepsilon},\mathrm{d}p_{11}\right)\ldots\hat{V}\left(\frac{s_{1n_1}}{\varepsilon},\mathrm{d}p_{1n_1}\right)\cdots\hat{V}\left(\frac{s_{N1}}{\varepsilon},\mathrm{d}p_{N1}\right)\right.\right.$$
$$\left.\left.\times\ \ldots\hat{V}\left(\frac{s_{Nn_N}}{\varepsilon},\mathrm{d}p_{Nn_N}\right)\right]\right|.$$

The right-hand side can be estimated essentially in the same way as in (4.6) and (4.7), and we obtain that

$$|J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)| \leqq \frac{\|\hat{\phi}_0\|^N_\infty}{n_1!\ldots n_N!}C^n_*\|a\|^n_\infty$$

for any $n_1,\ldots,n_N$ such that $n_1+\cdots+n_N = 2n$. The right-hand side is summable over n_j-s. Thus, we conclude that the conclusion of Proposition 4.6 holds. □

Proposition 4.6 leads to the following.

Corollary 4.7. *We have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)]^N = \sum_{n_1,\ldots,n_N=0}^{\infty}\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi). \tag{4.33}$$

Hence, it remains only to evaluate the individual limits of $J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)$ as $\varepsilon\downarrow 0$.

Computation of $\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)$

In order to rewrite $\lim_{\varepsilon\downarrow 0}J^{\varepsilon}_{n_1,\ldots,n_N}(t,\xi)$ in a form more convenient for the subsequent analysis, we will once again use the variables τ_j, with $s_i = \sum_{j=i}^n\tau_j$ to express $\hat{\zeta}_\varepsilon(t,\xi)$, so the phase function $\tilde{G} = \tilde{A} - \tilde{B}$ with $\tilde{A}$ and $\tilde{B}$ as in (4.9)–(4.11). Then the domain of integration in the τ-variables is the set

$$D_n(t) = \{(\tau_1,\ldots,\tau_n):\ \tau_j \geqq 0 \text{ for all } 1\leqq j\leqq n \text{ and } \tau_1+\cdots+\tau_n\leqq t\}.$$

We will also use the spectral representation of the stationary field $V(t,x)$:

$$V(t,x) = \int \mathrm{e}^{i(\omega t+p\cdot x)}\frac{\hat{V}(\mathrm{d}\omega,\mathrm{d}p)}{(2\pi)^{d+1}},$$

where $\hat{V}(\mathrm{d}\omega,\mathrm{d}p)$ is a Gaussian spectral measure with the structure measure given by

$$\mathbb{E}[\hat{V}(\mathrm{d}\omega,\mathrm{d}p)\hat{V}^*(\mathrm{d}\omega',\mathrm{d}p')] = (2\pi)^{d+1}\delta(\omega-\omega')\delta(p-p')\frac{2\mu a(p)|p|^{2\beta}}{(\omega^2+\mu|p|^{4\beta})|p|^{2\alpha+d-2}}.$$

We can now transform expression (2.3) into

$$\hat{\zeta}_n^{\varepsilon}(t,\xi) = \left(\frac{\gamma}{i\varepsilon}\right)^n \int_{D_n(t)} \mathrm{d}\tau^{(n)} \int \frac{\hat{V}(\mathrm{d}\omega_1,\mathrm{d}p_1)\hat{V}(\mathrm{d}\omega_2,\mathrm{d}p_2)\dots\hat{V}(\mathrm{d}\omega_n,\mathrm{d}p_n)}{(2\pi)^{(n+1)d}}$$
$$\times\hat{\phi}_0(\xi - p_1 - p_2 - \cdots - p_n)$$
$$\times \mathrm{e}^{i[\omega_1(\tau_1+\cdots+\tau_n)+\omega_2(\tau_2+\cdots+\tau_n)+\cdots+\omega_n\tau_n]/\varepsilon}\mathrm{e}^{i\tilde{G}_n(\tau^{(n)},\mathbf{p}^{(n)})/\varepsilon}, \tag{4.34}$$

and further rewrite (4.34) in the following way:

$$\hat{\zeta}_n^{\varepsilon}(t,\xi) = \mathrm{e}^t \left(\frac{\gamma}{i\varepsilon}\right)^n \int_0^{+\infty}\dots\int_0^{+\infty} \mathrm{d}\tau_0\dots\mathrm{d}\tau_n$$
$$\times\int \delta(t-\tau_0-\cdots-\tau_n)\mathrm{e}^{i\sum_{j=1}^n \tau_j(\sum_{k=1}^j \omega_k)/\varepsilon}$$
$$\times\frac{\hat{V}(\mathrm{d}\omega_1,\mathrm{d}p_1)\hat{V}(\mathrm{d}\omega_2,\mathrm{d}p_2)\dots\hat{V}(\mathrm{d}\omega_n,\mathrm{d}p_n)}{(2\pi)^{n(d+1)}}$$
$$\times\hat{\phi}_0(\xi-p_1-\cdots-p_n)\mathrm{e}^{-\sum_{j=0}^n\tau_j}\mathrm{e}^{i\tilde{G}_n(\tau^{(n)},\mathbf{p}^{(n)})/\varepsilon}.$$

Since

$$\delta(t-\tau_0-\cdots\tau_n) = \int_{\mathbb{R}} \mathrm{e}^{-iz(t-\tau_0-\cdots\tau_n)}\frac{\mathrm{d}z}{2\pi},$$

integrating out the τ variables we obtain:

$$\frac{1}{i^n}\int_0^{\infty}\dots\int_0^{\infty}\mathrm{d}\tau_0\,..\,\mathrm{d}\tau_n\mathrm{e}^{-(1-iz)(\tau_0+\cdots+\tau_n)}\mathrm{e}^{i\sum_{m=1}^n\tau_m(\sum_{j=1}^m[(\xi\cdot p_j)+\omega_j]-Q_m(\mathbf{p}^{(n)})/2)/\varepsilon}$$
$$=\frac{1}{1-iz}\prod_{m=1}^n\left\{i+z+\frac{1}{2\varepsilon}\left[2\sum_{j=1}^m[(\xi\cdot p_j)+\omega_j]-Q_m(\mathbf{p}^{(n)})\right]\right\}^{-1},$$

so that (4.34) becomes

$$\hat{\zeta}_n^{\varepsilon}(t,\xi) = \mathrm{e}^t\left(\frac{\gamma}{\varepsilon}\right)^n\int\mathrm{d}z\mathrm{e}^{-izt}\int\frac{\hat{V}(\mathrm{d}p_1,\mathrm{d}\omega_1)\hat{V}(\mathrm{d}p_2,\mathrm{d}\omega_2)\dots\hat{V}(\mathrm{d}p_n,\mathrm{d}\omega_n)}{(2\pi)^{n(d+1)+1}}$$
$$\times\hat{\phi}_0(\xi-p_0-\cdots-p_n)$$
$$\times\frac{1}{1-iz}\left\{\prod_{m=1}^n\left[z+\frac{1}{2\varepsilon}\left[2\sum_{j=1}^m[(\xi\cdot p_j)+\omega_j]-Q_m(p)\right]+i\right]\right\}^{-1}. \tag{4.35}$$

This expression for $\hat{\zeta}_n^{\varepsilon}(t,\xi)$ will be the starting point for our analysis of $J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)$, that is, $J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)$ is given by

$$
\begin{aligned}
J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi) &= \frac{(-1)^n \mathrm{e}^{tN}}{(2\pi)^{2n(d+1)+1}}\left(\frac{\gamma}{\varepsilon}\right)^{2n}\int\dots\int\prod_{j=1}^{N}\left\{\frac{\exp\{-itz_j\}}{1-iz_j}\right\}\mathrm{d}z_1\dots\mathrm{d}z_N \\
&\times\int\mathbb{E}\Big[\hat{V}(\mathrm{d}\omega_{11},\mathrm{d}p_{11})\dots\hat{V}(\mathrm{d}\omega_{1n_1},\mathrm{d}p_{1n_1})\dots\hat{V}(\mathrm{d}\omega_{N1},\mathrm{d}p_{N1}) \\
&\times\dots\hat{V}(\mathrm{d}\omega_{N,n_N},\mathrm{d}p_{Nn_N})\Big] \\
&\times\hat{\zeta}_0(\xi-p_{11}-\dots-p_{1n_1})\dots\hat{\zeta}_0(\xi-p_{N1}-\dots-p_{Nn_N}) \\
&\times\prod_{j=1}^{N}\left\{\prod_{m=1}^{n_j}\left[z_j+\frac{1}{2\varepsilon}\left\{2\sum_{k=1}^{m}[(\xi\cdot p_{jk})+\omega_{jk}]-Q_m(p_j)\right\}+i\right]\right\}^{-1},
\end{aligned}
\tag{4.36}
$$

with $2n=\sum_{j=1}^{N}n_j$. We evaluate the expectation above using pairings and get

$$
\begin{aligned}
J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi) &= \frac{(-1)^n \mathrm{e}^{tN}}{(2\pi)^{n(d+1)+1}}\left(\frac{\gamma}{\varepsilon}\right)^{2n}\sum_{\mathcal{F}}\int\mathrm{d}\mathbf{p}_1\dots\mathrm{d}\mathbf{p}_N\int\mathrm{d}\omega_1\dots\mathrm{d}\omega_N \\
&\times\prod_{(jk,j'm)\in\mathcal{F}}\frac{2\mu|p_{jk}|^{2\beta+2-2\alpha-d}a(p_{jk})}{\omega_{jk}^2+\mu^2|p_{jk}|^{4\beta}}\delta(p_{jk}+p_{j'm})\delta(\omega_{jk}+\omega'_{j'm}) \\
&\times\hat{\zeta}_0(\xi-p_{11}-\dots-p_{1n_1})\dots\hat{\zeta}_0(\xi-p_{N1}-\dots-p_{Nn_N}) \\
&\times\int\mathrm{d}\mathbf{z}\prod_{j=1}^{N}\left\{\frac{\exp\{-itz_j\}}{1-iz_j}\prod_{m=1}^{n_j}\left[z_j+\frac{1}{2\varepsilon}\right.\right. \\
&\left.\left.\times\left[2\sum_{k=1}^{m}(\xi\cdot p_{jk}+\omega_{jk})-Q_m(p_j)\right]+i\right]^{-1}\right\}.
\end{aligned}
\tag{4.37}
$$

Here $\mathrm{d}\mathbf{p}_m := \mathrm{d}p_{m1}\dots\mathrm{d}p_{mn_m}$ and, once again, the summation extends over all pairings formed over elements that are pairs of integers (jk), $j=1,\dots,N$, $k=1,\dots,n_j$.

We change variables, setting $p'=p/\varepsilon^{1/(2\beta)}$, $\omega'=\omega/\varepsilon$ and using the relation $\gamma=\varepsilon^{\kappa}$ we get, after dropping the primes

$$
\begin{aligned}
J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi) &= \frac{(-1)^n \mathrm{e}^{tN}}{(2\pi)^{n(d+1)+1}}\sum_{\mathcal{F}}\int\mathrm{d}\mathbf{p}_1\dots\mathrm{d}\mathbf{p}_N\int\mathrm{d}\omega_1\dots\mathrm{d}\omega_N \\
&\times\prod_{(jk,j'm)\in\mathcal{F}}\frac{2\mu|p_{jk}|^{2\beta+2-2\alpha-d}a(\varepsilon^{1/(2\beta)}p_{jk})}{\omega_{jk}^2+\mu^2|p_{jk}|^{4\beta}}\delta(p_{jk}+p_{j'm})\delta(\omega_{jk}+\omega'_{j'm}) \\
&\times\hat{\zeta}_0(\xi-\varepsilon^{1/(2\beta)}(p_{11}+\dots+p_{1n_1}))\dots\hat{\zeta}_0(\xi-\varepsilon^{1/(2\beta)}(p_{N1}+\dots+p_{Nn_N}))
\end{aligned}
$$

$$\times \int \mathrm{d}\mathbf{z} \prod_{j=1}^{N} \left\{ \frac{\exp\{-itz_j\}}{1-iz_j} \prod_{m=1}^{n_j} \left[z_j + \frac{1}{2} \left[2 \sum_{k=1}^{m} (\xi \cdot p_{jk} \varepsilon^{1/(2\beta)-1} + \omega_{jk}) \right.\right.\right.$$
$$\left.\left.\left. - Q_m(p_j)\varepsilon^{1/\beta-1} \right] + i \right]^{-1} \right\}. \tag{4.38}$$

One can majorize the integrand above by an integrable function, when $\alpha + \beta > 1$, as we did in the proof of Proposition 4.4, and obtain that:

$$\lim_{\varepsilon \downarrow 0} J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n \mathrm{e}^{tN} (\hat{\zeta}_0(\xi))^N (2\mu a(0))^n}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F} \in \mathfrak{F}(2n)} \int \mathrm{d}\mathbf{p}_1 \dots \mathrm{d}\mathbf{p}_N$$
$$\times \int \mathrm{d}\omega_1 \dots \mathrm{d}\omega_N \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{2\beta+2-2\alpha-d}}{\omega_{jk}^2 + \mu^2 |p_{jk}|^{4\beta}} \delta(p_{jk} + p_{j'm}) \delta(\omega_{jk} + \omega'_{j'm})$$
$$\times \int \mathrm{d}\mathbf{z} \prod_{j=1}^{N} \left\{ \frac{\exp\{-itz_j\}}{1-iz_j} \prod_{m=1}^{n_j} \left[z_j + \sum_{k=1}^{m} (\zeta(\beta)(\xi \cdot p_{jk}) + \omega_{jk}) + i \right] \right\}^{-1}. \tag{4.39}$$

Recall that $\zeta(\beta) = 1$ for $\beta = 1/2$ and $\zeta(\beta) = 0$ for $\beta < 1/2$. Now, we need to relate the right side of (4.39) to the fractional Brownian motion. We do it separately for $\beta < 1/2$ and $\beta = 1/2$.

The case $\beta < 1/2$**.** When $\beta < 1/2$ the limit in (4.39) equals

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n \mathrm{e}^{tN} (2\mu a(0))^n [\hat{\zeta}_0(\xi)]^N}{(2\pi)^{n(d+1)+1}}$$
$$\times \sum_{\mathcal{F} \in \mathfrak{F}(2n)} \int \mathrm{d}\mathbf{p}_1 \dots \mathrm{d}\mathbf{p}_N \int \mathrm{d}\omega_1 \dots \mathrm{d}\omega_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{2\beta+2-2\alpha-d} \delta(p_{jk} + p_{j'm}) \delta(\omega_{jk} + \omega'_{j'm})}{\omega_{jk}^2 + \mu^2 |p_{jk}|^{4\beta}} \int \mathrm{d}\mathbf{z}$$
$$\times \prod_{j=1}^{N} \left\{ \frac{\exp\{-itz_j\}}{1-iz_j} \prod_{m=1}^{n_j} \left[z_j + \sum_{k=1}^{m} \omega_{jk} + i \right]^{-1} \right\}. \tag{4.40}$$

Integrating out the z and ω variables and reverting back to the s-variables time we obtain that

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n a^n(0) [\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}} \int \mathrm{d}\mathbf{p}_1 \dots \mathrm{d}\mathbf{p}_N$$
$$\times \int_{\Delta_{n_1}(t)} \cdots \int_{\Delta_{n_N}(t)} \mathrm{d}\mathbf{s}_1 \dots \mathrm{d}\mathbf{s}_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{\mathrm{e}^{-\mu |p_{jk}|^{2\beta} |s_{jk} - s_{j'm}|}}{|p_{jk}|^{2\alpha+d-2}} \delta(p_{jk} + p_{j'm}). \tag{4.41}$$

Integrating out, also, the p variables gives

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0)K_1(\alpha,\beta,\mu)]^n}{(2\pi)^{nd}}[\hat{\zeta}_0(\xi)]^N \sum_{\mathcal{F}} \times \int_{\Delta_{n_1}(t)} \dots \int_{\Delta_{n_N}(t)} \mathrm{d}\mathbf{s}_1 \dots \mathrm{d}\mathbf{s}_N |s_{jk} - s_{j'm}|^{(\alpha-1)/\beta} \tag{4.42}$$

It remains now to relate $\bar{J}_{n_1,\dots,n_N}(t,\xi)$ to the fractional Brownian motion and to sum all these terms. Note that the function

$$f(s_1,\dots,s_{2n}) := \sum_{\mathcal{F}} \prod_{\widehat{km}\in\mathcal{F}} |s_k - s_m|^{2\alpha-2}$$

is symmetric in all of its arguments, that is, $f(s_1,\dots,s_{2n}) = f(s_{\pi(1)},\dots,s_{\pi(2n)})$, where π is an arbitrary permutation of $\{1, 2. \dots, 2n\}$. Using this fact we can rewrite (4.42) in the form

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0)K_1(\alpha,\beta,\mu)]^n[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd} n_1!\dots n_N!} \times \sum_{\mathcal{F}} \underbrace{\int_0^t \dots \int_0^t}_{n_1\text{-times}} \mathrm{d}\mathbf{s}_1 \dots \underbrace{\int_0^t \dots \int_0^t}_{n_N\text{-times}} \mathrm{d}\mathbf{s}_N \prod_{(jk,j'm)\in\mathcal{F}} |s_{jk} - s_{j'm}|^{2\alpha-2}. \tag{4.43}$$

Recall that, as (4.21),

$$\sum_{\mathcal{F}} \prod_{(jk,j'm)\in\mathcal{F}} |s_{jk} - s_{j'm}|^{2\kappa-2} = c_\kappa^{2n} \mathbb{E}\left[\prod_{j=1}^N \prod_{m=1}^{n_j} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_{jm}s_{jm}}}{|k_{jm}|^{\alpha-1/2}} w(\mathrm{d}k_{jm})\right],$$

where $w(\mathrm{d}k)$ is a Gaussian white noise and $c_\kappa > 0$ is given by (4.22). We get

$$\begin{aligned}\bar{J}_{n_1,\dots,n_N}(t,\xi) &= \frac{(-1)^n[\hat{\zeta}_0(\xi)]^N (a(0)K_1(\alpha,\beta,\mu)c_\kappa^2)^n}{(2\pi)^{nd} n_1!\dots n_N!} \\ &\times \underbrace{\int_0^t \dots \int_0^t}_{n_1\text{-times}} \mathrm{d}\mathbf{s}_1 \dots \underbrace{\int_0^t \dots \int_0^t}_{n_N\text{-times}} \mathrm{d}\mathbf{s}_N \mathbb{E}\left[\prod_{j=1}^N \prod_{m=1}^{n_j} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_{jm}s_{jm}}}{|k_{jm}|^{\alpha-1/2}} w(\mathrm{d}k_{jm})\right] \\ &= \frac{[\hat{\zeta}_0(\xi)]^N \mathbb{E}[iD_\xi^{1/2} B_\kappa(t)]^{n_1+\dots+n_N}}{n_1!\dots n_N!}.\end{aligned} \tag{4.44}$$

Performing the integrations with respect to s_i and then subsequently the summation over $n_1,\dots,n_N$, we obtain

$$\sum_{n_1=0,\dots,n_N=0}^{+\infty} \bar{J}_{n_1,\dots,n_N}(t,\xi) = [\hat{\zeta}_0(\xi)]^N \mathbb{E}\left[\exp\left\{iND_\xi^{1/2} B_\kappa(t)\right\}\right], \tag{4.45}$$

where $B_\kappa(t)$ is the fractional Brownian motion with the Hurst exponent κ and variance 1.

The case $\beta = 1/2$. The computation for $\beta = 1/2$ is very similar to that for $\beta < 1/2$. Here the limit in (4.39) equals

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n(2\mu)a^n(0)[\hat{\zeta}_0(\xi)]^N e^{tN}}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1\dots d\mathbf{p}_N \int d\omega_1\dots d\omega_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{3-2\alpha-d}\delta(p_{jk}+p_{j'm})\delta(\omega_{jk}+\omega'_{j'm})}{\omega_{jk}^2+\mu^2|p_{jk}|^2}$$
$$\times \int d\mathbf{z} \prod_{j=1}^N \left\{ \frac{\exp\{-itz_j\}}{1-iz_j} \prod_{m=1}^{n_j} \left[z_j + \sum_{k=1}^m (\xi\cdot p_{jk}+\omega_{jk}) + i \right]^{-1} \right\}. \tag{4.46}$$

Integrating out the z and ω variables and reverting to s coordinates for time, we obtain that

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n a^n(0)[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1\dots d\mathbf{p}_N$$
$$\times \int_{\Delta_{n_1}(t)} \cdots \int_{\Delta_{n_N}(t)} d\mathbf{s}_1\dots d\mathbf{s}_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} |p_{jk}|^{2-2\alpha-d} e^{-\mu|p_{jk}||s_{jk}-s_{j'm}|+i\xi\cdot p_{jk}(s_{jk}-s_{j'm})}\delta(p_{jk}+p_{j'm}). \tag{4.47}$$

Integrating out the p variables we obtain that

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0)K_2(\xi;\mu)]^n[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}}$$
$$\times \int_{\Delta_{n_1}(t)} \cdots \int_{\Delta_{n_N}(t)} d\mathbf{s}_1\dots d\mathbf{s}_N |s_{jk}-s_{j'm}|^{2(\alpha-1)}, \tag{4.48}$$

with the constant K_2, as in (1.16). From here on we conduct the calculation as in the previous case. This finishes the proof of Proposition 4.5. □

4.3. The limit of absolute moments and convergence in law

The principal objective of this section is to establish the following.

Proposition 4.8. *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left\{|\hat{\zeta}_\varepsilon(t,\xi)|^{2N}\right\} = |\hat{\phi}_0(\xi)|^{2N} \tag{4.49}$$

for all integers $N \geqq 1$.

Convergence in law Proposition 4.8 in turn, in combination with (4.26), implies that

$$\lim_{\varepsilon\downarrow 0}\ \mathbb{E}\left\{[\hat{\zeta}_\varepsilon(t,\xi)]^N\hat{\zeta}_\varepsilon(t,\xi)^{*M}\right\} = \begin{cases} [\hat{\phi}_0(\xi)]^{N-M}|\hat{\phi}_0(\xi)|^{2M}\mathbb{E}e^{i(N-M)\sqrt{D(\xi)}B_\kappa(t)} & \text{when } N\geqq M \\ [\hat{\phi}_0(\xi)]^{*(M-N)}|\hat{\phi}_0(\xi)|^{2N}\mathbb{E}e^{i(N-M)\sqrt{D(\xi)}B_\kappa(t)} & \text{when } M\geqq N, \end{cases} \tag{4.50}$$

for all integers $N, M \geqq 1$. Let us denote $\hat{\zeta}(t,\xi) := \hat{\phi}_0(\xi)e^{i\sqrt{D(\xi)}B_\kappa(t)}$. We can conclude the convergence of $\hat{\zeta}_\varepsilon(t,\xi) \Rightarrow \hat{\zeta}(t,\xi)$ in law, provided it can be proved that the moments of the random vector $\mathbf{Z} := (\mathrm{Re}\,\hat{\zeta}(t,\xi), \mathrm{Im}\,\hat{\zeta}(t,\xi))$ determine its law.

It can easily be checked that

$$\sum_{n=1}^{+\infty}\left\{\mathbb{E}|\hat{\zeta}(t,\xi)|^{2n}\right\}^{-1/(2n)} \geqq |\hat{\phi}_0(\xi)|\sum_{n=1}^{+\infty}1 = +\infty. \tag{4.51}$$

Using Carleman's criterion for well-posedness of the moment problem on the real line, see for example [26], we can conclude from (4.51) that the moments of $\eta_1\mathrm{Re}\,\hat{\zeta}(t,\xi) + \eta_2\mathrm{Im}\,\hat{\zeta}(t,\xi)$, computed for an arbitrary $(\eta_1,\eta_2)\in\mathbb{R}^2$, determine its law. This, in turn, determines the characteristic function of $\mathbf{Z}$. In consequence, we conclude that the moments of the random vector determine its law, thus also the law of $\hat{\zeta}(t,\xi)$ on $\mathbb{C}$. This finishes the proof of Theorem 1.2 modulo the proof of Proposition 4.8. □

The proof of Proposition 4.8 The proof is very similar to that of Proposition 4.5. Evaluating the expectation in (4.49) we conclude that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left\{|\hat{\zeta}_\varepsilon(t,\xi)|^{2N}\right\} = \sum_{j=1}^{2}\sum_{l=1}^{N}\sum_{n_{jl}=0}^{\infty}\lim_{\varepsilon\downarrow 0}H_\varepsilon(t,\xi;(n_{jl})), \tag{4.52}$$

with

$$\begin{aligned} H_\varepsilon(t,\xi;\mathbf{n}) = {} & e^{2Nt}\left(\frac{\gamma}{\varepsilon}\right)^{2n}\sum_{\mathcal{F}}\int\prod_e dp_e\int\prod_e d\omega_e \\ & \times\prod_{j=1}^{2}\prod_{l=1}^{N}\hat{\phi}_{0j}\left(\xi - \sum_{e:\mathfrak{j}(e)=j,\,\mathfrak{l}(e)=l}p_e\right) \\ & \times\prod_{(e,e')\in\mathcal{F}}\frac{2\mu|p_e|^{2\beta+2-2\alpha-d}a(p_e)}{\omega_e^2+\mu^2|p_e|^{4\beta}}\delta(p_e+(-1)^{\mathfrak{j}(e)+\mathfrak{j}(e')}p_e') \\ & \times\delta(\omega_e+(-1)^{\mathfrak{j}(e)+\mathfrak{j}(e')}\omega_e')\int d\mathbf{z}\prod_{j=1}^{2}\prod_{l=1}^{N} \end{aligned}$$

$$\times\left\{\frac{\exp\left\{(-1)^j itz_l^{(j)}\right\}}{1+(-1)^j iz_l^{(j)}}\prod_{k=1}^{n_{jl}}\left[z_l^{(j)}+\frac{1}{2\varepsilon}\left[2\sum_{m=1}^{k}(\xi\cdot p_m^{(jl)}+\omega_m^{(jl)})\right.\right.\right.$$

$$\left.\left.\left.-Q_k(p^{(j,l)})\right]+(-1)^{j-1}i\right]^{-1}\right\}. \tag{4.53}$$

The summation above extends over all pairings $\mathcal{F}$ made of vertices $e=(j,l,k)$, $j=1,2$, $l=1,\dots,N$, $k=1,\dots,n_{jl}$. The difference with the non-absolute moments case considered in Proposition 4.5 is in the factor $(-1)^j$ that appears in the last line above. The rest is identical.

As before, we change variables, setting $p'=p/\varepsilon^{1/(2\beta)}$, $\omega'=\omega/\varepsilon$ and using the relation $\gamma=\varepsilon^\kappa$ we pass to the limit, as $\varepsilon\downarrow 0$. In case $\beta<1/2$ and $\alpha+\beta>1$ the limiting expression equals

$$\bar{H}(t,\xi;\mathbf{n}):=\lim_{\varepsilon\downarrow 0}H_\varepsilon(t,\xi;\mathbf{n})=\mathrm{e}^{2Nt}|\phi_0(\xi)|^{2N}\sum_{\mathcal{F}}\int\prod_e dp_e\int\prod_e d\omega_e$$

$$\times\prod_{(e,e')\in\mathcal{F}}\frac{2\mu|p_e|^{2\beta+2-2\alpha-d}a(0)}{\omega_{\mathrm{e}}^2+\mu^2|p_e|^{4\beta}}\delta(p_e+(-1)^{\mathrm{j}(e)+\mathrm{j}(e')}p_e')\delta(\omega_e+(-1)^{\mathrm{j}(e)+\mathrm{j}(e')}\omega_e')$$

$$\times\int d\mathbf{z}\prod_{j=1}^{2}\prod_{l=1}^{N}\left\{\frac{\exp\left\{(-1)^j itz_l^{(j)}\right\}}{1+(-1)^j iz_l^{(j)}}\prod_{k=1}^{n_{jl}}\left[z_l^{(j)}+\sum_{m=1}^{k}\omega_m^{(jl)}+(-1)^{j-1}i\right]^{-1}\right\} \tag{4.54}$$

We integrate z and ω variables in the same way as has been done in Section 4.2, see (4.41)–(4.44), and obtain

$$\bar{H}(t,\xi;\mathbf{n})=|\hat{\phi}_0(\xi)|^{2N}\mathbb{E}\left\{\prod_{j=1}^{2}\frac{\left[(-1)^{j-1}iD_\xi^{1/2}B_\kappa(t)\right]^{n_{j,1}+\cdots+n_{j,N}}}{n_{j,1}!\dots n_{j,N}!}\right\}. \tag{4.55}$$

The only difference in the above expression compared to the corresponding one for the non-absolute moments is in the factor $(-1)^{j-1}$ in the numerator, above. In consequence, we obtain now

$$\sum_{\mathbf{n}}\bar{H}(t,\xi;\mathbf{n})=|\hat{\phi}_0(\xi)|^{2N}\mathbb{E}\left[\exp\left\{iND_\xi^{1/2}B_\kappa(t)\right\}\exp\left\{-iND_\xi^{1/2}B_\kappa(t)\right\}\right]$$

$$=|\hat{\phi}_0(\xi)|^{2N} \tag{4.56}$$

and (4.49) follows. The case $\beta=1/2$ is very similar and is therefore omitted. □

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