The Harnack inequality for second-order parabolic equations with divergence-free drifts of low regularity

Mihaela Ignatova* Igor Kukavica† Lenya Ryzhik‡

April 23, 2013

Abstract

We establish the Harnack inequality for advection-diffusion equations with divergence-free drifts of low regularity. While our previous work [IKR] considered the elliptic case, here we treat the more challenging parabolic problem by adapting the classical Moser technique to parabolic equations with drifts with regularity lower than the scale-invariant spaces.

1 Introduction

In this paper, we address the qualitative properties of solutions to the parabolic equation

$$u_t - \Delta u + b \cdot \nabla u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where b is a given divergence free vector field of low regularity, and Ω is a space-time domain. The study of such equations with non-smooth drifts $b(x,t)$ is motivated by the need to understand the qualitative and quantitative properties of nonlinear partial differential equations, where the drift depends on the solution u and its first derivatives and for which we often do not have a priori bounds available except in some very low regularity spaces. Advection-diffusion equations of the form (1.1) often arise in applications with the additional divergence-free condition $\operatorname{div} b = 0$, in particular, in problems involving incompressible fluids. Several important recent papers have addressed regularity of the solutions of the linear advection-diffusion equations with very little smoothness assumptions on the divergence free drift [CV1, CV2, FV, KNSS, NU, SSSZ, Z]. Here, we study this problem for the parabolic equation (1.1) with a divergence-free "supercritical" drift b. Criticality here referes to the following property: the usual parabolic rescaling $x \to \lambda x$, $t \to \lambda^2 t$ leaves the norm of the drift term in the equation invariant in a space $b \in L_t^q L_x^p$ if $2/q + n/p = 1$. Accordingly, we say that the drift is critical if this relation holds, is subcritical if $2/q + n/p < 1$ and is supercritical if $2/q + n/p > 1$.

*Department of Mathematics, Stanford University, Stanford CA 94305, e-mail: mihaelai@.stanford.edu

†Department of Mathematics, University of Southern California, Los Angeles, CA 90089, e-mail: kukavica@usc.edu

‡Department of Mathematics, Stanford University, Stanford CA 94305, e-mail: ryzhik@math.stanford.edu

Our main result is the Harnack-type inequality for parabolic advection-diffusion equations with "supercritical" drifts. We use the notation

$$Q^*_R(x_0,t_0)=\{(x,t)\in\mathbb{R}^{n+1}\colon |x-x_0|<R,\ t_0<t<t_0+R^2\} \tag{1.2}$$

for the parabolic cylinder centered at the bottom and

$$Q_R(x_0,t_0)=\{(x,t)\in\mathbb{R}^{n+1}\colon |x-x_0|<R,\ t_0-R^2<t<t_0\} \tag{1.3}$$

for the parabolic cylinder centered at the top, while we denote $Q_R=Q_R(0,0)$.

Theorem 1.1. *Let u be a nonnegative Lipschitz solution to the parabolic equation*

$$u_t-\Delta u+b\cdot\nabla u=0 \quad\text{in}\quad \Omega, \tag{1.4}$$

that is,

$$\int_\Omega(\partial_t u)\varphi+\int_\Omega(\partial_j u)(\partial_j\varphi)+\int_\Omega b_j(\partial_j u)\varphi=0 \tag{1.5}$$

for any Lipschitz function $\varphi\geq 0$ in Ω and $\varphi=0$ in Ω^c. Assume that $b\in L^{\bar q}(\Omega)\cap L^\infty L^2(\Omega)$ with $n/2+1<\bar q\leq n+2$ and $\operatorname{div} b=0$ *in the sense of distributions. Then for any $Q_{2R}\subset\Omega$,*

$$\sup_{Q_{R/2}(0,-3R^2)} u\leq \Big(C+C(R^{1-(n+2)/\bar q}\|b\|_{L^{\bar q}})^{1/(2-(n+2)/\bar q)}\Big)^{C(n)/p_0}\inf_{Q_R} u, \tag{1.6}$$

where $p_0=1/(CM_R^C)$ and $M_R=1+(R^{1-n/2}\|b\|_{L^\infty L^2})^2+R^{1-(n+2)/\bar q}\|b\|_{L^{\bar q}}$.

Here, and elsewhere in this paper, the symbol C denotes a large constant which depends on the parameters $\bar q$ and n, and on the domain $\Omega\subset\mathbb{R}^{n+1}$. Also, we denote the anisotropic Lebesque spaces by $L^pL^q(\Omega)=L^p_tL^q_x(\Omega)$, and in the case when $p=q$, by $L^q(\Omega)=L^q_{x,t}(\Omega)$.

The qualitative properties of solutions to the equation (1.1) have been extensively studied in the past. In particular, Harnack's inequality for the second order parabolic equation

$$u_t-\partial_i(a_{ij}(x,t)\partial_j u)=0$$

in the self-adjoint form, with measurable strongly elliptic coefficients a_{ij} was obtained in the seminal work of Moser [M] for subctricical drifts and no lower order terms. In [L], Liebermann established the Harnack inequality in the case of non-zero lower-order coefficients, when the drift belongs to a subcritical space.

Recently, Nazarov and Ural'tseva proved in [NU] that the assumptions on the divergence free drift b can be significantly relaxed to allow it to lie in the scale invariant (critical) Morrey spaces $M^{n/q+2/l-1}_{l,q}$ for all q and l satisfying $1\leq n/q+2/l<2$. Seregin et al. (c.f. [SSSZ]) established the Harnack inequality when b belongs to $L^\infty(BMO^{-1})$, which is also a critical (scale-invariant) space. In our previous paper [IKR], we obtained a Harnack inequality for elliptic equations with supercritical divergence-free drifts. The purpose of the present paper is to relax the assumptions on the drift in the parabolic case to lie in a supercritical space. Note that the approach from [IKR] does not apply here and the proof in the parabolic case is different.

The paper is organized as follows. In Section 2, we establish the local boundedness of nonnegative Lipschitz subsolutions to (1.1) by using Moser's iteration. This result of independent interest was previously obtained in [NU]. However, the bound (2.2) with an explicit dependence on the parameters is needed for establishing the validity of Theorem 1.1, thus we provide our proof here for completeness. The rest of the paper, Section 3, is devoted to the lower bound of the infimum of Lipschitz supersolutions to (1.1), stated in Lemma 3.1. We proceed by deriving consecutive estimates on the nonnegative supersolution $w = \log_+(u/K)$, where the constant K is determined in the initial step (c.f. Lemma 3.2) and depends on the values of the supersolution u to (1.1). Here, we follow the approach of Liebermann [L]. We emphasize that this initial step requires the additional assumption on the drift $b \in L^\infty L^2(\Omega)$ which was not needed in the elliptic case (see [IKR]). In Lemma 3.3, we establish an estimate which allows us to bootstrap the initial bounds on w from Lemma 3.2 to higher L^σ-norms for any $\sigma \in [1, (n+2)/n)$. Using Lemma 3.3, we also obtain a bound on $\|\nabla w\|_{L^2}$ in Subsection 3.3, which is essential for estimating higher norms. Then, the aforementioned estimates on all the higher norms are deduced by using Moser's iteration technique (see Subsection 3.4). The lower bound on the infimum then follows from the auxiliary assertion in Lemma 3.4. Our main result is a consequence of Lemmas 2.1 and 3.1.

Acknowledgments. I.K. was supported in part by the NSF grant DMS-1009769, L.R. was supported in part by the NSF grant DMS-0908507, and both M.I and L.R. were supported in part by NSF FRG grant DMS-115893.

2 Local boundedness

In this section, we show that any nonnegative Lipschitz subsolution of (1.1) is locally bounded when the divergence free drift belongs to the anisotropic Lebesgue spaces $L^l L^{\bar{q}}(\Omega)$ for all l and $\bar{q}$ satisfying $1 \leq 2/l + n/\bar{q} < 2$.

Lemma 2.1. *Assume that u is a nonnegative Lipschitz subsolution to the equation*

$$u_t - \Delta u + b \cdot \nabla u = 0 \tag{2.1}$$

with $b \in L^l L^{\bar{q}}(\Omega)$ for $1 \leq 2/l + n/\bar{q} < 2$ and $\operatorname{div} b \leq 0$ in the sense of distributions. Then for any $Q_R \subset \Omega$, $p > 0$, and $0 < \theta < \tau < 1$

$$\sup_{Q_{\theta R}} u \leq C \left(1 + \left(R^{1-2/l-n/\bar{q}} \|b\|_{L^l(L^{\bar{q}}(\Omega))}\right)^{1/(2-2/l-n/\bar{q})}\right)^{(n+2)/p} R^{-(n+2)/p} \|u\|_{L^p(Q_{\tau R})}, \tag{2.2}$$

where $C = C(n, p, l, \bar{q}, \theta, \tau)$ is a positive constant.

Proof of Lemma 2.1. Let u be a nonnegative Lipschitz subsolution of (2.1) in Ω, that is,

$$\int_\Omega (\partial_t u)\varphi + \int_\Omega (\partial_j u)(\partial_j \varphi) + \int_\Omega b_j (\partial_j u)\varphi \leq 0 \tag{2.3}$$

for any Lipschitz function $\varphi \geq 0$ in Ω and $\varphi = 0$ in Ω^c.

We assume without loss of generality that $R = 1$. We will use in (2.3) test functions of the form

$$\varphi = \left(\frac{\beta}{2} + 1\right) u^{\beta+1}\eta^{2\gamma}\chi_{\{t\leq T\}},$$

with a Lipschitz cut-off function η in Q_τ, such that $0 \leq \eta \leq 1$, and the constants $\beta > 0$ and $\gamma > 0$ to be set later – we will let $\beta \to +\infty$ while γ will remain fixed. This gives, for $T \in (-\tau^2, 0)$

$$\begin{aligned}\left(\frac{\beta}{2} + 1\right)\int_{Q_\tau}(\partial_t u)(u^{\beta+1})\eta^{2\gamma}\chi_{\{t\leq T\}} + \left(\frac{\beta}{2} + 1\right)\int_{Q_\tau}(\partial_j u)[\partial_j(u^{\beta+1})]\eta^{2\gamma}\chi_{\{t\leq T\}} \\ + \left(\frac{\beta}{2} + 1\right)\int_{Q_\tau} u^{\beta+1}[\partial_j u][\partial_j(\eta^{2\gamma})]\chi_{\{t\leq T\}} + \left(\frac{\beta}{2} + 1\right)\int_{Q_\tau} b_j u^{\beta+1}[\partial_j u]\eta^{2\gamma}\chi_{\{t\leq T\}} \leq 0.\end{aligned} \tag{2.4}$$

Set $w = u^{\beta/2+1}$, so that

$$\partial_j w = \left(\frac{\beta}{2} + 1\right) u^{\beta/2}\partial_j u.$$

Using (2.4), we get, integrating the first term by parts in time:

$$\begin{aligned}\frac{1}{2}\int_{B_\tau} w^2\eta^{2\gamma}\Big|_{t=T} + \frac{\beta+1}{\beta/2+1}\int_{Q_\tau}|\nabla w|^2\eta^{2\gamma}\chi_{\{t\leq T\}} \leq -2\gamma\int_{Q_\tau}(\partial_j w)w\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\leq T\}} \\ - \int_{Q_\tau} b_j(\partial_j w)w\eta^{2\gamma}\chi_{\{t\leq T\}} + \gamma\int_{Q_\tau} w^2\eta^{2\gamma-1}(\partial_t\eta)\chi_{\{t\leq T\}}.\end{aligned} \tag{2.5}$$

Here, we have utilized the fact that $\eta(x, -\tau^2) = 0$. For the first term in the right side of (2.5) we have

$$-2\gamma\int_{Q_\tau}(\partial_j w)w\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\leq T\}} = \gamma\int_{Q_\tau} w^2\left(\eta^{2\gamma-1}\Delta\eta + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2\right)\chi_{\{t\leq T\}}, \tag{2.6}$$

while for the second term:

$$\begin{aligned}-\int_{Q_\tau} b_j(\partial_j w)w\eta^{2\gamma}\chi_{\{t\leq T\}} &= \frac{1}{2}\int_{Q_\tau}(\partial_j b_j)w^2\eta^{2\gamma}\chi_{\{t\leq T\}} + \gamma\int_{Q_\tau} b_j w^2\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\leq T\}} \\ &\leq \gamma\int_{Q_\tau} b_j w^2\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\leq T\}},\end{aligned} \tag{2.7}$$

since $\operatorname{div} b \leq 0$.

Next, let $\gamma_0 = 2/l + n/\bar{q}$. Then, by assumption, we have $\gamma_0 \in [1, 2)$. We also choose $\gamma = 1/(2-\gamma_0)$, so that $\gamma\gamma_0 = 2\gamma - 1$. By Hölder's inequality we have the following estimate for the right side in (2.7):

$$\begin{aligned}\int_{Q_\tau} b_j w^2\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\leq T\}} &\leq \int_{Q_\tau}|b_j||w\eta^\gamma|^{\gamma_0}|w|^{2-\gamma_0}|\partial_j\eta|\chi_{\{t\leq T\}} \\ &\leq \|b\|_{L^l_t L^{\bar{q}}_x}\|w\eta^\gamma\chi_{t\leq T}\|^{\gamma_0}_{L^s_t L^r_x}\|w|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\leq T}\|^{2-\gamma_0}_{L^2_{t,x}}.\end{aligned} \tag{2.8}$$

Here s and r are determined by

$$\frac{1}{\bar{q}} + \frac{\gamma_0}{r} + \frac{2-\gamma_0}{2} = 1,$$

and

$$\frac{1}{l} + \frac{\gamma_0}{s} + \frac{2-\gamma_0}{2} = 1.$$

It is easy to verify that $2/s + n/r = n/2$ – this is how γ_0 was chosen. Now, Young's and the interpolation inequality

$$\|f\|_{L_t^s L_x^r} \le C\|f\|_{L_t^\infty L_x^2}^{1-\alpha} \|\nabla f\|_{L_{t,x}^2}^{\alpha} \tag{2.9}$$

with $2/s + n/r = n/2$ and $\alpha = n/2 - n/r$, applied to the right side of (2.8), imply

$$\begin{aligned} \int_{Q_\tau} b_j w^2 \eta^{2\gamma-1} (\partial_j \eta) \chi_{\{t\le T\}} &\le \epsilon \|w\eta^\gamma \chi_{t\le T}\|_{L_t^s L_x^r}^2 + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|w|\nabla\eta|^{1/(2-\gamma_0)} \chi_{t\le T}\|_{L_{t,x}^2}^2 \qquad (2.10) \\ &\le \frac{1}{2}\left(\|w\eta^\gamma \chi_{t\le T}\|_{L_t^\infty L_x^2}^2 + \|\nabla(w\eta^\gamma)\chi_{t\le T}\|_{L_{t,x}^2}^2 \right) + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|w|\nabla\eta|^{1/(2-\gamma_0)} \chi_{t\le T}\|_{L_{t,x}^2}^2. \end{aligned}$$

By (2.5), (2.6), and (2.10), we obtain, for any $-\tau^2 < T < 0$:

$$\begin{aligned} &\int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T) + \int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 \chi_{\{t\le T\}} \qquad (2.11) \\ &\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta|\chi_{\{t\le T\}} + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2\chi_{\{t\le T\}} + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta|\chi_{\{t\le T\}} \\ &\quad + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\le T}\|_{L_{t,x}^2}^2 + \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\chi_{t\le T}\|_{L_t^\infty L_x^2}^2. \end{aligned}$$

As this inequality holds for all $-\tau^2 < T < 0$, we may take the supremum over T to eliminate the $L_t^\infty L_x^2$-norm in the right side. Namely, from (2.11), we have

$$\begin{aligned} \sup_{T\in[-\tau^2,0]} \int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T) &\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta| \\ &\quad + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|_{L_{t,x}^2}^2 + \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\|_{L_t^\infty L_x^2}^2 \qquad (2.12) \end{aligned}$$

and

$$\begin{aligned} \int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 &\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta| \\ &\quad + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|_{L_{t,x}^2}^2 + \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\|_{L_t^\infty L_x^2}^2. \qquad (2.13) \end{aligned}$$

Adding the last two estimates and absorbing the $L_t^\infty L_x^2$-norm, we obtain

$$\begin{aligned} &\sup_{-\tau^2\le T\le 0} \int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T) + \int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 \qquad (2.14) \\ &\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta| \\ &\quad + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|_{L_{t,x}^2}^2, \end{aligned}$$

with an increased constant $C > 0$. By the interpolation inequality (2.9), used on the left side of (2.14) with $r = s = 2(n+2)/n$, and $\alpha = n/(n+2)$, used together with Young's inequality, we get the following estimate:

$$\begin{aligned} \|u^{\beta/2+1}\eta^{\gamma}\|_{L^{2\chi}(Q_\tau)} \leq C \left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| \right)^{1/2} + C \left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 \right)^{1/2} \\ + C \left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta| \right)^{1/2} + C\|b\|_{L^l L^{\bar{q}}}^{1/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|_{L^2}, \end{aligned} \tag{2.15}$$

with $\chi = (n+2)/n$.

We will now use (2.15) iteratively. We take a decreasing sequence $r_i > 0$, and at each step choose the cut-off function $\eta \in C_0^\infty(\Omega)$ such that

$$\eta \equiv 1 \text{ in } Q_{r_{i+1}}, \text{ and } \eta \equiv 0 \text{ in } Q_{r_i}^c,$$

and

$$|\nabla\eta| \leq \frac{C}{r_i - r_{i+1}}, \quad |\Delta\eta| \leq \frac{C}{(r_i - r_{i+1})^2}, \quad |\partial_t\eta| \leq \frac{C}{(r_i - r_{i+1})^2}.$$

Then (2.15) gives

$$\|u^{\beta/2+1}\|_{L^{2\chi}(Q_{r_{i+1}})} \leq \frac{C}{r_i - r_{i+1}} \|u^{\beta/2+1}\|_{L^2(Q_{r_i})} + \frac{C\|b\|_{L^l L^{\bar{q}}(Q_{r_i})}^{1/(2-\gamma_0)}}{(r_i - r_{i+1})^{1/(2-\gamma_0)}} \|u^{\beta/2+1}\|_{L^2(Q_{r_i})}. \tag{2.16}$$

Let us choose β_i in (2.16) so that $\chi^i = \beta_i/2 + 1$. In addition, we set

$$r_i = \theta + \frac{(\tau - \theta)}{2^i} \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

so that $r_i - r_{i+1} = (\tau - \theta)/2^{i+1}$. Thus, we obtain

$$\begin{aligned} \|u\|_{L^{2\chi^{i+1}}(Q_{r_{i+1}})} &\leq \Big(\frac{C2^{i+1}}{\tau - \theta} + \frac{C2^{(i+1)/(2-\gamma_0)}}{(\tau - \theta)^{1/(2-\gamma_0)}} \|b\|_{L^l L^{\bar{q}}(Q_{r_i})}^{1/(2-\gamma_0)} \Big)^{1/\chi^i} \|u\|_{L^{2\chi^i}(Q_{r_i})} \\ &\leq C^{1/\chi^i} 2^{(i+1)/(\gamma_1\chi^i)} \Big((\tau - \theta)^{-1} + \Big((\tau - \theta)^{-1} \|b\|_{L^l L^{\bar{q}}(Q_{r_i})} \Big)^{1/(2-\gamma_0)} \Big)^{1/\chi^i} \|u\|_{L^{2\chi^i}(Q_{r_i})}, \end{aligned} \tag{2.17}$$

where $\gamma_1 = \min\{2 - \gamma_0, 1\}$. By iteration, starting from $i = 0$, we conclude that the estimate (2.2) holds for $p \geq 2$.

Now, let $p \in (0, 2)$. The previous argument has shown that

$$\begin{aligned} \sup_{Q_\theta} u &\leq C \left((\tau - \theta)^{-1} + \Big((\tau - \theta)^{-1} \|b\|_{L^l L^{\bar{q}}(Q_\tau)} \Big)^{1/(2-\gamma_0)} \right)^{(n+2)/2} \|u\|_{L^2(B_\tau)} \\ &\leq C \left((\tau - \theta)^{-1} + \Big((\tau - \theta)^{-1} \|b\|_{L^l L^{\bar{q}}(Q_\tau)} \Big)^{1/(2-\gamma_0)} \right)^{(n+2)/2} \|u\|_{L^\infty(Q_\tau)}^{1-p/2} \|u\|_{L^p(Q_\tau)}^{p/2}, \end{aligned} \tag{2.18}$$

which implies

$$\sup_{Q_\theta} u \leq \frac{1}{2}\|u\|_{L^\infty(Q_\tau)} + C\left((\tau-\theta)^{-1} + \left((\tau-\theta)^{-1}\|b\|_{L^l L^{\bar q}(Q_\tau)}\right)^{1/(2-\gamma_0)}\right)^{(n+2)/p} \|u\|_{L^p(Q_\tau)}. \tag{2.19}$$

Now, the iteration argument of [HL, Lemma 4.3] can be applied to complete the proof of Lemma 2.1 for $0 < p < 2$. □

3 The lower bound

The goal of this section is to establish a lower bound of the infimum of a Lipschitz supersolution to (2.1), given in Lemma 3.1 below. Then, from Lemmas 2.1 and 3.1, we obtain the Harnack inequality

$$\sup_{Q_{R/2}(0,-3R^2)} u \leq \left(C + C(R^{1-(n+2)/\bar q}\|b\|_{L^{\bar q}})^{1/(2-(n+2)/\bar q)}\right)^{C(n)/p_0} \inf_{Q_R} u \tag{3.1}$$

for any Lipschitz solutions u to (2.1), where $p_0 = 1/(CM_R^C)$ and

$$M_R = 1 + (R^{1-n/2}\|b\|_{L^\infty L^2})^2 + R^{1-(n+2)/\bar q}\|b\|_{L^{\bar q}}. \tag{3.2}$$

Recall (see (1.2) and (1.3)) that we use the notation $Q_R^*(x_0,t_0)$ for the cylinder centered at the bottom and $Q_R(x_0,t_0)$ for the cylinder centered at the top, and $Q_R = Q_R(0,0)$.

Lemma 3.1. *Assume that u is a nonnegative Lipschitz supersolution to (2.1), and $b \in L^{\bar q}(\Omega) \cap L^\infty L^2(\Omega)$ with $n/2+1 < \bar q \leq n+2$ and $\operatorname{div} b = 0$ in the sense of distributions. Then there exists a small positive number $p_0 = p_0(n, \bar q, R, M_R)$ such that*

$$\left(CR^{-n-2}\int_{Q_R^*(0,-4R^2)} u^{p_0}\right)^{1/p_0} \leq \exp\left(1 + (R^{1-(n+2)/\bar q}\|b\|_{L^{\bar q}})^{1/(2-(n+2)/\bar q)}\right)^{C(n)} \inf_{Q_R} u, \tag{3.3}$$

with M_R given by (3.2).

We establish the proof of Lemma 3.1 in several steps, successively improving the estimate. We will primarily work with the function

$$v = \log(u/K),$$

with a constant K to be determined. If u is a supersolution to (2.1), then v is also a supersolution to (2.1). More precisely, v satisfies the inequality

$$|\nabla v|^2 \leq v_t - \Delta v + b\cdot\nabla v, \qquad \text{in} \quad \Omega. \tag{3.4}$$

We will obtain various bounds on $w = v_+$ below.

3.1 A bound on $\int w^\alpha$ for $\alpha \in (0,1)$.

We begin with the following initial estimate on w. Note that the constant K we choose in (3.5) below does depend on the solution $u(x,t)$.

Lemma 3.2. *Let $\eta(x) = C(1-|x|^2/(9R^2))_+$ be normalized so that $\int_{\mathbb{R}^n} \eta^2(x)\,dx = 1$, and set*

$$K = \exp\left(\int_{B_{3R}} \eta^2(x) \log u(x, 4R^2)\,dx\right). \tag{3.5}$$

Then for $\alpha \in (0,1)$ we have

$$\int_{Q^*_{2R}} w^\alpha \,dx\,dt \le CM_0 R^{n+2} \tag{3.6}$$

with $M_0 = 1 + (R^{1-n/2}\|b\|_{L^\infty L^2})^2$.

Proof of Lemma 3.2. Again, without loss of generality, we assume that $R=1$. We multiply (3.4) by the cut-off $\eta^2(x)$ and integrate over $B_3 \times (t_1, t_2)$ with $0 \le t_1 < t_2 \le 4$ in order to obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B_3} |\nabla v(x,t)|^2 \eta^2(x)\,dx\,dt &\le \int_{B_3} v(x,t_2)\eta^2(x)\,dx - \int_{B_3} v(x,t_1)\eta^2(x)\,dx \\ &+ 2\int_{t_1}^{t_2}\int_{B_3} (\partial_j v(x,t))\eta(x)(\partial_j \eta(x))\,dx\,dt + \int_{t_1}^{t_2}\int_{B_3} b_j(x,t)(\partial_j v(x,t))\eta^2(x)\,dx\,dt.\end{aligned} \tag{3.7}$$

After rearranging the terms and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{B_3} v(x,t_1)\eta^2(x)\,dx - \int_{B_3} v(x,t_2)\eta^2(x)\,dx + \int_{t_1}^{t_2}\int_{B_3} |\nabla v(x,t)|^2\eta^2(x)\,dx\,dt \\ &\le 2\int_{t_1}^{t_2}\int_{B_3} (\partial_j v(x,t))\eta(x)(\partial_j\eta(x))\,dx\,dt + \int_{t_1}^{t_2}\int_{B_3} b_j(x,t)(\partial_j v(x,t))\eta^2(x)\,dx\,dt \\ &\le \frac{1}{2}\int_{t_1}^{t_2}\int_{B_3} |\nabla v(x,t)|^2\eta^2(x)\,dx\,dt + C\|\nabla\eta(x)\|^2_{L^2(B_3\times(t_1,t_2)))} \\ &\qquad + C\|b\|^2_{L^\infty_t L^2_x(B_3\times(t_1,t_2))}\|\eta\|^2_{L^2_t L^\infty_x(B_3\times(t_1,t_2))}.\end{aligned} \tag{3.8}$$

Absorbing the first term on the far right, (3.8) leads to

$$\begin{aligned}&\int_{B_3} v(x,t_1)\eta^2(x)\,dx - \int_{B_3} v(x,t_2)\eta^2(x)\,dx + \frac{1}{2}\int_{t_1}^{t_2}\int_{B_3} |\nabla v(x,t)|^2\eta^2(x)\,dx\,dt \\ &\le C\left(1 + \|b\|^2_{L^\infty L^2(\Omega)}\right)(t_2 - t_1),\end{aligned} \tag{3.9}$$

since $0 \le \eta \le 1$. Now, we set $M_0 = 1 + \|b\|^2_{L^\infty L^2(\Omega)}$. Using weighted Poincaré's inequality (c.f. [Lieberman, Lemma 6.12]) in the left side of (3.9), we get

$$\begin{aligned}&\int_{B_3} v(x,t_1)\eta^2(x)\,dx - \int_{B_3} v(x,t_2)\eta^2(x)\,dx \\ &+ \frac{1}{C}\int_{t_1}^{t_2}\int_{B_3}\left|v(x,t) - \int_{B_3} v(x,t)\eta^2(x)\,dx\right|^2 \eta^2(x)\,dx\,dt \le CM_0(t_2 - t_1).\end{aligned} \tag{3.10}$$

For the rest of the proof we may proceed as in the proof of Lemma 6.21 from Lieberman. Consider the function

$$p(x,t) = v(x,t) - CM_0 t,$$

defined as a translation of v in time by the factor coming from the right side of (3.10). Note that the constant K in (3.5) was chosen so that

$$\int_{B_3} v(x,4)\eta^2(x)\,dx = 0, \tag{3.11}$$

and (3.10) and (3.11) imply that

$$\int_{B_3} v(x,t)\eta^2(x)\,dx \le CM_0(4-t), \qquad \text{for all } 0 \le t \le 4. \tag{3.12}$$

As $\eta(x)$ is uniformly positive for $|x| \le 2$, we deduce the upper bound

$$|\{(x,t) \in Q_2^* \colon p(x,t) > \mu\}| \le \frac{|Q_2^*|}{C\mu}, \tag{3.13}$$

on the size of the level sets of p that holds for any $\mu \ge 1$. This leads to the bound

$$\begin{aligned}\int_{\{(x,t)\in Q_2^* \colon p(x,t)>1\}} p^\alpha\,dx\,dt &= \alpha \int_1^\infty \mu^{\alpha-1}|\{(x,t) \in Q_2^* \colon p(x,t) > \mu\}|\,d\mu \\ &\le \frac{\alpha|Q_2^*|}{C}\int_1^\infty \mu^{\alpha-2}\,d\mu = C|Q_2^*|\end{aligned} \tag{3.14}$$

since $\alpha \in (0,1)$. We conclude the proof of (3.6) by noticing that the function w satisfies

$$w^\alpha \le Cp^\alpha + CM_0^\alpha$$

if $p \ge 1$ and $w^\alpha \le C + CM_0^\alpha$ if $p < 1$. $\square$

3.2 Bound on $\int w^\sigma$ for any $\sigma \in [1,(n+2)/n)$.

From now on, without loss of generality, we assume that $R = 1$. As before, we will work with $w = \log_+(u/K)$ with a constant K defined as in (3.5). The function $w = v_+$ is a supersolution to the equation for v, that is,

$$|\nabla w|^2 \le w_t - \Delta w + b \cdot \nabla w, \tag{3.15}$$

since it is a maximum of two supersolutions, $v_1 = v(x,t)$ and $v_2 \equiv 0$. We need the following bound that will bootstrap bounds for the L^α-norms with $\alpha \in (0,1)$ we have obtained in Lemma 3.2, to higher norms.

Lemma 3.3. *For any $\sigma \in [1,(n+2)/n)$ and any $\alpha \in (0,1)$, we have*

$$\|w\|_{L^\sigma(Q_1^*)} \le C(1+\|b\|_{L^{\bar q}})^C \|w\|_{L^\alpha(Q_2^*)}, \tag{3.16}$$

where $C = C(\alpha,\sigma,n,\bar q)$.

Proof of Lemma 3.3. Let η be a Lipschitz cut-off in Q_2^* with $0 \le \eta \le 1$ – note that unlike in the proof of Lemma 3.2 we use a cut-off that also depends on time. We multiply (3.15) by the function

$$(w+1)^{2\beta}\eta^{2\gamma}\chi_{\{t\ge T\}}$$

with $\beta \in (-1/2,0)$, $\gamma > 1$ to be determined, and $T \in (0,4)$, and integrate over Q_2^* to obtain

$$\begin{aligned} &\frac{1}{2\beta+1}\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int_{Q_2^*}|\nabla w|^2(w+1)^{2\beta}\eta^{2\gamma}\chi_{\{t\ge T\}} \\ &\quad\le 2\beta\int_{Q_2^*}|\nabla w|^2(w+1)^{2\beta-1}\eta^{2\gamma}\chi_{\{t\ge T\}} + 2\gamma\int_{Q_2^*}(\partial_j w)(w+1)^{2\beta}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\ge T\}} \\ &\quad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\ge T\}} - \frac{2\gamma}{2\beta+1}\int_{Q_2^*}(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_t\eta)\chi_{\{t\ge T\}}. \end{aligned} \tag{3.17}$$

Here we have used the condition $\operatorname{div} b = 0$. The first term on the right side is negative since $\beta \in (-1/2,0)$, while integration by parts in the second term on the right gives

$$\begin{aligned} 2\gamma\int(\partial_j w)(w+1)^{2\beta}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\ge T\}} &= \frac{2\gamma}{2\beta+1}\int \partial_j\left((w+1)^{2\beta+1}\right)\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\ge T\}} \\ = -\frac{2\gamma(2\gamma-1)}{2\beta+1}\int(w+1)^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2\chi_{\{t\ge T\}} &- \frac{2\gamma}{2\beta+1}\int(w+1)^{2\beta+1}\eta^{2\gamma-1}(\Delta\eta)\chi_{\{t\ge T\}}. \end{aligned} \tag{3.18}$$

This, together with (3.17) leads to

$$\begin{aligned} &\frac{1}{2\beta+1}\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int_{Q_2^*}(w+1)^{2\beta}|\nabla w|^2\eta^{2\gamma}\chi_{\{t\ge T\}} \\ &\quad\le -\frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\ge T\}} \\ &\qquad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}\partial_t\eta + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}\Delta\eta\right)\chi_{\{t\ge T\}}. \end{aligned} \tag{3.19}$$

We may use the inequality

$$|\nabla((w+1)^{\beta+1/2}\eta^\gamma)|^2 \le 2(\beta+1/2)^2(w+1)^{2\beta-1}|\nabla w|^2\eta^{2\gamma} + 2\gamma^2(w+1)^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2 \tag{3.20}$$

in the left side of (3.19). In addition, as $w > 0$, we have $(w+1)^{2\beta-1} \le (w+1)^{2\beta}$, which altogether gives

$$\begin{aligned} &\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}|_{t=T} + \int_{Q_2^*}|\nabla((w+1)^{\beta+1/2}\eta^\gamma)|^2\chi_{\{t\ge T\}} \\ &\quad\le C\gamma\int_{Q_2^*}|b_j|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta|\chi_{\{t\ge T\}} \\ &\quad + C\gamma^2\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right)\chi_{\{t\ge T\}}. \end{aligned} \tag{3.21}$$

An application of the interpolation inequality (2.9) with $r = s = 2(n+2)/n$ leads to

$$\begin{aligned}
\|(w+1)^{\beta+1/2}\eta^{\gamma}\|^2_{L^{2(n+2)/n}} &\le C\|(w+1)^{\beta+1/2}\eta^{\gamma}\|^2_{L^\infty L^2} + C\|\nabla((w+1)^{\beta+1/2}\eta^{\gamma})\|^2_{L^2} \qquad (3.22)\\
&\le C\gamma\int_{Q_2^*} |b_j|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta|\chi_{\{t\ge T\}}\\
&+ C\gamma^2\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right)\chi_{\{t\ge T\}}.
\end{aligned}$$

Next, we may estimate the drift term in (3.22) with the help of Hölder's inequality as follows

$$\begin{aligned}
C\gamma\int_{Q_2^*}|b|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| &= C\gamma\int_{Q_2^*}|b|(w+1)^{(2\beta+1)(1-\lambda)}(w+1)^{(2\beta+1)\lambda}\eta^{2\gamma-1}|\nabla\eta| \qquad (3.23)\\
&\le C\gamma\|b\|_{L^{\bar q}}\|(w+1)^{(2\beta+1)(1-\lambda)}\|_{L^{1/(1-\lambda)}}\|(w+1)^{(2\beta+1)\lambda}\eta^{2\gamma-1}\|_{L^{(n+2)/(n\lambda)}}\|\nabla\eta\|_{L^\infty},
\end{aligned}$$

where

$$\frac{1}{\bar q} + 1 - \lambda + \frac{n\lambda}{n+2} = 1.$$

Therefore, λ is given by $\lambda = (n+2)/(2\bar q)$ and $\lambda \in [1/2, 1)$, as $1 \le (n+2)/\bar q < 2$ by assumption. Using Young's inequality, this leads to

$$\begin{aligned}
C\gamma\int_{Q_2^*}|b|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| &\le C\gamma\|b\|_{L^{\bar q}}\|(w+1)^{2\beta+1}\|^{1-\lambda}_{L^1}\|(w+1)^{2\beta+1}\eta^{(2\gamma-1)/\lambda}\|^{\lambda}_{L^{\frac{n+2}{n}}}\|\nabla\eta\|_{L^\infty}\\
&\le \frac{1}{2}\|(w+1)^{\beta+1/2}\eta^{(2\gamma-1)/(2\lambda)}\|^2_{L^{2(n+2)/n}} + (C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty})^{1/(1-\lambda)}\|(w+1)^{\beta+1/2}\|^2_{L^2}. \qquad (3.24)
\end{aligned}$$

Setting $\gamma = 1/(2(1-\lambda)) \ge 1$ so that $(2\gamma-1)/2\lambda = \gamma$ and using (3.22) and (3.24), we obtain

$$\begin{aligned}
\|(w+1)^{\beta+\frac12}\eta^{\gamma}\|^2_{L^{\frac{2(n+2)}{n}}} &\le \frac{1}{2}\|(w+1)^{\beta+\frac12}\eta^{\gamma}\|^2_{L^{\frac{2(n+2)}{n}}} + (C\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty})^{2\gamma}\|(w+1)^{\beta+\frac12}\|^2_{L^2}\\
&+C\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right). \qquad (3.25)
\end{aligned}$$

The first term on the right can be absorbed into the left side:

$$\begin{aligned}
\|(w+1)^{\beta+\frac12}\eta^{\gamma}\|^2_{L^{\frac{2(n+2)}{n}}(Q_2^*)} &\le (C\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty})^{2\gamma}\|(w+1)^{\beta+\frac12}\|^2_{L^2}\\
&+ C\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right). \qquad (3.26)
\end{aligned}$$

We will now once again use an iteration procedure, applied to a decreasing sequence of parabolic cylinders Q_{r_i} with $r_{i+1} < r_i$. Choosing the cut-off η such that $\eta \equiv 1$ in $Q^*_{r_{i+1}}$ and $\eta \equiv 0$ in $(Q^*_{r_i} \cup Q_{r_i})^c$, we have, from (3.26):

$$\|(w+1)^{\beta+\frac12}\|^2_{L^{\frac{2(n+2)}{n}}(Q^*_{r_{i+1}})} \le \left(\frac{C\gamma\|b\|_{L^{\bar q}}}{r_i - r_{i+1}}\right)^{2\gamma}\|(w+1)^{\beta+\frac12}\|^2_{L^2(Q^*_{r_i})} + \frac{C}{(r_i - r_{i+1})^2}\|(w+1)^{\beta+\frac12}\|^2_{L^2(Q^*_{r_i})}$$

or equivalently

$$\|(w+1)^{2\beta+1}\|_{L^{(n+2)/n}(Q^*_{r_{i+1}})} \le C(r_i - r_{i+1})^{-2\gamma}(\|b\|^{2\gamma}_{L^{\bar q}}+1)\|(w+1)^{2\beta+1}\|_{L^1(Q^*_{r_i})}, \tag{3.27}$$

since $\gamma \ge 1$. Set $\chi = (n+2)/n$, pick $\alpha \in (0,1)$, and consider $\sigma \in [1,(n+2)/n)$. Possibly increasing σ and decreasing α we may assume that $\sigma = \chi^j\alpha$ with $j \in \mathbb{N}$. We will use (3.27) with

$$\beta_i = \frac{\chi^i\alpha - 1}{2},$$

for $i = 0,\dots,j$ so that $2\beta_0+1 = \alpha$ and $2\beta_j + 1 = \sigma$, and $r_i = 1 + 2^{-i}$. Then (3.27) implies the recursive relation

$$\|w+1\|_{L^{\chi^{i+1}\alpha}(Q^*_{r_{i+1}})} \le C2^{2\gamma(i+1)/\chi^i}(\|b\|^{2\gamma}_{L^{\bar q}}+1)^{1/\chi^i}\|w+1\|_{L^{\chi^i\alpha}(Q^*_{r_i})} \tag{3.28}$$

and a finite number of iteration gives (3.16). □

3.3 An estimate for $\int |\nabla w|^2$

The next step is to obtain bounds on $\|\nabla w\|_{L^2}$. Recall that w satisfies

$$|\nabla w|^2 \le w_t - \Delta w + b\cdot\nabla w, \tag{3.29}$$

Multiplying (3.29) by $\eta^2\chi_{\{t\ge T\}}$ and integrating over Q_2^* gives

$$\int_{B_2} w\eta^2\Big|_{t=T} + \int_{Q_2^*}|\nabla w|^2\eta^2 \le 2\int_{Q_2^*}(\partial_j w)\eta(\partial_j\eta) - 2\int_{Q_2^*} b_j w\eta\partial_j\eta - 2\int_{Q_2^*} w\eta\partial_t\eta, \tag{3.30}$$

where we used $\operatorname{div} b = 0$. After estimating the right side, we get

$$\begin{aligned}\int_{B_2} w\eta^2\Big|_{t=T} + \int_{Q_2^*}|\nabla w|^2\eta^2 &\le C\|\nabla\eta\|^2_{L^2} + C\|b\|_{L^{\bar q}}\|w\eta\|_{L^{\bar q^*}}\|\nabla\eta\|_{L^\infty} + C\|\eta_t\|_{L^\infty}\|w\eta\|_{L^1} \\ &\le C(1+\|b\|^2_{L^\infty L^2} + \|b\|_{L^{\bar q}})^C = CM^C,\end{aligned} \tag{3.31}$$

where $M = 1 + \|b\|^2_{L^\infty L^2} + \|b\|_{L^{\bar q}}$. In the last inequality, we used Lemmas 3.2 and 3.3 with $\sigma = \bar q^*$ and $\sigma = 1$, respectively, where $\bar q^* < (n+2)/n$, as $1/\bar q + 1/\bar q^* = 1$ and $(n+2)/2 < \bar q \le n+2$.

Note that with the bound (3.31) in hand we may extend the argument in the proof of Lemma 3.3 to include $\beta \in [0,1/2]$. Namely, in that proof we have considered $\beta \in (-1/2, 0)$ and dropped the first term in the right side of (3.17) simply because β was negative. Now, we can rely on (3.31) to bound this term in (3.17). The rest of the argument in the proof of Lemma 3.3 did not rely on the negativity of β. As the aforementioned term in (3.17) involves the product $|\nabla w|^2(w+1)^{2\beta-1}$ while (3.31) estimates $|\nabla w|^2$, we would still need the restriction $\beta \le 1/2$.

3.4 Bound on $\int w^{2\beta+1}$ for $\beta \geq 1/2$

We now extend the bound for

$$\int w^{2\beta+1}$$

to $\beta \geq 1/2$. As in the proof of Lemma 3.3, we let η be a Lipschitz cut-off in Q_2^* with $0 \leq \eta \leq 1$. This time, we multiply (3.15) by the function

$$w^{2\beta}\eta^{2\gamma}\chi_{\{t\geq T\}},$$

with $\beta \geq 1/2$ and $T \in (0,4)$, and integrate over Q_2^*, using the divergence-free condition on b:

$$\begin{aligned}\frac{1}{2\beta+1}\int\limits_{B_2} w^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int\limits_{Q_2^*} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} &\leq 2\beta\int\limits_{Q_2^*}|\nabla w|^2 w^{2\beta-1}\eta^{2\gamma} + 2\gamma\int\limits_{Q_2^*}(\partial_j w)w^{2\beta}\eta^{2\gamma-1}\partial_j\eta \\ &- \frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j w^{2\beta+1}\eta^{2\gamma-1}\partial_j\eta - \frac{2\gamma}{2\beta+1}\int_{Q_2^*} w^{2\beta+1}\eta^{2\gamma-1}\partial_t\eta. \end{aligned} \tag{3.32}$$

For the first term in the right side of (3.32), we use the inequality

$$2\beta|w|^{2\beta-1} \leq \frac{1}{2}|w|^{2\beta} + (4\beta)^{2\beta-1}, \tag{3.33}$$

and for the second:

$$\begin{aligned} 2\gamma\int(\partial_j w)w^{2\beta}\eta^{2\gamma-1}\partial_j\eta &= \frac{2\gamma}{2\beta+1}\int\partial_j(w^{2\beta+1})\eta^{2\gamma-1}\partial_j\eta \\ &= -\frac{2\gamma(2\gamma-1)}{2\beta+1}\int w^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2 - \frac{2\gamma}{2\beta+1}\int w^{2\beta+1}\eta^{2\gamma-1}\Delta\eta. \end{aligned} \tag{3.34}$$

Together with (3.32) this gives

$$\begin{aligned} \frac{1}{2\beta+1}\int\limits_{B_2} w^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \frac{1}{2}\int\limits_{Q_2^*}|w|^{2\beta}|\nabla w|^2\eta^{2\gamma} &\leq (4\beta)^{2\beta-1}\int_{Q_2^*}|\nabla w|^2\eta^{2\gamma} \\ -\frac{2\gamma}{2\beta+1}\int\limits_{Q_2^*} b_j w^{2\beta+1}\eta^{2\gamma-1}\partial_j\eta - \frac{2\gamma}{2\beta+1}\int\limits_{Q_2^*} & w^{2\beta+1}\left(\eta^{2\gamma-1}\partial_t\eta + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}\Delta\eta\right). \end{aligned} \tag{3.35}$$

Applying the estimate (3.33) for the second term on the left side of (3.35), we obtain

$$\begin{aligned} &\frac{1}{2\beta+1}\int_{B_2} w^{2\beta+1}\eta^{2\gamma}|_{t=T} + 2\beta\int_{Q_2^*}|w|^{2\beta-1}|\nabla w|^2\eta^{2\gamma} \\ &\leq 2(4\beta)^{2\beta-1}\int_{Q_2^*}|\nabla w|^2\eta^{2\gamma} + \frac{2\gamma}{2\beta+1}\int_{Q_2^*}|b_j|w^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta| \\ &+ \frac{2\gamma}{2\beta+1}\int_{Q_2^*} w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right). \end{aligned} \tag{3.36}$$

Next, we multiply (3.36) by $(2\beta+1)$ and use the inequality

$$|\nabla(|w|^{\beta+1/2}\eta^{\gamma})|^2 \le 2(\beta+1/2)^2|w|^{2\beta-1}|\nabla w|^2\eta^{2\gamma} + 2\gamma^2|w|^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2 \tag{3.37}$$

for the left side to get

$$\begin{aligned}\int\limits_{B_2} w^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int\limits_{Q_2^*} |\nabla(|w|^{\beta+1/2}\eta^{\gamma})|^2 &\le C(4\beta)^{2\beta}\int\limits_{Q_2^*}|\nabla w|^2\eta^{2\gamma} + C\gamma\int\limits_{Q_2^*}|b_j|w^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta| \\ &+ C\gamma^2\int\limits_{Q_2^*} w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right).\end{aligned} \tag{3.38}$$

We use the interpolation inequality (2.9) with $r = s = 2(n+2)/n$ to write

$$\begin{aligned}\|w^{\beta+1/2}\eta^{\gamma}\|^2_{L^{2(n+2)/n}} &\le C\|w^{\beta+1/2}\eta^{\gamma}\|^2_{L^\infty L^2} + C\|\nabla(w^{\beta+1/2}\eta^{\gamma})\|^2_{L^2} \\ &\le C(4\beta)^{2\beta}\int_{Q_2^*}|\nabla w|^2\eta^{2\gamma} + C\gamma\int_{Q_2^*}|b_j|w^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta| \\ &+ C\gamma^2\int_{Q_2^*} w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right).\end{aligned} \tag{3.39}$$

First, we note that unlike in the proof of Lemma 3.3 we now have the uniform estimate (3.31) for the gradient:

$$\int_{Q_2^*}|\nabla w|^2 \le CM^C. \tag{3.40}$$

Next, we estimate the drift term in (3.39) similarly to what we did in the proof of Lemma 3.3. Namely, we may write

$$\begin{aligned}C\gamma\int_{Q_2^*}|b|w^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| &= C\gamma\int_{Q_2^*}|b|w^{(2\beta+1)(1-\lambda)}w^{(2\beta+1)\lambda}\eta^{2\gamma-1}|\nabla\eta| \\ &\le C\gamma\|b\|_{L^{\bar q}}\|w^{(2\beta+1)(1-\lambda)}\|_{L^{1/(1-\lambda)}}\|w^{(2\beta+1)\lambda}\eta^{2\gamma-1}\|_{L^{(n+2)/(n\lambda)}}\|\nabla\eta\|_{L^\infty},\end{aligned} \tag{3.41}$$

with $\lambda = (n+2)/(2\bar q) \in [1/2, 1)$. An application of Young's inequality gives

$$\begin{aligned}C\gamma\int_{Q_2^*}|b|w^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| &\le C\gamma\|b\|_{L^{\bar q}}\|w^{2\beta+1}\|^{1-\lambda}_{L^1}\|w^{2\beta+1}\eta^{(2\gamma-1)/\lambda}\|^{\lambda}_{L^{(n+2)/n}}\|\nabla\eta\|_{L^\infty} \\ &\le \frac{1}{2}\|w^{\beta+1/2}\eta^{(2\gamma-1)/(2\lambda)}\|^2_{L^{2(n+2)/n}} + \left(C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty}\right)^{1/(1-\lambda)}\|w^{\beta+1/2}\|^2_{L^2}.\end{aligned} \tag{3.42}$$

As before, choosing $\gamma = 1/(2(1-\lambda))$, we may absorb the first term in the right side of (3.42) into the left side of (3.39). Thus, we obtain

$$\begin{aligned}\|(w\eta)^{\beta+1/2}\|^2_{L^{2(n+2)/n}} &\le CM^C(4\beta)^{2\beta} + \left(C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty}\right)^{2\gamma}\|w^{\beta+1/2}\|^2_{L^2} \\ &+ C\gamma^2\int_{Q_2^*} w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right).\end{aligned} \tag{3.43}$$

We are ready to do the iteration process. We set $r_i = 1 + 2^{-i}$ for $i = 0, 1, 2, \ldots$ and choose the cut-off η such that $\eta \equiv 1$ in $Q^*_{r_{i+1}}$ and $\eta \equiv 0$ in $(Q^*_{r_i} \cup Q_{r_i})^c$. Then (3.43) gives at each iteration step:

$$\|w^{\beta+1/2}\|^2_{L^{2(n+2)/n}(Q^*_{r_{i+1}})} \leq CM^C (4\beta)^{2\beta} + \left(\frac{C\gamma}{r_i - r_{i+1}}\|b\|_{L^{\bar{q}}}\right)^{2\gamma} \|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})} \tag{3.44}$$
$$+\frac{C\gamma^2}{(r_i - r_{i+1})^2}\|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})} \leq CM^C(4\beta)^{2\beta} + \left(\frac{C\gamma}{r_i - r_{i+1}}\right)^{2\gamma} (\|b\|^{2\gamma}_{L^{\bar{q}}} + 1)\|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})},$$

since $\gamma \geq 1$. Thus, we have the following relation between consecutive scales:

$$\|w^{\beta+1/2}\|^2_{L^{2(n+2)/n}(Q^*_{r_{i+1}})} \leq CM^C \left(\frac{C\gamma}{r_i - r_{i+1}}\right)^{2\gamma} \left((4\beta)^{2\beta} + \|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})}\right). \tag{3.45}$$

As in the proof of Lemma 3.3 we will use it with $\beta_i = (\chi^i - 1)/2$ where $\chi = (n+2)/n$ but this time we may allow β (and thus i) to be arbitrarily large. We obtain

$$\|w\|^{\chi^i}_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq CM^C 2^{\gamma(i+1)} \left((2\chi^i)^{\chi^i} + \|w\|^{\chi^i}_{L^{\chi^i}(Q^*_{r_i})}\right) \leq CM^C 2^{\gamma(i+1)} \left(2\chi^i + \|w\|_{L^{\chi^i}(Q^*_{r_i})}\right)^{\chi^i}, \tag{3.46}$$

for $i = 0, 1, 2, \ldots$ Iterating the inequality

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq (CM)^{C/\chi^i} 2^{2\gamma(i+1)/\chi^i} \left(2\chi^i + \|w\|_{L^{\chi^i}(Q^*_{r_i})}\right), \tag{3.47}$$

obtained from (3.45) by taking $1/\chi^i$ power on both sides, we get

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq CM^C \left(\chi^{i+1} + \|w\|_{L^1(Q^*_1)}\right). \tag{3.48}$$

By Lemma 3.2 and Lemma 3.3, we have

$$\|w\|_{L^1(Q^*_1)} \leq CM^C \|w\|_{L^\alpha(Q^*_2)} \leq CM^C \tag{3.49}$$

which together with (3.48) implies

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq CM^C \chi^{i+1}. \tag{3.50}$$

Thus, we may conclude

$$\left(\int_{Q^*_1} w^{2\beta+1}\right)^{1/(2\beta+1)} \leq CM^C(2\beta+1) \tag{3.51}$$

for all $\beta > 0$, and

$$\int_{Q^*_1} \frac{(p_0 w)^{2\beta+1}}{(2\beta+1)!} \leq (Cp_0 M^C e)^{2\beta+1} \leq \frac{1}{2^{2\beta+1}} \tag{3.52}$$

provided $p_0 = (2CM^C e)^{-1}$. The last inequality leads to the estimate

$$\int_{Q_R^*} \left(\frac{u}{K}\right)^{p_0} \leq CR^{n+2}, \tag{3.53}$$

where the constant K is defined in (3.5) and

$$M_R = 1 + (R^{1-n/2}\|b\|_{L^\infty L^2})^2 + R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}}.$$

We apply Lemma 3.2 and (3.53) to the translated in time cylinder $Q_R^*(0, -4R^2)$ and obtain

$$\int_{Q_R^*(0,-4R^2)} \left(\frac{u}{K}\right)^{p_0} \leq CR^{n+2} \tag{3.54}$$

with $K = \exp(\int_{B_{3R}} \eta^2(x) \log u(x, 0)\, dx)$.

If u is a supersolution to (2.1), then $\log_+(K/u)$ is a subsolution to (2.1). The last ingredient in the proof of Lemma 3.1 is the following result.

Lemma 3.4. *We have*

$$\sup_{Q_R} \log_+ \left(\frac{K}{u}\right) \leq C\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)}, \tag{3.55}$$

where

$$K = \exp\left(\int_{B_{3R}} \eta^2(x) \log u(x, 0)\, dx\right). \tag{3.56}$$

Proof of Lemma 3.4. We apply Lemma 2.1 for the positive subsolution $\log_+(K/u)$ to (2.1) with $p \in (0, 1)$ to obtain

$$\sup_{Q_R} \log_+ \frac{K}{u} \leq C\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{(n+2)/p} R^{-(n+2)/p} \left\|\log_+ \frac{K}{u}\right\|_{L^p(Q_{2R})}. \tag{3.57}$$

Now, let $v = \log(u/K)$ with K given by (3.56). We have $v = -\log(K/u)$ and $\log_+(K/u) = \log_-(u/K)$. The choice of K implies that

$$\int_{B_{3R}} \eta^2(x) v(x, 0)\, dx = 0.$$

We may proceed as in the proof of Lemma 3.2 to conclude

$$\left\|\log_+ \frac{K}{u}\right\|_{L^p(Q_{2R})} \leq CR^{(n+2)/p}, \tag{3.58}$$

which, combined with (3.57) proves (3.55). □

Lemma 3.4 is, actually, an upper bound on K, or a lower bound on $\inf_{Q_R} u$:

$$K \le C \exp\left(\left(1 + (R^{1-(n+2)/\bar{q}} \|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)}\right) \inf_{Q_R} u, \tag{3.59}$$

which together with (3.53) gives

$$\left(CR^{-n-2} \int_{Q_R^*(0,-4R^2)} u^{p_0}\right)^{1/p_0} \le K \le C \exp\left(1 + (R^{1-(n+2)/\bar{q}} \|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)} \inf_{Q_R} u. \tag{3.60}$$

Thus, the proof of Lemma 3.1 is complete. □

References

[BKNR] H. Berestycki, A. Kiselev, A. Novikov, and L. Ryzhik, *The explosion problem in a flow*, J. Anal. Math. **110** (2010), 31–65.

[CV1] L.A. Caffarelli and A.F. Vasseur, *The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics*, Discrete Contin. Dyn. Syst. Ser. S **3** (2010), no. 3, 409–427.

[CV2] L.A. Caffarelli and A.F. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. **171**, 2010, 1903–1930.

[CKRZ] P. Constantin, A. Kiselev, L. Ryzhik and A. Zlatos, *Diffusion and mixing in a fluid flow,* Ann. Math., **68**, 2008, 643–674.

[DG] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[FV] S. Friedlander and V. Vicol, *Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), no. 2, 283–301.

[G] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[HL] Q. Han and F. Lin, *Elliptic partial differential equations*, second ed., Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences, New York, 2011.

[IKR] M. Ignatova, I. Kukavica, and L.Ryzhik, *The Harnack inequality for second-order elliptic equations with divergence-free drift*, Communications in Math. Sciences (to appear).

[K] I. Kukavica, *On the dissipative scale for the Navier-Stokes equation*, Indiana Univ. Math. J. **48** (1999), no. 3, 1057–1081.

[KNSS] G. Koch, N. Nadirashvili, G.A. Seregin, and V. Šverák, *Liouville theorems for the Navier-Stokes equations and applications*, Acta Math. **203** (2009), no. 1, 83–105.

[L] G. M. Lieberman, "Second Order Parabolic Partial Differential Equations", World Sci. Publishing Co., NJ, 1996

[M] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134.

[N] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954.

[NU] A.I. Nazarov and N.N. Ural'tseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, Algebra i Analiz **23** (2011), no. 1, 136–168.

[S] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), no. fasc. 1, 189–258.

[SSSZ] G. Seregin, L. Silvestre, V. Šverák, and A. Zlatoš, *On divergence-free drifts*, J. Differential Equations **252** (2012), no. 1, 505–540.

[Z] Q.S. Zhang, *A strong regularity result for parabolic equations*, Comm. Math. Phys. **244** (2004), no. 2, 245–260.