THE PARABOLIC WAVE APPROXIMATION AND TIME REVERSAL

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Abstract

We analyze the phenomenon of super-resolution in time-reversal acoustics. A signal recorded by an array of transducers and sent back reversed in time approximately refocuses on the original source point. The refocusing resolution is improved in an inhomogeneous medium when the pulse has rich frequency content. We show that the back-propagated signal is self-averaging in this situation. This explains the statistical stability of the time-reversal refocusing and super-resolution.

1 Introduction

In time-reversal acoustics a signal is propagated from a source to an array of receivers-transducers, where it is recorded and then sent back reversed in time. The re-transmitted signal propagates back through the same medium and refocuses approximately on the source because of time-reversibility of the wave equation. The refocusing is approximate since the array of transducers (also called a time reversal mirror, or TRM) has a finite size which causes information loss. The refocusing phenomenon of the time-reversed signal has many applications in medicine, underwater sound, wireless communications etc., and has been studied experimentally in many settings. Recent popular reviews [4, 5] contain detailed description of various possible applications. One application is kidney stone destruction, when the signal reflected from a kidney stone is time-reversed, amplified and sent back. It refocuses on the stone and destroys it. Time-reversal for a fixed frequency amounts to the phase-conjugation method which has been studied extensively in optics [9]. However, it was observed

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experimentally that refocusing in time-reversal of broad band signals occurs much more effectively than for a single frequency phase conjugation. One of the purposes of this paper is to explain this difference.

It was also observed in experiments that refocusing effect is sharper when the medium is inhomogeneous, or has reflecting boundaries. In a homogeneous medium the width of the refocusing peak is given by the Fresnel zone. Its size is of the order of $\lambda L/a$. Here λ is the wavelength, L is the propagation distance to the time reversal mirror, and a is the size of TRM. Furthermore, spurious focusing spots arise at the higher Fresnel zone locations. If the medium is inhomogeneous then both the focusing resolution of the back propagated signal may be better than the Fresnel limit and spurious ghosts disappear. For instance, in underwater acoustics a signal with wavelength about 1m propagated over a distance about 10km may refocus after time-reversal at a TRM of size 50m on a spot of the size of 1m, while Fresnel zone is of the order of 100m [2, 8, 6].

One of the reasons for this super-resolution in a random medium is multipathing [3], which effectively increases the size of the TRM because the arriving signal has traveled through a larger part of the medium than in the homogeneous case. However, this requires statistical averaging, while in actual experiments compression is observed without any averaging (especially in the kidney stone treatment) if the signal has rich frequency content.

We suggest here that the key for the statistical stability of the time-reversed signal is decorrelation of different frequencies that occurs for every realization of the random medium and does not require averaging. This idea was first proposed and investigated numerically and analytically by one of the authors, P. Blomgren and H. Zhao in a companion paper [1]. We present here a detailed theory leading to super-resolution in a random medium based on frequency decorrelation.

We assume that the size of the mirror a is smaller than the overall propagation distance L, so that parabolic, or beam approximation is valid [11]. Then wave amplitude at frequency $\omega = c_0 k$ is described by the Schrödinger parabolic equation

$$2ik\psi_z + \Delta_{\mathbf{x}}\psi + k^2(n^2 - 1)\psi = 0, \quad \mathbf{x} = (x, y). \tag{1}$$

Here $n(x, y, z)$ is the index of refraction of the medium, and waves are propagating mostly in the direction z. We believe that all our results may be extended to the full wave equation, this will be pursued elsewhere. The parabolic approximation provides a simple setup, where time-reversal super-resolution may be analyzed without additional technical complications. The wavelength λ is assumed to be much smaller than a and L. The correlation length of the fluctuations is larger than the wave length. The latter requirement is needed for the validity of the parabolic approximation. Fluctuations are assumed to be weak. More precisely, we let $\varepsilon = \lambda/L$ be a small parameter, and consider the fluctuations of the index of refraction in the following two related scalings:

$$n(x,y,z) = 1 + \sqrt{N\varepsilon}\mu\left(\frac{x}{N\varepsilon}, \frac{y}{N\varepsilon}, \frac{z}{N\varepsilon}\right) \tag{2}$$

with $1 \ll N \ll 1/\varepsilon$, or

$$n(x,y,z) = 1 + \sqrt{\delta}\mu\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{z}{\delta}\right) \tag{3}$$

with $\varepsilon \ll \delta \ll 1$. The first scaling is studied in Section 2, and the second in Section 3. The main result of this paper is that the back-propagated and time reversed signal is self-averaging. More precisely, it converges to its average in probability in both scalings above, as $\varepsilon \to 0$ and $N \to \infty$ in the first scaling (2), and as $\varepsilon \to 0$ and $\delta \to 0$ in the second (3).

The paper is organized as follows. In Section 2 we first outline the derivation of the parabolic approximation. Then, in order to establish the self-averaging of the back-propagated signal, we relate it to the Wigner distribution (denoted W) of a pair of Green's functions for a rescaled version of (1). The Wigner distribution is a function of a position and wave vector. It is a convenient tool for a study of waves in a random medium [10]. In the first scaling (2) we obtain a transport equation for W as $\varepsilon \to 0$ for a fixed N and at a fixed frequency. Passing then to the large N limit we obtain a diffusion equation for W in the wave vector space, which allows us to get an explicit expression for the effective aperture of a Gaussian time-reversal mirror. The role of frequency decorrelation in pulse stabilization and super-resolution is discussed in detail in Section 2.4. This part of the paper is a detailed development of the analysis outlined in [1]. The second scaling (3) is studied in Section 3. We first obtain a Liouville equation for the Wigner distribution in the limit $\varepsilon \to 0$ and fixed $\delta > 0$. Passing to the limit $\delta \to 0$ we recover diffusion in wave vectors. The Liouville equation provides a convenient setup for the study of the decay of correlations of various frequencies and at different positions, which is done in

Section 3.3. This leads to self-averaging of broad band time-reversed signals, which explains the super-resolution phenomenon. We summarize our results in Section 4.

2 Loss of correlation in forward scattering in a random medium

We develop here in detail the theory of time-reversal in a random medium outlined in [1].

2.1 The parabolic approximation

We start with the wave equation

$$\frac{1}{c^2(x,y,z)}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0. \tag{4}$$

Taking its Fourier transform in time we get the Helmholtz equation

$$\Delta \hat{u} + k^2 n^2(x,y,z)\hat{u} = 0 \tag{5}$$

with

$$\hat{u}(x,y,z;\omega) = \int_{\mathbb{R}} dt e^{-i\omega t} u(t,x,y,z).$$

Here $k = \omega/c_0$ is the wave number, c_0 is a reference speed, $c(x,y,z)$ is the propagation speed, and $n(x,y,z) = c_0/c(x,y,z)$ is the index of refraction. Throughout the paper we define the Fourier transform by

$$\hat{f}(\mathbf{k}) = \int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})$$

so that

$$f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{f}(\mathbf{k}).$$

The time reversal mirror is typically much smaller than the distance to the source. This puts the problem in the narrow beam regime, and the parabolic approximation may be used as follows. We write the solution of the Helmholtz equation in the form

$$\hat{u}(x,y,z;k) = e^{ikz}\psi(x,y,z;k) \tag{6}$$

and obtain

$$2ik\psi_z + \psi_{zz} + \Delta_{\mathbf{x}}\psi + k^2(n^2 - 1)\psi = 0, \quad \mathbf{x} = (x, y).$$

We assume that the function ψ is slowly varying in z so that

$$k|\psi_z| \gg |\psi_{zz}|. \tag{7}$$

Then we get a parabolic (paraxial) initial value problem for ψ:

$$\begin{aligned} &2ik\psi_z + \Delta_{\mathbf{x}}\psi + k^2(n^2 - 1)\psi = 0, \\ &\psi(0, \mathbf{x}) = \psi_0(\mathbf{x}). \end{aligned} \tag{8}$$

The parabolic approximation is not valid near the point source where the beam geometry is not appropriate. Moreover, it may not be used when the typical size of inhomogeneities l is comparable to the wave length λ of the signal. When $kl = O(1)$ the variations of $n(\mathbf{x}, z)$ will produce oscillations in ψ in the z direction on the scale $l \sim 1/k$. That will make the two terms in (7) comparable and violate the validity of the parabolic approximation.

2.2 The back-propagated signal in the high-frequency limit

In order to study the effect of random inhomogeneities on the back propagated signal we consider the regime of high frequency and weak fluctuations. More precisely, we consider the following scaling:

(i) The wavelength λ is short compared to the propagation distance L, and we let $\varepsilon = \lambda/L \ll 1$ be a small parameter;

(ii) The fluctuations of the index of refraction are weak and isotropic:

$$\langle (n-1)^2 \rangle = O(\varepsilon).$$

Here and below $\langle \cdot \rangle$ denotes averaging with respect to realizations of a random medium.

(iii) The correlation length l of the fluctuations is comparable to the wave length: $l \sim \lambda$.

If the fluctuations are strong or anisotropic then backscattering is important and and parabolic approximation may not be used. Our last assumption makes the interaction between waves and inhomogeneities strong. However, it violates condition (7) for the validity of the parabolic approximation. We will ignore that for the moment but at the end of this section we will pass to the beam approximation limit of the transport equation which will restore the validity of the paraxial regime. This important issue will be addressed in detail in Section 3, where we will show how the beam equation may be obtained in one step from the parabolic approximation bypassing the transport regime altogether and without violation of (7).

Under assumptions made above we write

$$n(z,\mathbf{x}) = 1 + \sqrt{\varepsilon}\mu(z,\mathbf{x}).$$

The random process $\mu(z,\mathbf{x})$ is assumed to be a stationary isotropic mean zero random process:

$$\langle \mu(z,\mathbf{x})\rangle = 0, \quad \langle \mu(z,\mathbf{x})\mu(z+z',\mathbf{x}+\mathbf{x}')\rangle = R(z',\mathbf{x}') \tag{9}$$

with the power spectrum

$$\hat{R}(\Omega,\mathbf{q}) = \int dz d\mathbf{x} e^{-i\Omega z - i\mathbf{q}\cdot\mathbf{x}} R(z,\mathbf{x}). \tag{10}$$

The correlation function $R(z',\mathbf{x}')$ depends only on $|z'|$ and $|\mathbf{x}'|$.

We rescale the spatial variables $\mathbf{x} \to \mathbf{x}/\varepsilon$, $z \to z/\varepsilon$ and dropping the term $\varepsilon\mu^2/2$ that is of a higher order, we obtain the high-frequency version of (8)

$$ik\varepsilon\psi^\varepsilon_z + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi^\varepsilon + k^2\sqrt{\varepsilon}\mu(\frac{\mathbf{x}}{\varepsilon},\frac{z}{\varepsilon})\psi^\varepsilon = 0. \tag{11}$$

The initial data should be rescaled accordingly:

$$\psi^\varepsilon(0,\mathbf{x}) = \psi^\varepsilon_0(\mathbf{x}) = \psi_0\left(\frac{\mathbf{x}}{\varepsilon}\right) \tag{12}$$

so that the initial pulse enters the random medium at position $\mathbf{x} = 0$.

The time-reversed and back-propagated filed is constructed in the frequency space as follows. The Green's function $G_\varepsilon(z,\mathbf{x};\xi)$ with a point source at $(0,\xi)$, $\xi = (\xi_1,\xi_2)$ satisfies (11) with the initial data

$$G_\varepsilon(0,\mathbf{x};\xi) = \delta(\mathbf{x}-\xi). \tag{13}$$

The recorded wave field at the time reversal mirror at distance L from the source along the z axis is given then by

$$\psi^\varepsilon(L,\mathbf{x}) = \int d\eta G_\varepsilon(L,\mathbf{x};\eta)\psi_0^\varepsilon(\eta). \tag{14}$$

Then the part of this signal recored at the mirror is sent back reversed in time. Time reversal $t \to -t$ is equivalent to the change $\omega \to -\omega$, or $k \to -k$. Therefore the back-propagated signal may be approximated in the parabolic approximation as

$$\hat{u}^B = e^{-ikz}\psi^B(x,y,z;k)$$

in the unscaled variables, as in (6). The amplitude $\psi^{\varepsilon,B}$ solves the backward Schrödinger equation after rescaling:

$$-ik\varepsilon\psi_z^{\varepsilon,B} + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi^{\varepsilon,B} + k^2\sqrt{\varepsilon}\mu(\frac{\mathbf{x}}{\varepsilon},\frac{z}{\varepsilon})\psi^{\varepsilon,B} = 0 \tag{15}$$

with the end condition

$$\psi^{\varepsilon,B}(L,\mathbf{x};k) = \psi^\varepsilon(L,\mathbf{x};-k). \tag{16}$$

The identities

$$\int d\mathbf{x} G_\varepsilon(L,\mathbf{x};\xi;k)\bar{G}_\varepsilon(L,\mathbf{x};\xi_0) = \delta(\xi-\xi_0)$$

and

$$\bar{G}_\varepsilon(L,\mathbf{x};\xi;k) = G_\varepsilon(L,\mathbf{x};\xi;-k)$$

imply that the backpropagated field at the plane $z=0$ may be written as

$$\begin{aligned}\psi^{\varepsilon,B}(0,\varepsilon\xi) &= \int d\mathbf{x}\bar{G}_\varepsilon(L,\mathbf{x};\varepsilon\xi;-k)\psi^\varepsilon(L,\mathbf{x};-k)\chi_A(\mathbf{x})\\ &= \iint d\mathbf{x}d\eta G_\varepsilon(L,\mathbf{x};\varepsilon\xi;k)\bar{G}_\varepsilon(L,\mathbf{x};\eta;k)\psi_0^\varepsilon(\eta;-k)\chi_A(\mathbf{x}).\end{aligned}$$

We choose the observation point $\mathbf{x} = \varepsilon\xi$ close to the source point $\mathbf{x} = 0$ with separation being comparable to the wavelength. The signal received at distances much larger than the wave length will be negligible. Here χ_A is the aperture function of the mirror, which may be the characteristic function of the mirror, that occupies region A in the plane $z = L$:

$$\chi_A(\mathbf{x}) = \begin{cases} 1 & \mathbf{x}\in A \\ 0 & \mathbf{x}\notin A \end{cases}.$$

However, we may consider more general aperture functions, when mirrors are more complicated, for instance, χ_A may be a Gaussian:

$$\chi_A(\mathbf{x}) = \frac{1}{2\pi a^2} e^{-|\mathbf{x}|^2/2a^2}.$$

We will analyze our problem by the invariant imbedding method with respect to the position of the time-reversal mirror. For that purpose it is convenient to introduce

$$\Psi^{\varepsilon,B}(L, \varepsilon\xi) = \psi^{\varepsilon,B}(0, \varepsilon\xi)$$

with the first argument in Ψ reflecting the position of the time reversal mirror. Recall that the incoming pulse for the wave equation is real and thus $\psi(\eta, -k) = \bar{\psi}(\eta, k)$. Then the back-propagated signal is given by

$$\Psi^{\varepsilon,B}(L, \varepsilon\xi) = \iint d\mathbf{x} d\eta G_\varepsilon(L, \mathbf{x}; \varepsilon\xi) \bar{G}_\varepsilon(L, \mathbf{x}; \eta) \bar{\psi}_0^\varepsilon(\eta) \chi_A(\mathbf{x}). \tag{17}$$

The back-propagated signal in time near the source is given by

$$u^{\varepsilon,B}(t, \varepsilon\xi, 0) = \int \frac{d\omega}{2\pi} e^{i\omega t} \Psi^{\varepsilon,B}(L, \varepsilon\xi; \omega/c_0),$$

where we put in explicitly the dependence of Ψ^B on wave number $k = \omega/c_0$. Here the time t is measured after subtracting the travel time from the source to the time-reversal mirror and back. As we shall see, the statistical properties of the signal in time are very different from that for individual frequencies.

It is convenient to rewrite the field $\Psi^{\varepsilon,B}$ as follows, using the special form of the initial data (12)

$$\begin{aligned} \Psi^{\varepsilon,B}(L, \varepsilon\xi) &= \varepsilon^2 \iint d\eta d\mathbf{x} \chi_A(\mathbf{x}) \bar{\psi}_0(\eta) G_\varepsilon(L, \mathbf{x}; \varepsilon\xi) \bar{G}_\varepsilon(L, \mathbf{x}; \varepsilon\eta) \\ &= \iint d\eta d\mathbf{y} \chi_A(\mathbf{y}) \bar{\psi}_0(\eta) \int d\mathbf{p} W_\varepsilon(L, \mathbf{y}, \mathbf{p}; \xi, \eta). \end{aligned} \tag{18}$$

We introduced here the Wigner distribution of two Green's functions with sources at $\varepsilon\xi$ and $\varepsilon\eta$, respectively:

$$W_\varepsilon(\mathbf{z}, \mathbf{x}, \mathbf{p}; \xi, \eta) = \varepsilon^2 \int_{\mathbb{R}^2} \frac{d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{y}} G_\varepsilon(\mathbf{z}, \mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}; \varepsilon\xi) G_\varepsilon(\mathbf{z}, \mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}; \varepsilon\eta). \tag{19}$$

The scaling factor ε^2 that appears in (18) and (19) because we have normalized the incoming pulse to have amplitude $O(1)$ makes the limit of the Wigner distribution as $\varepsilon \to 0$ at $z = 0$ non-trivial. More precisely, we have for any test function $\phi(\mathbf{x})$:

$$\begin{aligned}\int d\mathbf{x}\phi(\mathbf{x})W_\varepsilon(0,\mathbf{x},\mathbf{p};\xi,\eta) &= \varepsilon^2 \int \frac{d\mathbf{x}d\mathbf{y}e^{i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^2}\phi(\mathbf{x})\delta(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}-\varepsilon\xi)\\ &\quad\times\delta(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}-\varepsilon\eta)\\ &= \int \frac{d\mathbf{x}d\mathbf{y}}{(2\pi)^2}e^{i\mathbf{p}\cdot(\eta-\xi)+i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})/\varepsilon}\delta(\mathbf{x})\delta(\mathbf{y})\phi\left(\frac{\varepsilon(\xi+\eta)}{2}+\frac{\mathbf{x}+\mathbf{y}}{2}\right)\\ &= e^{i\mathbf{p}\cdot(\eta-\xi)}\phi\left(\frac{\varepsilon(\xi+\eta)}{2}\right).\end{aligned}$$

Therefore the initial data for the Wigner distribution is

$$W_\varepsilon(0,\mathbf{x},\mathbf{p}) = \frac{e^{i\mathbf{p}\cdot(\eta-\xi)}}{(2\pi)^2}\delta(\mathbf{x}-\frac{\varepsilon(\xi+\eta)}{2}). \tag{20}$$

Note that we consider the Wigner distribution of Green's functions with sources at two different points. Therefore it needs not be real. It becomes real only if we take the same point on the screen and on the mirror, that is, if the two source points coincide: $\eta = \xi$. The exponential factor in (20) is extremely important. It carries the phase information that is crucial in the time-reversal refocusing phenomenon, since it takes place mostly due to phase cancellations.

2.3 The transport equation

Relation (18) shows that the backscattered field may be represented in terms of the Wigner distribution, which is a well-studied object in the context of the high-frequency limit in a random medium of the type we are considering [10]. The Wigner distribution satisfies the following initial value problem, which can be derived from the rescaled Schrödinger equation (11) for the Green's function:

$$\begin{aligned}k\frac{\partial W_\varepsilon}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W_\varepsilon &= \frac{k^2}{i\sqrt{\varepsilon}}\int\frac{d\mathbf{q}d\omega}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon+i\omega z/\varepsilon}\hat{\mu}\left(\omega,\mathbf{q}\right)\\ &\quad\times\left[W_\varepsilon\left(\mathbf{z},\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2}\right)-W_\varepsilon\left(\mathbf{z},\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2}\right)\right]\end{aligned}$$

with the initial data (20). Here $\hat{\mu}(\omega,\mathbf{q})$ is the Fourier transform of $\mu(z,\mathbf{x})$ both in z and $\mathbf{x}$. An asymptotic theory based on a straightforward multiple scales expansion (see [10] and references therein) shows that in the high frequency limit $\varepsilon \to 0$, the averaged Wigner distribution $\langle W_\varepsilon \rangle$ converges to the solution $\langle W(z, \mathbf{x}, \mathbf{p}) \rangle$ of the following transport equation

$$
\begin{aligned}
k\frac{\partial \langle W \rangle}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} \langle W \rangle &= k^3 \int\limits_{\mathbb{R}^2} \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(\frac{\mathbf{p}^2 - \mathbf{q}^2}{2k}, \mathbf{p} - \mathbf{q}\right) \\
&\quad \times \left[\langle W \rangle(z, \mathbf{x}, \mathbf{q}) - \langle W \rangle(z, \mathbf{x}, \mathbf{p})\right] \\
\langle W \rangle(0, \mathbf{x}, \mathbf{p}) &= \frac{e^{i\mathbf{p}\cdot(\eta - \xi)}}{(2\pi)^2} \delta(\mathbf{x}).
\end{aligned} \tag{21}
$$

Here $R(\Omega, \mathbf{q})$ is the power spectrum of μ defined by (10). We note that for every realization of the random medium the Wigner distribution W_ε has a weak limit W even without averaging. However, the limit of the unaveraged Wigner distribution may not be described in simple terms as a stochastic process to the best of our knowledge.

2.4 Pulse stabilization

As established by physical experiments discussed in the Introduction and references given there, time reversal of very narrow band signals is not statistically stable and refocusing is not very sharp. This reflects itself in the fact that for a fixed frequency $\omega = c_0 k$ only the average $\langle W_\varepsilon \rangle$ over realizations of μ has a nice limit, but not W_ε itself. The same is true for the back-propagated field $\Psi^{\varepsilon,B}(L, \varepsilon\xi; k)$ by virtue of relation (18). However, as we will discuss in detail in Section 3, the Wigner distributions for different wave numbers k_1, k_2 become decorrelated in the high-frequency limit:

$$
\langle W_\varepsilon(z, \mathbf{x}_1, \mathbf{p}_1; k_1) W_\varepsilon(z, \mathbf{x}_2, \mathbf{p}_2; k_2) \rangle \to \langle W(z, \mathbf{x}_1, \mathbf{p}_1; k_1) \rangle \langle W(z, \mathbf{x}_2, \mathbf{p}_2; k_2) \rangle \tag{22}
$$

and thus

$$
\langle \Psi^{\varepsilon,B}(L, \varepsilon\xi; k_1) \Psi^{\varepsilon,B}(L, \varepsilon\xi; k_2) \rangle - \langle \Psi^B(L, \varepsilon\xi; k_1) \rangle \langle \Psi^B(L, \varepsilon\xi; k_2) \rangle \to 0
$$

as $\varepsilon \to 0$, with

$$
\Psi^B(L, \varepsilon\xi; k) = \iint d\eta d\mathbf{y} \chi_A(\mathbf{y}) \bar{\psi}_0(\eta) \int d\mathbf{p} W(L, \mathbf{y}, \mathbf{p}; \xi, \eta).
$$

Therefore we have

$$\langle u^{\varepsilon,B}(t,\varepsilon\mathbf{x},0)^2\rangle = \langle\left(\int \frac{d\omega}{2\pi} e^{i\omega t}\Psi^{\varepsilon,B}(L,\varepsilon\xi,\frac{\omega}{c_0})d\omega\right)^2\rangle$$
$$= \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} e^{i\omega_1 t+i\omega_2 t}\langle\Psi^{\varepsilon,B}(L,\varepsilon\xi,\frac{\omega_1}{c_0})\Psi^{\varepsilon,B}(L,\varepsilon\xi,\frac{\omega_2}{c_0})\rangle$$
$$\to \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} e^{i\omega_1 t+i\omega_2 t}\langle\Psi^{B}(L,\varepsilon\xi,\frac{\omega_1}{c_0})\rangle\langle\Psi^{B}(L,\varepsilon\xi,\frac{\omega_2}{c_0})\rangle.$$

Therefore, we obtain

$$\langle u^{\varepsilon,B}(t,\varepsilon\mathbf{x},L)^2\rangle - \langle u^{\varepsilon,B}(t,\varepsilon\mathbf{x},L)\rangle^2 \to 0 \tag{23}$$

and hence Chebyshev's inequality implies that

$$u^{\varepsilon,B}(t,\varepsilon\mathbf{x},L) - \langle u^{\varepsilon,B}(t,\varepsilon\mathbf{x},L)\rangle \to 0$$

in probability as $\varepsilon \to 0$. That means that the received time-reversed and back-propagated signal is self-averaging and has a deterministic limit. We postpone the detailed discussion of the decorrelation property (22) until Section 3. We show in Section 3 that the Wigner distribution decorrelates for different wave vectors even at the same frequency, and analyze this decorrelation in some detail. The important difference between one-frequency and broad band time-reversal is that in the former case only decorrelation for different wave vectors plays a role, while in the latter both decorrelation for different wave vectors and for different frequencies contribute to puls stabilization. We will also show that decorrelation for different wave vectors occurs only for wave vectors separated by a fairly large distance which is not a sufficiently effective mechanism to provide pulse stabilization.

2.5 Beam approximation

The transport equation (21) was derived in the scaling when the correlation length of the inhomogeneities is comparable to the wave length. This, as we noted before, violates the validity of the parabolic approximation. In order to reconcile the two theories we assume that the correlation length $l = N\lambda$ with $N \gg 1$, and rescale $\mu_N(z,\mathbf{x}) = \sqrt{N}\mu(z/N,\mathbf{x}/N)$, as in (2). Then the rescaled correlation function is given by

$$R_N(z',\mathbf{x}') = N\langle\mu(\frac{z}{N},\frac{\mathbf{x}}{N})\mu(\frac{z+z'}{N},\frac{\mathbf{x}+\mathbf{x}'}{N})\rangle = NR(\frac{z'}{N},\frac{\mathbf{x}'}{N})$$

and the power spectrum becomes

$$\hat{R}_N(\Omega, \mathbf{q}) = N \int dz d\mathbf{x} e^{-i\Omega z - i\mathbf{q}\cdot\mathbf{x}} R(\frac{z}{N}, \frac{\mathbf{x}}{N}) = N^4 \hat{R}(N\Omega, N\mathbf{q}).$$

We may now perform the transport limit, letting first $\varepsilon \to 0$, and then consider the large N limit of the transport equation. The operator on the right side of the transport equation (21) takes now the form

$$\begin{aligned}
& N^4 k^3 \int \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(N\frac{\mathbf{p}^2 - \mathbf{q}^2}{2k}, N(\mathbf{p}-\mathbf{q})\right) [\langle W\rangle(z,\mathbf{x},\mathbf{q}) - \langle W\rangle(z,\mathbf{x},\mathbf{p})] \\
& = N^2 k^3 \int \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(\frac{(\mathbf{q}\cdot(2\mathbf{p} + \frac{\mathbf{q}}{N}))}{2k}, \mathbf{q}\right) \left[\langle W\rangle(z,\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{N}) - \langle W\rangle(z,\mathbf{x},\mathbf{p})\right] \\
& = \frac{k^3}{2} \int \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(\frac{1}{k}(\mathbf{q}\cdot\mathbf{p}), \mathbf{q}\right) q_i q_j \frac{\partial^2 \langle W\rangle(z,\mathbf{x},\mathbf{p})}{\partial p_i \partial p_j} \\
& + \frac{k^2}{2} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{\partial \hat{R}}{\partial \Omega}\left(\frac{1}{k}(\mathbf{q}\cdot\mathbf{p}), \mathbf{q}\right) |\mathbf{q}|^2 q_i \frac{\partial \langle W\rangle(z,\mathbf{x},\mathbf{p})}{\partial p_i} + O(\frac{1}{N}) \\
& = \frac{k^3}{2} \frac{\partial}{\partial p_i} \left[\int \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(\frac{1}{k}(\mathbf{q}\cdot\mathbf{p}), \mathbf{q}\right) q_i q_j \frac{\partial \langle W\rangle(z,\mathbf{x},\mathbf{p})}{\partial p_j}\right] + O(\frac{1}{N}).
\end{aligned}$$

We used here the isotropy of fluctuations $\hat{R}(\Omega, \mathbf{q})$ depends only on $|\Omega|$ and $|\mathbf{q}|$. Thus in the leading order we obtain a diffusion equation in the $\mathbf{p}$ variable:

$$k\frac{\partial \langle W\rangle}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\langle W\rangle = \frac{k^3}{2}\frac{\partial}{\partial p_i}\left[D_{ij}(\mathbf{p})\frac{\partial\langle W\rangle}{\partial p_j}\right] \tag{24}$$

with the diffusion matrix given by

$$D_{ij}(\mathbf{p}) = \int_{\mathbb{R}^2} \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}\left(\frac{1}{k}(\mathbf{q}\cdot\mathbf{p}), \mathbf{q}\right) q_i q_j. \tag{25}$$

2.6 Application to super-resolution in time-reversal

The diffusion equation (24) is the basis of a simple explanation of the super-resolution phenomenon. Let us assume for simplicity that the power spectrum $\hat{R}(\Omega, \mathbf{q})$ is independent of Ω. Then the diffusion coefficient $D_{ij}(\mathbf{p}) = D\delta_{ij}$ with

$$D = \frac{1}{2}\int_{\mathbb{R}^2} \frac{d\mathbf{q}}{(2\pi)^2} \hat{R}(\mathbf{q}) |\mathbf{q}|^2$$

independent of $\mathbf{p}$. Equation (24) may be now solved explicitly by taking the Fourier transform in $\mathbf{x}$ and $\mathbf{p}$ and using the method of characteristics. Let

$$\hat{W}(z, \mathbf{q}, \mathbf{y}) = \int d\mathbf{x} d\mathbf{p} e^{-i\mathbf{q}\cdot\mathbf{x} - i\mathbf{p}\cdot\mathbf{y}} \langle W\rangle(z, \mathbf{x}, \mathbf{p})$$

be the Fourier transform of $\langle W\rangle$. We solve (24) to get

$$\int d\mathbf{p}\langle W\rangle(z,\mathbf{x},\mathbf{p}) = \int \frac{d\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}}\hat{W}(z,\mathbf{q},0)$$
$$= \left(\frac{k}{2\pi z}\right)^2 e^{ik(\mathbf{x}\cdot(\xi-\eta))/z - Dk^2(\eta-\xi)^2 z/6}.$$

Then the average time-reversed and back-propagated field is given by

$$\Psi^B(L,\varepsilon\xi;k) = \int d\eta d\mathbf{y}\chi_A(\mathbf{y})\bar{\psi}_0(\eta)\int d\mathbf{p}\langle W(L,\mathbf{y},\mathbf{p};\xi,\eta)\rangle$$
$$= \left(\frac{k}{2\pi L}\right)^2 \int d\eta\hat{\chi}_A\left(\frac{k(\eta-\xi)}{L}\right)\bar{\psi}_0(\eta)e^{-Dk^2(\eta-\xi)^2 L/6}.$$

We see that the effect of the random medium is the additional factor $e^{-Dk^2\eta^2 z/6}$.

Let us consider a model case when the aperture function is given by a normalized Gaussian

$$\chi_A(\mathbf{y}) = \frac{1}{2\pi a}e^{-|\mathbf{y}|^2/(2\pi a^2)}, \quad \hat{\chi}_A(\mathbf{q}) = e^{-a^2|\mathbf{q}|^2/2}$$

and the initial pulse is a delta-function: $\psi_0(\eta) = \delta(\eta)$. Then we obtain

$$\Psi^B(L,\varepsilon\xi;k) = \left(\frac{k}{2\pi L}\right)^2 e^{-a^2k^2\xi^2/(2L^2) - Dk^2 z\xi^2/6}.$$

Therefore the effective aperture in the presence of randomness is given by

$$a_e = \sqrt{a^2 + \frac{DL^3}{3}}. \tag{26}$$

A remarkable feature of this expression is that the effective aperture is independent of frequency, and thus the same expression is true in the time domain. Recall that we have obtained it in the beam approximation, thus we must have

$$\frac{a_e}{L} \ll 1,$$

therefore it may applied only as long as randomness is not too strong:

$$D \ll \frac{L^2 - a^2}{L^3}.$$

3 The beam approximation and random geometrical optics

We show in this section an alternative way to derive the diffusion equation (24) in a regime when the correlation length of the fluctuations is much larger than the wave length, which corresponds to random geometrical optics. That scaling allows to avoid violating the validity of the parabolic approximation in an intermediate step as happened in the transport regime. We consider the regime when the wavelength λ is short compared to both the propagation distance L and the correlation length l. We let $\varepsilon = \lambda/L \ll 1$ be a small parameter as before. However, we consider a different type of randomness, motivated by the combination of the two scaling limits we performed in Section 2: weak fluctuations and narrow beam. Recall that the narrow beam scaling corresponded to randomness of the form $\sqrt{N\varepsilon}\mu(\mathbf{x}/(N\varepsilon), z/(N\varepsilon))$. We performed first the limit $\varepsilon \to 0$ and then $N \to \infty$. Instead of doing these two limits we consider random fluctuations of the form $\sqrt{\delta}\mu(\mathbf{x}/\delta)$ with $\varepsilon \ll \delta \ll 1$. We will first take the limit $\varepsilon \to 0$, and then let $\delta \to 0$. This scaling with $\delta \gg \varepsilon$ has an advantage that we do not violate condition (7) for the validity of the parabolic approximation. The first limit $\varepsilon \to 0$ at a fixed $\delta > 0$ corresponds to doing geometrical optics in a random medium. The second limit $\delta \to 0$ is weak fluctuations and large distance propagation in the geometrical optics regime which leads to the diffusion equation (24)

3.1 The Liouville equation

We rescale the spatial variables $\mathbf{x} \to \mathbf{x}/\varepsilon$ and $z \to z/\varepsilon$ and obtain under our assumption regarding the random fluctuations

$$ik\varepsilon\psi_z^\varepsilon + \frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi^\varepsilon + k^2\sqrt{\varepsilon}\mu(\mathbf{x}, z)\psi^\varepsilon = 0. \tag{27}$$

Here $\mu(\mathbf{x}, z)$ is a stationary isotropic random process as in (9). The time-reversed and back propagated field can still be expressed in terms of the Wigner distribution of a pair of Green's functions as in (18). The Green's function $G_\varepsilon^\delta(L, \mathbf{x}; \xi)$ is now the solution of (27) with initial data (13), and the Wigner distribution is defined by (19) with G_ε replaced by G_ε^δ. The Wigner equation for W_ε^δ now takes the form

$$k\frac{\partial W_\varepsilon^\delta}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W_\varepsilon^\delta = \frac{k^2}{i\varepsilon}\int \frac{d\mathbf{q}d\omega}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}+i\omega z}\hat{\mu}_\delta(\omega,\mathbf{q}) \times \left[W_\varepsilon\left(\mathbf{z},\mathbf{x},\mathbf{p}+\frac{\varepsilon\mathbf{q}}{2}\right) - W_\varepsilon\left(\mathbf{z},\mathbf{x},\mathbf{p}-\frac{\varepsilon\mathbf{q}}{2}\right)\right]$$

with the initial data (20). We denoted here $\mu_\delta(\mathbf{x},z) = \sqrt{\delta}\mu(\mathbf{x}/\delta, z/\delta)$. It is straightforward to show that as $\varepsilon\to 0$ with a fixed δ the Wigner distribution W_ε^δ converges weakly to a limit measure W^δ. The measure W^δ satisfies the Liouville equation

$$k\frac{\partial W^\delta}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^\delta + \frac{k^2}{\sqrt{\delta}}\nabla\mu\left(\frac{\mathbf{x}}{\delta},\frac{z}{\delta}\right)\cdot\nabla_{\mathbf{p}} W^\delta = 0 \tag{28}$$

with the initial data as in (21)

$$W(0,\mathbf{x},\mathbf{p};\xi,\eta) = \frac{e^{i\mathbf{p}\cdot(\eta-\xi)}}{(2\pi)^2}\delta(\mathbf{x}). \tag{29}$$

For a fixed $\delta > 0$ this equation describes the phase space version of the standard geometrical optics. In particular, one may recover the usual transport and eikonal equations for the amplitude A and phase S by making an ansatz $W = |A|^2\delta(\mathbf{p}-\nabla S)$. We have a different type of initial data, so standard geometrical optics would not be immediately applicable, while the Liouville equation still holds. and are interested in the limit of W^δ as $\delta\to 0$.

3.2 The weak fluctuations limit

We are interested in the limit of W^δ as $\delta\to 0$, which is weak fluctuations limit. This problem for equation (28) is known as the stochastic acceleration problem. It was studied rigorously in [7] for a potential μ that is independent of "time" z. The fast z dependence makes our situation significantly easier for a rigorous analysis. We will present a formal multiple scales asymptotic expansion that gives a quick way to get the limit of W^δ.

We introduce the fast scale variables $\tau = z/\varepsilon$ and $\mathbf{y} = \mathbf{x}/\varepsilon$ and consider an asymptotic expansion for W^δ of the form

$$W^\delta(z,\mathbf{x},\mathbf{p}) = W^{(1)}(z,\mathbf{x},\mathbf{p}) + \sqrt{\delta}W^{(1)}(z,\tau,\mathbf{x},\mathbf{y},\mathbf{p}) + \delta W^{(2)}(z,\tau,\mathbf{x},\mathbf{y},\mathbf{p}) + \dots$$

We assume that the leading order term $W^{(0)}$ is independent of the fast variables τ and $\mathbf{y}$ and that it is deterministic. We insert this expansion into (28) and obtain an equation for $W^{(1)}$:

$$k\frac{\partial W^{(1)}}{\partial \tau} + \mathbf{p}\cdot\nabla_{\mathbf{y}} W^{(1)} = -k^2 \nabla\mu\left(\mathbf{y},\tau\right)\cdot\nabla_{\mathbf{p}} W^{(0)}.$$

A formal solution of this equation is given by

$$W^{(1)}(z,\tau,\mathbf{x},\mathbf{y},\mathbf{p}) = -k^2 \int_{-\infty}^{0} \nabla\mu\left(\mathbf{y}+\mathbf{p}s,\tau+ks\right)\cdot\nabla_{\mathbf{p}} W^{(0)}(\mathbf{x},\mathbf{p}).$$

In the next order we obtain

$$k\frac{\partial W^{(0)}}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^{(0)} + k\frac{\partial W^{(2)}}{\partial \tau} + \mathbf{p}\cdot\nabla_{\mathbf{y}} W^{(2)} = -k^2 \nabla\mu\left(\mathbf{y},\tau\right)\cdot\nabla_{\mathbf{p}} W^{(1)}.$$

Averaging this equation eliminates the gradients of $W^{(2)}$ and we get

$$k\frac{\partial W^{(0)}}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^{(0)} = -k^2 \langle\nabla\mu\left(\mathbf{y},\tau\right)\cdot\nabla_{\mathbf{p}} W^{(1)}\rangle.$$

We insert the expression for $W^{(1)}$ into the above and obtain

$$\begin{aligned}
&-k^2\langle\nabla\mu\left(\mathbf{y},\tau\right)\cdot\nabla_{\mathbf{p}} W^{(1)}\rangle \\
&= k^4\langle\frac{\partial\mu\left(\mathbf{y},\tau\right)}{\partial y_j}\frac{\partial}{\partial p_j}\left[\int_{-\infty}^{0}\frac{\partial\mu\left(\mathbf{y}+\mathbf{p}s,\tau+ks\right)}{\partial y_m}\frac{\partial W^{(0)}(\mathbf{x},\mathbf{p})}{\partial p_m}\right]\rangle \\
&= -k^4\frac{1}{2}\frac{\partial}{\partial p_j}\left[\int_{-\infty}^{\infty} ds\frac{\partial^2 R(\mathbf{p}s,ks)}{\partial y_i\partial y_m}\frac{\partial W^{(0)}}{\partial p_m}\right] \\
&= \frac{k^3}{2}\frac{\partial}{\partial p_j}\left[\int\frac{d\mathbf{q}}{(2\pi)^2}\hat{R}(\mathbf{q},\frac{\mathbf{q}\cdot\mathbf{p}}{k})q_i q_m\frac{\partial W^{(0)}}{\partial p_m}.\right]
\end{aligned}$$

Then we get the diffusion equation for $W^{(0)}$:

$$k\frac{\partial W^{(0)}}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^{(0)} = \frac{k^3}{2}\frac{\partial}{\partial p_i}\left[D_{ij}(\mathbf{p})\frac{\partial W^{(0)}}{\partial p_j}\right] \tag{30}$$

with the diffusion matrix D_{ij} given by (25) as in (24). A rigorous version of the above formal calculation would show that only the average $\langle W^{\delta}\rangle$ converges to the solution $W^{(0)}$ of (30) as was already observed in the transport/beam approximation in Section 2. Therefore we recover the diffusion equation (24) obtained in Section 2, but now we bypass the intermediate step of the radiative transport, and go through random geometrical optics instead. The two paths from the parabolic approximation to diffusion in the wave vector space lead to the same result.

3.3 Pulse stabilization and decay of correlations

As we discussed in Section 2.4 pulse stabilization is closely related to decorrelation of the Wigner distribution for various frequencies. We analyze this decorrelation now and show that for nearby points it occurs even for the same frequency, and correlations persist only for short times.

We start by looking at the behavior of the higher moments of $W^\delta(z,\mathbf{x},\mathbf{p};k)$ at the same point:

$$W_m^\delta = \left[W^\delta(z,\mathbf{x},\mathbf{p};k)\right]^m.$$

Then W_m satisfies the same Liouville equation as W^δ:

$$k\frac{\partial W_m^\delta}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}W_m^\delta + \frac{k^2}{\sqrt{\delta}}\nabla\mu\left(\frac{\mathbf{x}}{\delta},\frac{z}{\delta}\right)\cdot\nabla_{\mathbf{p}}W_m^\delta = 0 \tag{31}$$

with the initial data

$$W_m^\delta(0,\mathbf{x},\mathbf{p}) = [W_0(\mathbf{x},\mathbf{p})]^m.$$

Here W_0 is the initial data for W^δ, which we take as a smooth approximation to (20). Then the same asymptotic analysis as in the previous Section shows that as $\delta\to 0$ the moments $\langle W_m^\delta\rangle$ converge to the solution W_m of the diffusion equation (30) with the initial data $W_m(0,\mathbf{x},\mathbf{p}) = [W_0(\mathbf{x},\mathbf{p})]^m$. In particular we have

$$W_m(z,\mathbf{x},\mathbf{p}) \neq [W^{(0)}(z,\mathbf{x},\mathbf{p})]^m$$

and thus W^δ does not have a deterministic limit, which means once again that for a fixed frequency the Wigner distribution is not self-averaging.

However, as we show now, the Wigner distribution even for nearby points becomes decorrelated on a short time scale even at the same frequency. It is convenient to introduce

$$\tilde{W}^\delta(z,\mathbf{x},\mathbf{p};k) = W^\delta(z,\mathbf{x},k\mathbf{p};k).$$

It satisfies the Liouville equation with k-dependence preserved only in the initial data:

$$\begin{aligned}
&\frac{\partial\tilde{W}^\delta}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{W}^\delta + \frac{1}{\sqrt{\delta}}\nabla\mu\left(\frac{\mathbf{x}}{\delta},\frac{z}{\delta}\right)\cdot\nabla_{\mathbf{p}}\tilde{W}^\delta = 0\\
&\tilde{W}_k(0,\mathbf{x},\mathbf{p};k) = W_0(\mathbf{x},k\mathbf{p};k).
\end{aligned}$$

Its solution is obtained by the method of characteristics as

$$\tilde{W}_k(z,\mathbf{x},\mathbf{p};k) = W_0(\mathbf{X}^\delta(0;x,\mathbf{p}),k\mathbf{P}^\delta(0;\mathbf{x}.\mathbf{p}))$$

with $\mathbf{X}^\delta$ and $\mathbf{P}^\delta$ being the solutions of

$$\frac{d\mathbf{X}^\delta}{ds} = \mathbf{P}^\delta, \qquad \mathbf{X}^\delta(t) = \mathbf{x}$$
$$\frac{d\mathbf{P}^\delta}{ds} = \frac{1}{\sqrt{\delta}}\nabla\mu\left(\frac{\mathbf{X}^\delta}{\delta}, \frac{z}{\delta}\right), \quad \mathbf{P}^\delta(t) = \mathbf{p}.$$

Therefore W^δ may be written as

$$W^\delta(z, \mathbf{x}, \mathbf{p}; k) = W_0(\mathbf{X}^\delta(0; x, \frac{\mathbf{p}}{k}), k\mathbf{P}^\delta(0; \mathbf{x}, \frac{\mathbf{p}}{k})) \tag{32}$$

As our calculation leading to (30) showed, and as may be shown rigorously as in [7], the process $\mathbf{P}^\delta$ converges as $\delta \to 0$ to a diffusion process $\mathbf{P}$ with generator as in (30), and $\mathbf{X}^\delta$ converges to its integral

$$\mathbf{X}(z) = \mathbf{x} + \int_0^t \mathbf{P}^\delta(s)ds.$$

As relation (32) shows, correlations of W^δ both at two different frequencies and at two different points in the $(\mathbf{x}, \mathbf{p})$–space may be described in terms of correlations of the processes $\mathbf{X}^\delta(t, \mathbf{x}, \mathbf{p}), \mathbf{P}^\delta(t, \mathbf{x}, \mathbf{p})$ starting at different points. Therefore we look now at the correlations of $(\mathbf{X}^\delta, \mathbf{P}^\delta)$ and show that they persist only for short times. That will allows us to conclude that W^δ decorrelates both for two different points and at different frequencies.

It is intuitively clear that if the process $(\mathbf{X}^\delta, \mathbf{P}^\delta)$ starts at two points $\mathbf{x}_1$ and $\mathbf{x}_2$ separated by distance large compared to the randomness scale δ then there is no reason to expect that the processes are correlated because they will sample uncorrelated media. Therefore we consider the starting points separated by distance of order δ in $\mathbf{x}$. It is also clear that the points should be close in $\mathbf{p}$–space, otherwise the distance between the particles in $\mathbf{x}$ will become much larger than the correlation length at distances $O(\delta)$ in z and the particles will decorrelated on this scale. Motivated by these considerations we introduce the second moment of W^δ

$$U^\delta(z, \mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}) = \tilde{W}^\delta(z, \mathbf{x}, \mathbf{p})\tilde{W}^\delta(z, \mathbf{x} + \delta\mathbf{y}, \mathbf{p} + \delta^\mu\mathbf{q})$$

with the parameter $\mu > 0$ to be chosen. It measures the initial particle separation in the wave vector space. We get the equation for U^δ:

$$\frac{\partial\tilde{U}^\delta}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}U^\delta + \delta^{\mu-1}\mathbf{q}\cdot\nabla_{\mathbf{y}}U^\delta + \frac{1}{\sqrt{\delta}}\nabla\mu\left(\frac{\mathbf{x}}{\delta}, \frac{z}{\delta}\right)\cdot\nabla_{\mathbf{p}}U^\delta$$
$$+ \delta^{-\mu-1/2}\left[\nabla\mu\left(\frac{\mathbf{x}}{\delta} + \mathbf{y}, \frac{z}{\delta}\right) - \nabla\mu\left(\frac{\mathbf{x}}{\delta}, \frac{z}{\delta}\right)\right]\cdot\nabla_{\mathbf{q}}U^\delta = 0$$

We rescale now also the spatial variables $z = \delta^\alpha \tau$ and $\mathbf{x} = \delta^\alpha \mathbf{x}'$ with $\alpha > 0$ to be chosen to get

$$\frac{\partial U^\delta}{\partial \tau} + \mathbf{p} \cdot \nabla_{\mathbf{x}'} U^\delta + \delta^{\alpha+\mu-1} \mathbf{q} \cdot \nabla_{\mathbf{y}} U^\delta + \delta^{\alpha-1/2} \nabla \mu \left(\frac{\mathbf{x}'}{\delta^{1-\alpha}}, \frac{\tau}{\delta^{1-\alpha}} \right) \cdot \nabla_{\mathbf{p}} U^\delta$$
$$+ \delta^{\alpha-\mu-1/2} \left[\nabla \mu \left(\frac{\mathbf{x}'}{\delta^{1-\alpha}} + \mathbf{y}, \frac{\tau}{\delta^{1-\alpha}} \right) - \nabla \mu \left(\frac{\mathbf{x}'}{\delta^{1-\alpha}}, \frac{\tau}{\delta^{1-\alpha}} \right) \right] \cdot \nabla_{\mathbf{q}} U^\delta = 0$$

We choose now parameters μ and α as follows. The diffusive scaling requires that $0 < 1 - \alpha$ and

$$\alpha - \mu - \frac{1}{2} = -\frac{1-\alpha}{2}$$

and so $\mu = \alpha/2 \leq 1/2$. We will also assume that $\mu \geq 1/3$ so that $\alpha + \mu - 1 \geq 0$. Thus the range of μ we consider is $1/3 \leq \mu < 1/2$. Then a formal multiple scales expansion very similar to that in Section 3.2 and with the fast variables $\eta = \tau/\delta^{1-2\mu}$ and $\mathbf{x}'' = \mathbf{x}'/\delta^{1-2\mu}$, shows that $\langle U^\delta \rangle \to U$. The function U solves the diffusion equation

$$\frac{\partial U}{\partial \tau} + \mathbf{p} \cdot \nabla_{\mathbf{x}'} U = \frac{\partial}{\partial q_i} \left(D_{ij}^{(2)}(\mathbf{y}, \mathbf{p}) \frac{\partial U}{\partial q_j} \right)$$

if $\mu > 1/3$, and

$$\frac{\partial U}{\partial \tau} + \mathbf{p} \cdot \nabla_{\mathbf{x}'} U + \mathbf{q} \cdot \nabla_{\mathbf{y}} U = \frac{\partial}{\partial q_i} \left(D_{ij}^{(2)}(\mathbf{y}, \mathbf{p}) \frac{\partial U}{\partial q_j} \right)$$

if $\mu = 1/3$. The diffusion coefficient $D_{ij}^{(2)}$ is given by

$$D_{ij}^{(2)}(\mathbf{y}, \mathbf{p}) = \int_{-\infty}^{\infty} \left[\frac{\partial^2 R(\mathbf{y} + \mathbf{p}s, s)}{\partial y_j \partial y_i} - \frac{\partial^2 R(\mathbf{p}s, s)}{\partial y_j \partial y_i} \right].$$

This result may interpreted as follows: we start two particles at distance $O(\delta)$ apart in the physical space and $O(\delta^\mu)$ apart in the wave vectors. Then the physical separation remains of order $O(\delta)$ while wave vectors diffuse apart on the time scale $O(\delta^{2\mu})$. The wave vector separation keeps growing until it reaches the size $O(\delta^{1/3})$. At this time the spatial separation also starts growing according to

$$\frac{d\mathbf{Y}}{dz} = \mathbf{Q}$$

and the particles separate in the physical space on the penetration scale $O(\delta^{2/3})$ in z inside the random medium. For larger distances inside the medium they are no longer correlated since their separation is larger than $O(\delta)$.

It is interesting to interpret our result in terms of the stochastic caustic theory [12]. The latter predicts that in a random medium that has correlation length a and fluctuations of size σ a caustic will form at distance $O(\sigma^{-2/3}a)$ along the ray. At this moment the ray angle deviation is of order $O(\sigma^{2/3})$ while position deviates $O(1)$ from its deterministic value. We have $\sigma = \sqrt{\delta}$ and $a = \delta$, so a caustic will form at $z = O(\delta^{-1/3}\delta) = O(\delta^{2/3})$, when rays deviate by $q = O(\delta^{1/3})$. This is exactly the scaling that we obtain for the decorrelation distance and wave vector separation at the decorrelation moment. Therefore, as one might expect, decorrelation in the phase space is closely related to the caustic formation, which is physically quite natural.

4 Concluding remarks

We have analyzed the time-reversal acoustics in a weakly random medium with predominantly forward scattering. We have considered two possible asymptotic regimes and have shown that both of them lead to self-averaging of the back-propagated time-reversed signal. This explains the statistical stability of the refocusing phenomenon in time-reversal experiments. We have also shown that the the super-resolution phenomenon may be explained in this asymptotic regime as an effective increase in the aperture of the time-reversal mirror because of multipathing effects. We considered the decay of correlations of waves propagating with different frequency, wave vectors and positions, and related the decorrelation to the stochastic caustic theory. Numerical results that motivated the theoretical results of this paper have been presented in [1].

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