Homogenization driven by a fractional Brownian motion: the shear layer case

Tomasz Komorowski* Alexei Novikov† Lenya Ryzhik‡

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Abstract

We consider a passive scalar in a periodic shear flow perturbed by an additive fractional noise with the Hurst exponent $H \in (0,1)$. We establish a diffusive homogenization limit for the tracer when the Hurst exponent $H \in (0,1/2)$. We also identify an intermediate range of times when the tracer behaves diffusively even when $H \in (1/2,1)$. The proof is based on an auxiliary limit theorem for an additive functional of a fractional Brownian motion.

1 Introduction

Standard periodic homogenization

Evolution of a passive tracer in a periodic flow perturbed by a white noise is a more or less classical problem by now [1, 7]. Consider trajectories generated by a stochastic differential equation

$$dZ_t = V(Z_t)dt + \sqrt{2}dw_t, \quad Z_0 = x \in \mathbb{R}^n. \tag{1.1}$$

Here, w_t is the standard Brownian motion, and the flow $V(x)$ is periodic, mean-zero and incompressible: $\nabla \cdot V = 0$, and

$$\int_{\mathbb{T}^n} V(x)dx = 0.$$

We are interested in the long time limit of trajectories. Accordingly, given a large time T, we introduce a small parameter $\varepsilon^2 = 1/T \ll 1$ and define the rescaled process $Z_{t,\varepsilon} = \varepsilon Z_{t/\varepsilon^2}$. The basic result of the "mundane" linear periodic homogenization theory is that the process $Z_{t,\varepsilon}$ converges in law, as $\varepsilon \to 0$ to a Brownian motion $\bar{B}_t = (\bar{B}_t^{(1)}, \dots, \bar{B}_t^{(n)})$ with the diffusivity matrix $\bar{\kappa}_{ij}$ that can be related to the flow $V(x)$ via the correctors χ_j, $j = 1, \dots, n$, i.e.

$$\mathbb{E}\bar{B}_t = 0, \quad \mathbb{E}[\bar{B}_t^{(i)}\bar{B}_t^{(j)}] = \bar{\kappa}_{ij}t,$$

and

$$\bar{\kappa}_{ij} = \delta_{ij} + \int_{\mathbb{T}^n} \nabla\chi_i(y) \cdot \nabla\chi_j(y)dy.$$

The correctors are (unique) mean-zero periodic solutions of the corrector equations

$$-\Delta\chi_j(y) + V(y) \cdot \nabla\chi_j(y) = u_j(y), \quad y \in \mathbb{T}^n, \quad j = 1, \dots, n. \tag{1.2}$$

*IMPAN, ul. Śniadeckich 8, 00-956 Warsaw, Poland, e-mail: komorow@hektor.umcs.lublin.pl

†Department of Mathematics, Pennsylvania State University, USA; anovikov@math.psu.edu

‡Department of Mathematics, Stanford University, Stanford, CA 94305, USA, e-mail: ryzhik@math.stanford.edu

Introducing fractional noise

What we would like to understand is how the temporal correlations in the noise will affect this picture. More precisely, let us assume that Z_t is driven not by the standard white noise but by the fractional noise:

$$dZ_t = V(Z_t)dt + \kappa dB_t, \quad Z_0 = x \in \mathbb{R}^n, \tag{1.3}$$

or, in the integral form:

$$Z_t = x + \int_0^t V(Z_s)ds + \kappa B_t. \tag{1.4}$$

Here B_t is the fractional Brownian motion with the Hurst exponent $H \in (0,1)$ – a Gaussian, continuous trajectory process with stationary increments such that

$$\mathbb{E}B_t = 0 \text{ and } \mathbb{E}B_t^2 = t^{2H},\ t \geq 0.$$

The law of a fractional Brownian motion is scale invariant, that is, the process $a^H B_{t/a}$ has an identical law with that of B_t for an arbitrary $a > 0$. In the special case when $H = 1/2$, B_t is the standard Brownian motion that we denote by w_t.

The two point correlation function of the fractional Brownian motion is

$$\mathrm{Cov}(B_t, B_s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right], \quad \forall\, t, s \geq 0, \tag{1.5}$$

and for any $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ the correlation matrix $C = [C_{ij}]$ for the increments

$$\Delta B_{t_i} := B_{t_i} - B_{t_{i-1}}, \quad i = 1, \ldots, n,$$

is

$$C_{ij} := \mathbb{E}(\Delta B_{t_i}, \Delta B_{t_j}) = \frac{1}{2}\left[|t_i - t_{j-1}|^{2H} + |t_{i-1} - t_j|^{2H} - |t_i - t_j|^{2H} - |t_{i-1} - t_{j-1}|^{2H}\right], \tag{1.6}$$

for $i, j = 1, \ldots, n$. That is, the increments of the fractional Brownian motion are independent only when $H = 1/2$, when the fractional Brownian motion is simply the standard Brownian motion.

The general question we are interested in is the long-time behavior of the solutions of (1.3): do the long time correlations generated by the noise persist, or do they disappear in the long time limit, and the limit is a regular diffusion? The latter may be surprising for several reasons: first, if $u = 0$ (that is, when there is no advection) then, obviously, $Z_t = B_t$ and increments of Z_t are correlated. Moreover, since B_t is not mixing, the correlations survive for a very long time. Second, even when $V \neq 0$ temporal correlations are constantly being fed into Z_t by the noise so a Markovian limit would be surprising. The third reason is that this does not happen in other systems in random media with long range correlations – the limits are not Markovian.

Passive tracers in slowly decorrelating velocity fields

In order to illustrate the very last point above, let us recall some basic results about passive tracers in random (not periodic!) velocity fields without any noise. That is, the randomness of the motion comes not from a white or fractional additive noise but from a random in space and time velocity field, and the particle motion is described by

$$\dot{Z}_t = \varepsilon V(t, Z_t), \quad Z_0 = x. \tag{1.7}$$

Here, $V(t,x) = (V_1(t,x), \ldots, V_n(t,x))$ is a mean zero, stationary in time and spatially homogeneous (in the statistical sense) random vector field, with a correlation function

$$R_{ij}(t,x) = \mathbb{E}[V_i(t+s, x+y)V_j(s,y)].$$

As the parameter $\varepsilon \ll 1$ is small so that for $t \sim O(1)$ we have $Z_t \approx x$, we are interested in the long time behavior of the trajectories. When the correlation matrix $R_j(t,x)$ decays rapidly, the Khasminskii Theorem says that $Z_{t,\varepsilon} := Z_{t/\varepsilon^2}$ converges in law, as $\varepsilon \to 0$, to a Brownian motion with the diffusivity matrix given by Kubo-Taylor formula:

$$D_{ij} = \frac{1}{2}\int_0^\infty [R_{ij}(s,0) + R_{ji}(s,0)]ds. \tag{1.8}$$

The basic idea behind this result is that the Lagrangian velocity decorrelates fast in time so that the particle "feels a CLT velocity", hence the limit is a Brownian motion.

The situation when the two-point correlation function $R_{ij}(t,x)$ decays slowly in x and t was analyzed by Fannjiang and one of the authors in [5] who looked at the regime when the diffusion matrix D_{ij} is infinite. The fact that the diffusion matrix is infinite indicates that "the particle is at infinity" by the time $t \sim \varepsilon^{-2}$, hence one expects a non-trivial limit on a shorter time scale. They considered a Gaussian velocity field $V(t,x)$ with the covariance

$$R_{ij}(t,x) = \mathbb{E}[V_i(t,x)V_j(0,0)] = \int_{\mathbb{R}^n} e^{\mathrm{i}k\cdot x} e^{-|k|^{2\beta}|t|}\hat{R}_{ij}(k)dk, \quad i,j = 1,\ldots,n, \tag{1.9}$$

with $\beta \geq 0$ and the spatial power spectrum given by

$$\hat{R}(k) = \frac{a(|k|)}{|k|^{2\alpha+d-2}}\left(I - \frac{k\otimes k}{|k|^2}\right). \tag{1.10}$$

The function $a(\cdot)$ is non-negative, bounded, measurable, supported in $[0, K_0]$ for some $K_0 > 0$ and continuous at 0 with $a(0) > 0$. In order to ensure that the spectrum is integrable at $k = 0$ so that $V(t,x)$ is a vector valued, stationary random field, we assume that $\alpha < 1$. Then the spatial correlations decay as $R(0,x) \sim |x|^{2\alpha-2}$ for large x, and the temporal correlations decay as $R(t,0) \sim t^{-(2-2\alpha)/(2\beta)}$ for large t. Therefore the effective diffusivity (1.8) is finite if $\alpha + \beta < 1$, and the convergence in law to a Brownian motion in this case has been established in [4].

It was shown in [5] that in the opposite regime $\alpha + \beta > 1$ (also with $\beta > 0$, $\alpha < 1$), when the diffusion matrix is infinite, the result is as follows. Because of the slow decay of the temporal correlations of the velocity field, the process Z_t becomes non-trivial on a shorter time scale $t \sim O(\varepsilon^{-2\gamma})$ with $\gamma = \beta/(\alpha + 2\beta - 1) < 1$. That is, the process $X_{t,\varepsilon} = X_{t/\varepsilon^{2\gamma}}$ in the limit $\varepsilon \to 0$ converges to a superdiffusive fractional Brownian motion B_t, with the Hurst exponent

$$H = \frac{1}{2} + \frac{\alpha + \beta - 1}{2\beta} \in (1/2, 1).$$

A passive tracer in a periodic shear flow with a fractional noise

In this paper, we consider the simplest example of a tracer advected by a periodic flow perturbed by a fractional additive noise: the case of a two-dimensional shear flow.

$$dX_t = v(Y_t)dt + \kappa dB_t^{(1)}, \qquad dY_t = \kappa dB_t^{(2)}. \tag{1.11}$$

Our main result is that in the long time limit X_t does behave diffusively for $H \in (0, 1/2)$ (sub-diffusive noise) – but also in a certain range of times even for $H > 1/2$ (super-diffusive noise). This is very different both from what one sees when $v = 0$ and the aforementioned results for a particle advected by a random flow with slowly decaying correlations.

We assume that the drift v is periodic: $v \in C(\mathbb{T})$ and has mean zero:

$$\langle v \rangle_{\mathbb{T}} := \int_{\mathbb{T}} v(x)dx = 0. \tag{1.12}$$

Here we use the convention that the torus is $\mathbb{T} = [0, 2\pi]$, with periodic boundary conditions. We also use the convention

$$\hat{v}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-\mathrm{i}kx} v(x)dx$$

for the Fourier coefficients of $v(x)$, so that the inverse Fourier transform is

$$v(x) = \sum_{k \in \mathbb{Z}_*} e^{\mathrm{i}kx} \hat{v}(k),$$

where $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$. The Fourier coefficients satisfy

$$\hat{v}(-k) = \hat{v}^*(k), \tag{1.13}$$

so the function $v(x)$ is real valued, and

$$\sum_{k \in \mathbb{Z}_*} |\hat{v}(k)| < +\infty. \tag{1.14}$$

Let $B_t^{(1)}$, $B_t^{(2)}$ be two independent standard fractional Brownian motions. Then, $\beta_t = (B_t^{(1)}, B_t^{(2)})$ is called a two dimensional standard fractional Brownian motion. Suppose that $\vec{V}(x, y) := [v(y), 0]$, diffusivity $\kappa > 0$ and that two dimensional process $Z_t = (X_t, Y_t)$ is the solution of equation

$$Z_t = Z_0 + \int_0^t \vec{V}(Z_s)ds + \kappa\beta_t, \; t \geq 0. \tag{1.15}$$

Then, obviously $Y_t = Y_0 + \kappa B_t^{(2)}$ and

$$X_t = X_0 + \int_0^t v(Y_0 + \kappa B_s^{(2)})ds + \kappa B_t^{(1)}.$$

First we prove that for $H \in (0, 1/2)$ the long-time behavior of the x-component is diffusive (the sub-diffusive y-component is "washed out" by the diffusive scaling)

Theorem 1.1 *Suppose that $v \in C(\mathbb{T})$ satisfies (1.12), $Z_0 = 0$, and β_t is a two dimensional standard fractional Brownian motion with the Hurst exponent $H \in (0, 1/2)$. Then, the scaled processes $Z_{t,\varepsilon} := \varepsilon Z_{t/\varepsilon^2}$ converge in law over $C[0, +\infty)$, as $\varepsilon \to 0+$, to the two dimensional process $W_t = (w_t, 0)$, where w_t is a mean-zero Brownian motion with the variance*

$$R(\kappa) := 2 \sum_{k \in \mathbb{Z} \setminus \{0\}_*} \frac{|\hat{v}(k)|^2}{|k|^{1/H}} \int_0^{+\infty} e^{-\kappa^2 \rho^{2H}/2} d\rho. \tag{1.16}$$

When $H \in (1/2, 1)$, the "very long time" behavior of the x-component can not be diffusive since the additive noise is super-diffusive. Nevertheless, when the noise is weak, one still sees the diffusive behavior on "intermediately long" time scales, and only after that the super-diffusion takes over. This is quantified by the next theorem.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1 hold except that the Hurst exponent H of β_t is assumed to belong to $(1/2,1)$. Then, the following hold:*

(i) if $\varepsilon,\kappa\to 0+$, and $\varepsilon\gg\kappa^{(1+1/(2H))/(2H-1)}$ then, $\kappa^{1/(2H)}Z_{t,\varepsilon}$ converge in law over $C[0,+\infty)$ to $W_t=(w_t,0)$, where w_t is a mean-zero Brownian motion, with variance $R_*=R(1)$,

(ii) if $\varepsilon,\kappa\to 0+$, and $\varepsilon\ll\kappa^{(1+1/(2H))/(2H-1)}$ then, $(\varepsilon^{2H-1}/\kappa)Z_{t,\varepsilon}$ converge in law over $C[0,+\infty)$ to a two dimensional, standard fractional Brownian motion β_t with the Hurst exponent H,

(iii) if $\varepsilon=\kappa^{(1+1/(2H))/(2H-1)}$, then $\kappa^{1/(2H)}Z_{t,\varepsilon}$ converge in law over $C[0,+\infty)$, as $\varepsilon,\kappa\to 0+$, to a Gaussian process $W_t+\beta_t$, where W_t is as in part (i), while β_t is an independent, two dimensional, standard fractional Brownian motion β_t with Hurst exponent H.

In other words, we have the following picture when κ is small and $H\in(1/2,1)$: for "large but not too large" times in the range $1\ll t\ll T_\kappa:=\kappa^{-(2+1/H)/(2H-1)}$ the x-component of the tracer behaves diffusively, for $t\sim T_\kappa$ it is a sum of independent fractional and standard Brownian motions, and, finally, for $t\gg T_\kappa$, the tracer behaves super-diffusively.

The behavior of X_t is completely different when the drift $v(x)$ is not periodic but localized – this case was recently considered in [6, 8]. Then the long time asymptotics is determined by the local time of the fractional Brownian motion on the support of the function $v(x)$. We refer the reader to the above papers for the precise results.

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2 Preliminaries on the fractional Brownian motion

We will need the following estimate for the correlation matrix in the proof of our main results.

Theorem 2.1 *For any $H\in(0,1)$ and an integer $n\ge 1$ there exists a constant $c_n>0$ such that*

$$C\xi\cdot\xi\ge c_n\sum_{j=1}^{n}(\Delta t_j)^{2H}\xi_j^2,\quad \forall\,\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n, \tag{2.1}$$

$\Delta t_j=t_j-t_{j-1}$, and all $0=t_0\le t_1\le\ldots\le t_n$.

Proof of Theorem 2.1. The result is stated without proof in [8] (see (2.2) in that paper), with a reference to [2]. We provide the details of the argument for the convenience of a reader.

Recall that a stochastic process X_t defined for $t\in(0,T)$ is called *locally non-deterministic*, see p. 70 of [2], if for any $m\ge 1$ and $t_1<\ldots<t_m$ belonging to $(0,T)$

$$\lim_{h\to+0}\inf_{t_m-t_1\le h}\frac{\mathrm{Var}\left(\Delta X_{t_m}|X_{t_1},\ldots,X_{t_{m-1}}\right)}{\mathrm{Var}\left(\Delta X_{t_m}\right)}=:\gamma_m>0, \tag{2.2}$$

where $\Delta X_{t_m}=X_{t_m}-X_{t_{m-1}}$. Here $\mathrm{Var}(\cdot|\cdot)$ and $\mathrm{Var}(\cdot)$ denote the conditional and unconditional variances, respectively. We shall also assume that $\mathrm{Var}\left(X_t-X_s\right)>0$ and $\mathrm{Var}\left(X_t\right)>0$ for all $t,s\in(0,T)$. By virtue of formula (2.11) of [2] and Lemma 2.1 of ibid., condition (2.2) is equivalent to

$$\lim_{h\to+0}\inf_{t_m-t_1\le h}\det\left[\Delta X_{t_i}\Delta X_{t_j}\right]\left\{\prod_{i=1}^{m}\mathrm{Var}\left(\Delta X_{t_i}\right)\right\}^{-1}=:\gamma_m'>0. \tag{2.3}$$

In order to show that fractional Brownian motion is locally non-deterministic, recall that it admits the following "harmonizable" representation, see (7.2.12) and (7.2.13) of [11]:

$$B_t = c_H \int_{\mathbb{R}} \frac{e^{\mathrm{i}t\tau} - 1}{i\tau} \frac{dw_\tau}{|\tau|^{H-1/2}}. \tag{2.4}$$

Here dw_τ is a complex valued white noise on $\mathbb{R}$ and

$$c_H := \frac{1}{\pi} H\Gamma(2H)\sin(\pi H).$$

From (2.4) we conclude that

$$\mathbb{E}B_t^2 = c_H^2 \int_{\mathbb{R}} \frac{|e^{\mathrm{i}t\tau} - 1|^2}{\tau^{2H+1}} d\tau = c_H^2 \int_{\mathbb{R}} |e^{\mathrm{i}t\tau} - 1|^2 \frac{1+\tau^2}{\tau^2} dF(\tau),$$

with

$$dF(\tau) = \frac{d\tau}{(1+\tau^2)\tau^{2H-1}}.$$

Since

$$\int_0^\infty \tau^2 dF(\tau) = \int_0^{+\infty} \frac{d\tau}{(1+\tau^2)\tau^{2H-3}} = +\infty,$$

by virtue of Theorem 4.1 of [2], there exists $T > 0$ such that the fractional Brownian motion B_t is locally non-deterministic on $(0, T)$. From this property we conclude the following

Lemma 2.2 *Let $H \in (0,1)$, then for any $n \geq 1$ there exists $c_n^* > 0$ such that*

$$c_n^* \left[\prod_{i=1}^n (t_i - t_{i-1})\right]^{2H} \leq \det[C_{ij}] \leq \left[\prod_{i=1}^n (t_i - t_{i-1})\right]^{2H}, \quad \forall\, 0 = t_0 \leq t_1 \leq \ldots \leq t_n. \tag{2.5}$$

Proof. The upper bound follows from the classical Hadamard inequality for the determinant of a symmetric, non-negative definite matrix, see Theorem 7.12, p. 218 of [13]. To conclude the lower bound we invoke the aforementioned property of local non-determinism of a fractional Brownian motion. According to this property, the lower bound in (2.5) holds for all $0 < t_1 < \ldots < t_n < h$, provided that $h > 0$ is chosen sufficiently small, cf. (2.3). The result for an arbitrary h follows by a simple scaling argument, using the scale-invariance of the law of a fractional Brownian motion. □

We now use Lemma 2.2 to finish the proof of Theorem 2.1. It suffices only to show that there exists $\gamma_n > 0$, depending only on n, sufficiently small so that the matrix

$$C(\gamma) := [C_{ij} - \gamma\delta_{ij}(\Delta t_j)^{2H}] \tag{2.6}$$

is positive definite for $\gamma \in [0, \gamma_n)$. We proceed by induction on n. By the Sylvester criterion of positive definiteness, it suffices only to show that there exists $\gamma_n > 0$ such that det $C(\gamma) > 0$ for all $\gamma \in [0, \gamma_n)$. Given a permutation σ of the set $\{1, \ldots, n\}$, we denote by $\mathrm{F}(\sigma)$ the cardinality of the set of the fixed points of σ. Note that

$$\det[C(\gamma)] = \sum_{k=0}^n \sum_{\sigma:\mathrm{F}(\sigma)=k} (-1)^{\mathrm{sgn}\sigma}(1-\gamma)^k C_{1\sigma(1)} \ldots C_{n\sigma(n)}. \tag{2.7}$$

Observe that the Cauchy-Schwartz inequality gives

$$|C_{1\sigma(1)} \ldots C_{n\sigma(n)}| = \prod_{j=1}^n \left|\mathbb{E}\left[\Delta B_{t_j} \Delta B_{t_{\sigma(j)}}\right]\right| \leq \prod_{j=1}^n (\Delta t_j)^{2H}.$$

We expand expression (2.7) in powers of γ. The result then follows from Lemma 2.2 and the above estimate, choosing γ sufficiently small. □

3 Proofs of Theorems 1.1 and 1.2

3.1 Additive functionals of a fractional Brownian motion

Theorems 1.1 and 1.2 both come from the following auxiliary result. Suppose that $v \in C(\mathbb{T})$ has mean-zero, $x \in \mathbb{T}$, and B_t is a standard fractional Brownian motion with the Hurst exponent $H \in (0,1)$. For a given $\kappa > 0$ define,

$$U_t = \int_0^t v(x + \kappa B_s) ds.$$

Theorem 3.1 *The scaled processes $U_{t,\varepsilon} := \varepsilon U_{t/\varepsilon^2}$ converge in law over $C[0,+\infty)$, as $\varepsilon \to 0+$, to a mean-zero Brownian motion, whose variance is given by* (1.16).

Note that this result covers all $H \in (0,1)$ and there is no distinction between the cases $H \in (0,1/2)$ and $H \in (1/2,1)$, as was the case for Theorems 1.1 and 1.2. In order to indicate where the variance in Theorem 3.1 comes from, which is the easy part of the proof, let us compute the asymptotics of the variance of U_t under the assumption that the initial position is uniformly distributed on the torus. Then,

$$\begin{aligned} &\mathbb{E}\int_{\mathbb{T}} dx \left\{ \int_0^T v\left(x + \kappa B_s\right) ds \right\}^2 \qquad (3.1) \\ &= \mathbb{E}\int_{\mathbb{T}} dx \left[\int_0^T \int_0^T \sum_{k_1,k_2 \in \mathbb{Z}} \hat{v}(k_1)\hat{v}(k_2) \exp\left\{ \mathrm{i}k_1 \left(x + \kappa B_{s_1}\right)\right\} \exp\left\{ \mathrm{i}k_2 \left(x + \kappa B_{s_2}\right)\right\} ds_1 ds_2 \right] \\ &= \int_0^T \int_0^T \sum_{k_1,k_2 \in \mathbb{Z}} \delta_0(k_1 + k_2)\hat{v}(k_1)\hat{v}(k_2)\mathbb{E}\left[\exp\left\{\mathrm{i}\kappa k_1 B_{s_1}\right\} \exp\left\{\mathrm{i}\kappa k_2 B_{s_2}\right\}\right] ds_1 ds_2. \end{aligned}$$

Here $\delta_\ell(k)$ is the Kronecker delta. The right side of (3.1) can be rewritten using (1.13) in the form

$$\begin{aligned} &\int_0^T \int_0^T \sum_{k \in \mathbb{Z}} |\hat{v}(k)|^2 \mathbb{E}\left[\exp\left\{\mathrm{i}\kappa k (B_{s_1} - B_{s_2})\right\}\right] ds_1 ds_2 \\ &\qquad = 2\int_0^T ds_1 \int_0^{s_1} \sum_{k \in \mathbb{Z}} |\hat{v}(k)|^2 \exp\left\{ -\frac{\kappa^2}{2} k^2 (s_1 - s_2)^{2H} \right\} ds_2. \end{aligned}$$

Passing to the limit $T \to +\infty$ we obtain

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}\int_{\mathbb{T}} dx \left[\int_0^T v\left(x + \kappa B_s\right) ds \right]^2 = 2 \sum_{k \in \mathbb{Z}_*} \frac{|\hat{v}(k)|^2}{|k|^{1/H}} \int_0^{+\infty} e^{-\kappa^2 \rho^{2H}/2} d\rho,$$

so that (1.16) holds.

Theorem 3.1 is a consequence of the following two results. The first concerns convergence of moments.

Proposition 3.2 *Suppose that $H \in (0,1)$, $\ell \ge 1$ and $0 = t_0 \le t_1 < \ldots < t_\ell$. Then, for any integers $n_1, \ldots, n_\ell \ge 1$ we have*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left\{ \prod_{j=1}^{\ell} \left(U_{t_j,\varepsilon} - U_{t_{j-1},\varepsilon}\right)^{n_j} \right\} = 0, \quad \textit{if at least one } n_j \textit{ is odd} \qquad (3.2)$$

and

$$\lim_{\varepsilon\to 0+}\mathbb{E}\left\{\prod_{j=1}^{\ell}\left(U_{t_j,\varepsilon}-U_{t_{j-1},\varepsilon}\right)^{n_j}\right\}=\prod_{j=1}^{\ell}\left\{(n_j-1)!![R(\kappa)(t_j-t_{j-1})]^{n_j/2}\right\},\quad \textit{if all } n_j \textit{ are even.} \tag{3.3}$$

The second result deals with the tightness of the processes $U_{t,\varepsilon}$, that, combined with the convergence of moments provides the weak convergence in law of the corresponding processes.

Proposition 3.3 *Suppose that $T>0$. Then, for any even integer $n\ge 2$ there exists $C>0$ such that*

$$\mathbb{E}\left(U_{t,\varepsilon}-U_{s,\varepsilon}\right)^n\le C(t-s)^{n/2},\quad \forall\, 0<s<t<T,\ \varepsilon\in(0,1]. \tag{3.4}$$

3.2 Proof of Propositions 3.2 and 3.3

To simplify the notation, we assume in this section that $\kappa=1$ and $x=0$. First, we prove the convergence of the moments of one point statistics. A simple calculation yields

$$\mathbb{E}\left[\int_0^{t/\varepsilon^2}v\left(B_s\right)ds\right]^n=n!\mathbb{E}\left\{\int\limits_{\Delta_n(t/\varepsilon^2)}\sum_{k_1,\ldots,k_n\in\mathbb{Z}}\prod_{p=1}^{n}\left[\hat v(k_p)\exp\left\{\mathrm{i}k_pB_{s_p}\right\}\right]d\mathbf{s}_{1n}\right\}.$$

Here $d\mathbf{s}_{1n}=ds_1\ldots ds_n$, and $\Delta_n(T):=\Delta_n(T,0)$ with $\Delta_n(T,S):=[T\ge s_n\ge\ldots\ge s_1\ge S]$ the simplex of times between S and T. The right side equals

$$n!\mathbb{E}\left\{\int_{\Delta_n(t/\varepsilon^2)}\sum_{k_1,\ldots,k_n\in\mathbb{Z}}\prod_{p=1}^{n}\left[\hat v(k_p)\exp\left\{\mathrm{i}k_{p,n}\Delta B_{s_p}\right\}\right]d\mathbf{s}_{1n}\right\}.$$

Here $\Delta B_{s_p}=B_{s_p}-B_{s_{p-1}}$, $p=1,\ldots,n$ and $s_0:=0$, $k_{p,n}:=k_p+\ldots+k_n$. Performing the expectation we obtain

$$\mathbb{E}\left[\int_0^{t/\varepsilon^2}v\left(B_s\right)ds\right]^n=n!\sum_{k_1,\ldots,k_n\in\mathbb{Z}}\prod_{p=1}^{n}\left[\hat v(k_p)\right]\int_{\Delta_n(t/\varepsilon^2)}\exp\left\{-\frac{1}{2}\sum_{p,q=1}^{n}C_{pq}k_{p,n}k_{q,n}\right\}d\mathbf{s}_{1n}, \tag{3.5}$$

with the matrix

$$C_{pq}=\mathbb{E}(\Delta B_{s_p}\Delta B_{s_q}). \tag{3.6}$$

Next, given $\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{Z}^n$ we set

$$m(\mathbf{k}):=\#[p:k_{p,n}=0].$$

We will show that: (1) for n odd the limit as $\varepsilon\to 0+$ of the right side of (3.5) vanishes, and (2) if n is even, the only terms that make a non-trivial contribution to (3.5) as $\varepsilon\to 0+$ come from $(k_1,\ldots,k_n)$ such that

$$k_1+k_2=k_3+k_4=\cdots=k_{n-1}+k_n=0.$$

The first step in this direction is the following lemma.

Lemma 3.4 *Suppose that* $\mathbf{k}$ *is such that* $m(\mathbf{k}) \le n/2$*. Then, there exists* $C > 0$ *such that*

$$\int_{\Delta_n(t/\varepsilon,s/\varepsilon)} \exp\left\{-\frac{1}{2}\sum_{p,q=1}^{n} C_{pq}k_{p,n}k_{q,n}\right\} d\mathbf{s}_{1n} \le C\left(\frac{t-s}{\varepsilon}\right)^{m(\mathbf{k})}, \quad \forall\, 0 < s < t. \tag{3.7}$$

Here C_{pq} *is given by* (3.6).

Proof. Denote by $\mathcal{M}(\mathbf{k})$ the set of those p for which $k_{p,n} \neq 0$. Its cardinality equals $n - m(\mathbf{k})$ and, according to Theorem 2.1 the expression in (3.7) can be estimated by

$$\begin{aligned} &\int_{\Delta_n(t/\varepsilon,s/\varepsilon)} \exp\left\{-\frac{c_n}{2}\sum_{p\in\mathcal{M}(\mathbf{k})} (\Delta s_p)^{2H}k_{p,n}^2\right\} d\mathbf{s}_{1n} \\ &\le \int_{\tilde{\Delta}_n(t/\varepsilon,s/\varepsilon)} d\tilde{\mathbf{s}}_{1n} \prod_{p\in\mathcal{M}(\mathbf{k})} \sup_{a\in\mathbb{R}} \int_{\mathbb{R}} \exp\left\{-\frac{c_n}{2}(s_p - a)^{2H}k_{p,n}^2\right\} ds_p. \end{aligned} \tag{3.8}$$

Here $d\tilde{\mathbf{s}}_{1n} = \prod_{p\notin\mathcal{M}(\mathbf{k})} ds_p$, and $\tilde{\Delta}_n(T,S) := [T \ge \tilde{s}_{m(\mathbf{k})} \ge \ldots \ge \tilde{s}_1 \ge S]$ is a reduced simplex of times. As $k_{p,n}^2 \ge 1$ for $p \in \mathcal{M}(\mathbf{k})$, the right hand side of (3.8) can be estimated by

$$\left(\frac{t-s}{\varepsilon}\right)^{m(\mathbf{k})}\left[\int_{\mathbb{R}} \exp\left\{-\frac{c_n}{2}s^{2H}\right\} ds\right]^{n-m(\mathbf{k})}$$

and (3.7) follows. □

The next lemma shows that a nontrivial contribution to the limit can only come from $(k_1, \ldots, k_n)$ with exactly $m(\mathbf{k}) = n/2$ as the terms with $m(\mathbf{k}) > [n/2]$ actually vanish identically.

Lemma 3.5 *Suppose that* $\mathbf{k}$ *is such that* $m(\mathbf{k}) > [n/2]$*. Then,*

$$\prod_{p=1}^{n} [\hat{v}(k_p)] = 0. \tag{3.9}$$

Proof. Assume that there exists p such that $k_{p,n} = 0$ and $k_{p+1,n} = 0$. Then, obviously, for such a p we have $k_p = 0$ and as a result (3.9) holds since $\hat{v}(0) = 0$ because of (1.12). If, on the other hand, there is no p such that both $k_{p,n} = 0$ and $k_{p+1,n} = 0$ then, as $m(\mathbf{k}) > [n/2]$, we must have that n is odd, and, moreover, $m(\mathbf{k}) = [n/2] + 1$, and $k_{1,n} = k_{3,n} = \cdots = k_{n,n} = 0$. However, the last equality says nothing but $k_n = 0$, which also implies (3.9). □

As an immediate consequence of the above two lemmas we conclude that (3.2) holds when $\ell = 1$ (one-point statistics) and n is odd since either $m(\mathbf{k}) < n/2$ or $m(\mathbf{k}) > [n/2]$ for all $\mathbf{k} = (k_1, \ldots, k_n)$ if any n is odd. In addition, while computing the limit, as $\varepsilon \to 0+$, of the expression in (3.5) when $n = 2m$ for some non-negative integer m, the only non-trivial contribution in comes from those terms of the series on the right hand side that correspond to $\mathbf{k}$ such that $m(\mathbf{k}) = m$. From Lemma 3.4 we conclude, therefore, Proposition 3.3.

The limit of the even moments

To prove (3.3), with $\ell = 1$, the limit for the even moments (still for the one-point statistics), we need to consider only the case $n = 2m$ and then

$$\lim_{\varepsilon\to+0} \varepsilon^{2m}\mathbb{E}\left[\int_0^{t/\varepsilon^2} v\left(B_s\right) ds\right]^{2m} = (2m)! \lim_{\varepsilon\to+0} \varepsilon^{2m} \sum_{\mathbf{k}:m(\mathbf{k})=m} \left[\prod_{p=1}^{2m} \hat{v}(k_p)\right]$$
$$\times \int_{\Delta_{2m}(t/\varepsilon)} \exp\left\{-\frac{1}{2}\sum_{p,q=1}^{2m} C_{pq} k_{p,n} k_{q,n}\right\} d\mathbf{s}_{1,2m}. \tag{3.10}$$

According to Lemmas 3.4 and 3.5, the only case when we have a non-trivial contribution to the limit is when $m(\mathbf{k}) = m$. Moreover, as explained in the proof of Lemma 3.5, we can not have $k_{p,2m} = k_{p+1,2m} = 0$ for any $p = 1, \ldots, 2m$. There are, thus, two possibilities either $k_{1,2m} = 0$, or not. In the first case we get

$$k_{2i-1,2m} = 0 \qquad \text{and} \qquad k_{2i,2m} \neq 0, \quad \forall\, i = 1, \ldots, m. \tag{3.11}$$

In the second one, we have

$$k_{2i-1,2m} \neq 0 \qquad \text{and} \qquad k_{2i,2m} = 0, \quad \forall\, i = 1, \ldots, m \tag{3.12}$$

but this leads to $k_{2m} = 0$, which makes the respective term on the right hand side of (3.10) vanish. Hence, we only need to consider the situation when $k_2 = -k_1, \ldots, k_{2m} = -k_{2m-1}$, and all k_p, $p = 1, \ldots, 2m$ are non-zero. In that case, the expression in the right side of (3.10) equals

$$(2m)! \lim_{\varepsilon\to+0} \varepsilon^{2m} \sum_{k_1,k_3,\ldots,k_{2m-1}} \left[\prod_{p=1}^{m} |\hat{v}(k_{2p-1})|^2\right] \mathbb{E}\mathcal{D}_{2m}\left(\frac{t}{\varepsilon^2}, 0\right) \tag{3.13}$$

where

$$\mathcal{D}_{2m}(T,t) := \int_{\Delta_{2m}(T,t)} \mathcal{E}_{2m}(\mathbf{s}_{1,2m}) d\mathbf{s}_{1,2m}, \tag{3.14}$$

and

$$\mathcal{E}_{2m}(\mathbf{s}_{1,2m}) := \exp\left\{-\mathrm{i}\sum_{p=1}^{m} k_{2p-1}\Delta B(s_{2p})\right\}.$$

Here, by convention $\Delta B_{s_1} := B_{s_1} - B_t$. Thanks to the convergence of the series and bound (3.7) we can interchange the limit with the summation in the expression (3.13). As a result, it equals

$$(2m)! \sum_{k_1,\ldots,k_{2m-1}\in\mathbb{Z}_*} \left[\prod_{p=1}^{m} |\hat{v}(k_{2p-1})|^2\right] \lim_{\varepsilon\to+0} \varepsilon^{2m}\mathbb{E}\mathcal{D}_{2m}\left(\frac{t}{\varepsilon^2}, 0\right). \tag{3.15}$$

Let

$$\Delta^{(1)}_{2m,\varepsilon}(T,t) := \left[T \geq s_{2m} \geq \ldots \geq s_1 \geq t,\, \forall\, i = 1, 2\ldots, m : \Delta s_{2i} \leq \log^r\left(\frac{1}{\varepsilon}\right)\right] \tag{3.16}$$

and

$$\Delta^{(2)}_{2m,\varepsilon}(T,t) := \left[T \geq s_{2m} \geq \ldots \geq s_1 \geq t,\, \exists\, i = 1, 2\ldots, m : \Delta s_{2i} > \log^r\left(\frac{1}{\varepsilon}\right)\right]$$

and $\Delta^{(i)}_{2m.\varepsilon}(T) = \Delta^{(i)}_{2m,\varepsilon}(T, 0)$, $i = 1, 2$. We write

$$(2m)!\varepsilon^{2m}\mathbb{E}\mathcal{D}_{2m}\left(\frac{t}{\varepsilon}, 0\right) = (2m)!\varepsilon^{2m}\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right) + (2m)!\varepsilon^{2m}\mathbb{E}\mathcal{D}^{(2)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right), \tag{3.17}$$

where $\mathcal{D}^{(i)}_{m,\varepsilon}(T, t)$, $i = 1, 2$ correspond to the integration over the regions $\Delta^{(i)}_{m,\varepsilon}(T, t)$, $i = 1, 2$ respectively. Using estimate (2.1), we see that, upon the choice of $r > 1/(2H)$,

$$(2m)! \lim_{\varepsilon\to+0} \varepsilon^{2m}\mathbb{E}\mathcal{D}^{(2)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right) = 0.$$

Hence, we consider only the limit corresponding to $\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}(t/\varepsilon, 0)$. Note that

$$\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right) = \int_0^{t/\varepsilon^2} \mathbb{E}\left[\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, s_2\right)\mathcal{J}_\varepsilon(s_2)\right] ds_2, \tag{3.18}$$

where

$$\mathcal{J}_\varepsilon(s_2) := \int_{(s_2-\log^r \varepsilon^{-1})\vee 0}^{s_2} \exp\left\{-\mathrm{i}k_1\Delta B(s_2)\right\} ds_1.$$

We write

$$\Delta B(s_{2p}) = \Delta B^\perp(s_{2p}) + \rho_p \Delta B(s_2),$$

where (cf. (3.6))

$$\rho_p := \frac{C_{2p,2}}{C_{2,2}},$$

and $C_{2,2} = (s_2 - s_1)^{2H}$. We have

$$\mathbb{E}[\Delta B^\perp(s_{2p})\Delta B(s_2)] = 0, \quad p = 2, \dots, m. \tag{3.19}$$

Recall that $\rho_p < 0$ for $p \geq 2$. From (3.19) and elementary properties of Gaussians, see e.g. Theorem of 10.1 of [10], we conclude that the vector $(\Delta B^\perp(s_{2m}), \dots, \Delta B^\perp(s_4))$ is independent of $\Delta B(s_2)$.

Lemma 3.6 *Given $r > 0$ there exists $C > 0$ that depends on r such that for all $\varepsilon \in (0, 0.9)$, and all $0 \leq s_1 \leq \ldots \leq s_{2m} \leq t/\varepsilon$ we have*

$$\sum_{k=2}^m |\rho_k| \leq \frac{Cm}{1 + (s_3 - s_2)^{2-2H}} \log^{r(2-H)} \varepsilon^{-1}. \tag{3.20}$$

Proof. It suffices to show that there exists $C > 0$ such that for all $t, h, s_1, s_2 \geq 0$, $s_1, s_2 \leq \log^r \varepsilon^{-1}$ we have

$$|\mathbb{E}\left[[B(t + s_2 + h + s_1) - B(t + s_2 + h)][B(t + s_2) - B(t)]\right]| \leq \frac{Cs_2^H}{(1 + h)^{2-2H}} \log^{r(2-H)}\left(\frac{1}{\varepsilon^{-1}}\right). \tag{3.21}$$

When $h \in (0, 10]$ we may simply use the Cauchy-Schwartz inequality to obtain

$$|\mathbb{E}\left[[B(t + s_2 + h + s_1) - B(t + s_2 + h)][B(t + s_2) - B(t)]\right]| \leq s_1^H s_2^H \leq s_2^H \log^{rH}\left(\frac{1}{\varepsilon}\right),$$

whence (3.21) holds. Next, for $h > 10$ note that

$$\begin{aligned}
&\mathbb{E}\left[[B(t+s_2+h+s_1)-B(t+s_2+h)][B(t+s_2)-B(t)]\right]\\
&=\frac{1}{2}[(h+s_1+s_2)^{2H}-(h+s_1)^{2H}+h^{2H}-(h+s_2)^{2H}]\\
&=H\left\{\int_0^{s_2}\left[(h+s_1+x)^{2H-1}-(h+x)^{2H-1}\right]dx\right\}\\
&=H(2H-1)\left\{\int_0^{s_2}dx\left[\int_0^{s_1}(h+y+x)^{2H-2}dy\right]\right\}.
\end{aligned}$$

Since $s_1, s_2 \le \log^r \varepsilon^{-1}$, we can estimate the double integral by

$$\frac{Cs_1s_2}{(1+h)^{2-2H}} \le \frac{Cs_2^H \log^r \varepsilon^{-1} \log^{r(1-H)} \varepsilon^{-1}}{(1+h)^{2-2H}} = \frac{Cs_2^H}{(1+h)^{2-2H}} \log^{r(2-H)}\left(\frac{1}{\varepsilon}\right),$$

and we have obtained (3.21). □

We now return to estimating $\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}(t/\varepsilon, 0)$. Let

$$\delta B := \Delta B(s_2) \sum_{p=2}^{m} k_{2p-1}\rho_p$$

be the projection of the exponent on the increment $\Delta B(s_2)$. We conclude from Lemma 3.6 that

$$\left\{\mathbb{E}(\delta B)^2\right\}^{1/2} \le \frac{C \log^{r(2-H)} \varepsilon^{-1}}{1+(s_3-s_2)^{2-2H}}. \tag{3.22}$$

We write, using the independence of $\Delta B^{\perp}(s_{2p})$ and δB:

$$\begin{aligned}
\mathbb{E}\mathcal{E}_{2m}(\mathbf{s}_{1,2m}) &= \mathbb{E}\left[\exp\left\{-\mathrm{i}\sum_{p=2}^{m} k_{2p-1}\Delta B^{\perp}(s_{2p})\right\}\right]\mathbb{E}\left[\exp\left\{-\mathrm{i}\Delta B(s_2)-\mathrm{i}\delta B\right\}\right]\\
&= \mathbb{E}\left[\exp\left\{-\mathrm{i}\sum_{p=2}^{m} k_{2p-1}\Delta B(s_{2p})+\mathrm{i}\delta B\right\}\right]\mathbb{E}\left[\exp\left\{-\mathrm{i}\Delta B(s_2)-\mathrm{i}\delta B\right\}\right].
\end{aligned}$$

Using an elementary estimate $|e^{i(z+h)} - e^{iz}| \le |h|$ together with (3.20) to eliminate δB from the exponent, we get

$$\begin{aligned}
&\left|\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2},0\right) - \int_0^{t/\varepsilon^2}\mathbb{E}\left[\mathcal{D}^{(1)}_{2m-2,\varepsilon}\left(\frac{t}{\varepsilon^2},s_2\right)\right]\mathbb{E}\left[\mathcal{J}(s_2)\right]ds_2\right|\\
&\le \int_{\Delta^{(1)}_{2m,\varepsilon}(t/\varepsilon,0)} \frac{C\log^{r(2-H)}\varepsilon^{-1}}{1+(s_3-s_2)^{2-2H}}d\mathbf{s}_{1,2m}.
\end{aligned} \tag{3.23}$$

We now eliminate the time variables $s_5, \dots, s_{2m}$ from the above integral using the definition of $\Delta^{(1)}_{2m,\varepsilon}(t/\varepsilon^2, 0)$ which implies that $|s_{2p} - s_{2p-1}| \le \log^r \varepsilon^{-1}$. Hence, eliminating $(m-2)$ odd indexed time variables gives us the volume $\log^{(m-2)r} \varepsilon^{-1}$. On the other hand, elimination of the even indexed time variables $s_6, \dots, s_{2m}$ gives us the volume $\varepsilon^{2(2-m)}$. In addition, we have $|s_2 - s_1| \le \log^r \varepsilon^{-1}$

and $|s_3-s_4|\le \log^r\varepsilon^{-1}$, so that elimination of s_1 and s_4 gives us an additional factor of $\log^{2r}\varepsilon^{-1}$. Altogether, we obtain, that there exists $C>0$ such that

$$\left|\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2},0\right)-\int_0^{t/\varepsilon^2}\mathbb{E}\left[\mathcal{D}^{(1)}_{2m-2,\varepsilon}\left(\frac{t}{\varepsilon^2},s_2\right)\right]\mathbb{E}\left[\mathcal{J}_\varepsilon(s_2)\right]ds_2\right|$$

$$\le C\varepsilon^{2(2-m)}\log^{rm}\varepsilon^{-1}\int\limits_0^{t/\varepsilon^2}ds_3\int\limits_0^{s_3}\frac{\log^{r(2-H)}\varepsilon^{-1}ds_2}{1+(s_3-s_2)^{2-2H}} \tag{3.24}$$

$$\le C\varepsilon^{2(2-m)}\log^{r(2-H+m)}\varepsilon^{-1}\int\limits_0^{t/\varepsilon^2}s_3^{2H-1}ds_3\le C\varepsilon^{2(2-2H-m)}\log^{r(2-H+m)}\varepsilon^{-1},\quad \forall\,\varepsilon\in(0,1).$$

We have, therefore, shown that

$$\lim_{\varepsilon\to+0}\left|\varepsilon^{2m}\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2},0\right)-\varepsilon^{2m}\int_0^{t/\varepsilon^2}\mathbb{E}\left[\mathcal{D}^{(1)}_{2m-2,\varepsilon}\left(\frac{t}{\varepsilon^2},s_2\right)\right]\mathbb{E}\left[\mathcal{J}_\varepsilon(s_2)\right]ds_2\right|=0.$$

Repeating this argument m times we obtain that

$$\lim_{\varepsilon\to+0}\left|\varepsilon^{2m}\mathbb{E}\left[\mathcal{D}_{2m}\left(\frac{t}{\varepsilon^2},0\right)\right]-\varepsilon^{2m}\left|\Delta_m\left(\frac{t}{\varepsilon^{2m}}\right)\right|\left\{\prod_{p=1}^m\mathbb{E}\left[\int_0^{+\infty}\exp\left\{-\mathrm{i}k_{2p-1}B(s)\right\}ds\right]\right\}\right|=0, \tag{3.25}$$

where $|\Delta_m(T)|$ is the volume of the simplex. The limit in (3.15) equals therefore

$$t^m\sum_{k_1,\ldots,k_{2m-1}\in\mathbb{Z}_*}\left[\prod_{p=1}^m\frac{|\hat v(k_{2p-1})|^2}{|k_{2p-1}|^{1/H}}\right]\frac{(2m)!}{m!}\left[\int_0^{+\infty}e^{-\frac12\rho^{2H}}d\rho\right]^m=(2m-1)!!(2R_*t)^m, \tag{3.26}$$

where $R_*:=R(1)$. This coincides with (3.3) when $\ell=1$.

In order to generalize the above argument to show the convergence of the moments corresponding to the multiple point statistics at times $0=t_0<t_1<\ldots<t_\ell$ we write

$$\varepsilon^{|n|}\mathbb{E}\Big\{\prod_{j=1}^\ell\Big[\int_{t_{j-1}/\varepsilon^2}^{t_j/\varepsilon^2}v\left(B_s\right)ds\Big]^{n_i}\Big\} \tag{3.27}$$

$$=n!\varepsilon^{|n|}\mathbb{E}\Big\{\sum_{\mathbf{k}\in\mathbb{Z}_*^{n_1}\times\ldots\times\mathbb{Z}_*^{n_\ell}}\prod_{j=1}^\ell\hat v(\mathbf{k}_j)\int\limits_{\Delta_n(t_\ell/\varepsilon^2,\ldots,t_1/\varepsilon^2)}\prod_{(j,p)}\exp\left\{\mathrm{i}k_p^{(j)}B_{s_p^{(j)}}\right\}d\mathbf{s}^{(j)}_{1n_i}\Big\}.$$

Here $\mathbf{k}:=(\mathbf{k}_1,\ldots,\mathbf{k}_\ell)$, $\mathbf{k}_j:=(k_1^{(j)},\ldots,k_{n_j}^{(j)})$, $n=(n_1,\ldots,n_\ell)$,

$$|n|=\sum_{j=1}^\ell n_j,\qquad n!:=\prod_{j=1}^\ell n_j!,\qquad \hat v(\mathbf{k}_j):=\prod_{p=1}^{n_j}\hat v(k_p^{(j)}) \tag{3.28}$$

and

$$\Delta_n(t_\ell/\varepsilon^2,\ldots,t_1/\varepsilon^2):=\Delta_{n_\ell}(t_\ell/\varepsilon^2,t_{\ell-1}/\varepsilon^2)\times\ldots\times\Delta_{n_\ell}(t_1/\varepsilon^2,0). \tag{3.29}$$

We introduce the lexicographical ordering of indices (j,p), that is, we say that (j,p) precedes (j',p'), which is denoted by $(j,p)\prec(j',p')$, if either $j<j'$ or $j=j'$ and $p\le p'$. Denote by $\iota(j,p)$ the

predecessor of the element if (j,p). If $(j,p)=(1,1)$ we define the predecessor as 0 and let $s_0:=0$. Given (j,p) we denote

$$k_{j,p}:=\sum_{(j,p)\prec(j',p')}k_{p'}^{(j')}.$$

Also, for $\mathbf{k}$ fixed we define $m_j(\mathbf{k})$ as the cardinality of p-s such that $k_{j,p}=0$, and set

$$m(\mathbf{k}):=\sum_{j=1}^{\ell}m_j(\mathbf{k}).$$

We denote by $\mathcal{M}_j(\mathbf{k})$ the set of those indices (j,p), for which $k_{j,p}\neq 0$. Its cardinality equals $n_j-m_j(\mathbf{k})$. The right side of (3.27) can be rewritten as

$$\sum_{\mathbf{k}\in\mathbb{Z}_*^{n_1}\times\ldots\times\mathbb{Z}_*^{n_\ell}}n!\varepsilon^{|n|}\prod_{j=1}^{\ell}\hat{v}(\mathbf{k}_j)\mathbb{E}\left\{\int\limits_{\Delta_n(t_\ell/\varepsilon^2,\ldots,t_1/\varepsilon^2)}\prod_{(j,p)}\exp\left\{\mathrm{i}k_{j,p}\Delta B_{s_p^{(j)}}\right\}d\mathbf{s}_{1n_j}^{(j)}\right\}\tag{3.30}$$

Here $\Delta B_{s_p^{(j)}}:=B_{s_p^{(j)}}-B_{s_{p'}^{(j')}}$, where (j',p') is the predecessor of (j,p). Following the argument in Lemma 3.5, we conclude that only the terms that correspond to $\mathbf{k}$ such that $m_j(\mathbf{k})\le[n_j/2]$ for all $j=1,\ldots,\ell$ do not vanish. As in Lemma 3.4, we can estimate the absolute value of the term of the series in (3.30) by

$$n!\varepsilon^{|n|}\left|\prod_{j=1}^{\ell}\hat{v}(\mathbf{k}_j)\right|\int\limits_{\tilde{\Delta}_n(t_\ell/\varepsilon^2,\ldots,t_1/\varepsilon^2)}\prod_{j=1}^{\ell}d\tilde{\mathbf{s}}_j\prod_{p\in\mathcal{M}_j(\mathbf{k})}\sup_{a\in\mathbb{R}}\int_{\mathbb{R}}\exp\left\{-\frac{c_n}{2}(s_p-a)^{2H}k_{j,p}^2\right\}ds_p$$

Here $d\tilde{\mathbf{s}}_j=\prod_{(j,p)\notin\mathcal{M}_j(\mathbf{k})}ds_p^{(j)}$, and $\tilde{\Delta}_n(t_\ell/\varepsilon^2,\ldots,t_1/\varepsilon^2)$ is the product of the respective reduced simplices of the form $\tilde{\Delta}_{n_j}(T,S):=[T\ge\tilde{s}_{m_j(\mathbf{k})}\ge\ldots\ge\tilde{s}_1\ge S]$. The limit of the above expression, as $\varepsilon\to 0+$, can be non-zero only in the case when $n_j=2m_j$ for some integer m_j and all $j=1,\ldots,\ell$. Therefore, it equals

$$n!\lim_{\varepsilon\to+0}\varepsilon^{|n|}\sum_{\mathbf{k}\in\mathbb{Z}_*^{n_1}\times\ldots\times\mathbb{Z}_*^{n_\ell}}\mathbb{E}\left\{\prod_{j=1}^{\ell}\left[\prod_{p=1}^{m_i}|\hat{v}(k_{2p-1}^{(i)})|^2\mathcal{D}_{2m}\left(\frac{t_j}{\varepsilon^2},\frac{t_{j-1}}{\varepsilon^2}\right)\right]\right\}.\tag{3.31}$$

Repeating the argument used to compute the limit in (3.15) we obtain that expression in (3.31) equals

$$R_*^n\prod_{i=1}^{\ell}(2m_i-1)!!(\Delta t_i)^{m_i}$$

and the conclusion of Proposition 3.2 follows.

3.3 The end of the proof of Theorem 1.1 and parts (i), (ii) of Theorem 1.2

The statement of Theorem 1.1 follows directly from Theorem 3.1, since, under the diffusive scaling the fractional Brownian motion term in the definition of Z_t vanishes.

As for part (i) of Theorem 1.2, observe first that the scaling of the fractional Brownian motion term appearing in the definition of Z_t yields

$$\kappa^{1+1/(2H)}\varepsilon\beta_{t/\varepsilon^2}\stackrel{d}{=}\kappa^{1+1/(2H)}\varepsilon^{1-2H}\beta_t.$$

This term tends to 0 in the regime of part (i). On the other hand, for the additive functional in the definition of Z_t we have

$$\kappa^{1/(2H)}U_{t,\varepsilon}=\kappa^{1/(2H)}\varepsilon\int_0^{t/\varepsilon^2}v(x+\kappa B^{(2)}_s)ds\stackrel{d}{=}\sigma\int_0^{t/\sigma^2}v(x+B_s)ds \tag{3.32}$$

where $\stackrel{d}{=}$ denotes equality of laws of the relevant random variables, $\sigma:=\varepsilon/\kappa^{1/(2H)}$ and B_t is a one dimensional standard fractional Brownian motion with the Hurst exponent H. Using Proposition 3.2, we conclude that in this regime of (ε,κ) the process appearing in the utmost right side of (3.32) converges in law to a Brownian motion, as claimed in the assertion of part (i) of Theorem 1.2.

Part (ii) follows a similar argument. This time, due to the constraint of the regime, the additive functional $(\varepsilon^{2H-1}/\kappa)U_{t,\varepsilon}$ tends to 0, while

$$(\varepsilon^{2H-1}/\kappa)\kappa\varepsilon\beta_{t/\varepsilon^2}\stackrel{d}{=}\beta_t$$

and the conclusion of this part of the theorem follows.

3.4 Proof of part (iii) of Theorem 1.2

We denote, as before, setting the starting point $(X_0,Y_0)=0$, without loss of generality:

$$U_t=\int_0^t v(B_s)ds \tag{3.33}$$

and

$$U_{t,\varepsilon}=\varepsilon U_{t/\varepsilon^2},\qquad B_{t,\varepsilon}:=\varepsilon^{2H}B_{t/\varepsilon^2}.$$

To prove part (iii) of the theorem it suffices to show that the scaled, vector valued processes $(U_{t,\varepsilon},B_{t,\varepsilon})$ converge in law over $C([0,+\infty),\mathbb{R}^2)$, as $\varepsilon\to0+$, to (w_t,B_t), where w_t is a mean zero Brownian motion with variance R_* and B_t is an independent fractional Brownian motion. Tightness is a consequence of tightness of the laws of each of the marginal processes. We only need to prove the convergence of finite dimensional distributions. This is a consequence of the following generalization of Proposition 3.2.

Proposition 3.7 *Suppose that $H\in(0,1)$, $\ell\ge1$ and $0=t_0\le t_1<\ldots<t_\ell$. Then, for any integers $n_1,\ldots,n_\ell\ge1$ and $\xi_1,\ldots,\xi_\ell\in\mathbb{R}$ we have*

$$\lim_{\varepsilon\to0+}\mathbb{E}\left\{\prod_{j=1}^{\ell}\left\{\left(U_{t_j,\varepsilon}-U_{t_{j-1},\varepsilon}\right)^{n_j}\exp\left\{\mathrm{i}\xi_j(B_{t_j,\varepsilon}-B_{t_{j-1},\varepsilon})\right\}\right\}\right\}=0,\quad \textit{if at least one } n_j \textit{ is odd} \tag{3.34}$$

and

$$\lim_{\varepsilon\to0+}\mathbb{E}\left\{\prod_{j=1}^{\ell}\left\{\left(U_{t_j,\varepsilon}-U_{t_{j-1},\varepsilon}\right)^{n_j}\exp\left\{\mathrm{i}\xi_j(B_{t_j,\varepsilon}-B_{t_{j-1},\varepsilon})\right\}\right\}\right\} \tag{3.35}$$

$$=\prod_{j=1}^{\ell}\left\{(n_j-1)!![R(t_j-t_{j-1})]^{n_j/2}\right\}\mathbb{E}\left\{\prod_{j=1}^{\ell}\exp\left\{\mathrm{i}\xi_j(B_{t_j}-B_{t_{j-1}})\right\}\right\},$$

if all n_j-s are even.

Indeed, consider for simplicity sake only the case $\ell = 1$. As a consequence of the proposition we conclude that any limiting law μ of $(U_{t_1,\varepsilon}, B^Y_{t_1,\varepsilon})$, as $\varepsilon \to 0+$, satisfies

$$\int_{\mathbb{R}^2} x^n e^{i\xi y}\mu(dx,dy) = \int_{\mathbb{R}^2} x^n e^{i\xi y}\Phi_*(dx,dy), \quad \forall\, n \geq 0,\, \xi \in \mathbb{R},$$

where $\Phi_*(dx,dy)$ is the law of $(w_{t_1}, B^Y_{t_1})$. From this we conclude, differentiating m times in ξ, that

$$\int_{\mathbb{R}^2} x^n y^m \mu(dx,dy) = \int_{\mathbb{R}^2} x^n y^m \Phi_*(dx,dy), \quad \forall\, n, m \geq 0. \tag{3.36}$$

Suppose that $\xi_1, \xi_2 \in \mathbb{R}$, $V := \xi_1 w_{t_1} + \xi_2 B_{t_1}$. It can easily be checked, due to the Gaussianity of Φ_*, that

$$\sum_{n=1}^{+\infty} \left\{\mathbb{E}V^{2n}\right\}^{-1/(2n)} = +\infty. \tag{3.37}$$

Using Carleman's criterion for well-posedness of the moment problem on the real line, see e.g. Theorem 1.10, p. 19 of [12] we can conclude from (3.37) that the moments of V determine its law, therefore (3.36) implies

$$\int_{\mathbb{R}^2} e^{\mathrm{i}(\xi_1 x + \xi_2 y)}\mu(dx,dy) = \int_{\mathbb{R}^2} e^{\mathrm{i}(\xi_1 x + \xi_2 y)}\Phi_*(dx,dy), \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}.$$

This obviously implies $\mu = \Phi_*$. Therefore, part (iii) of Theorem 1.2, indeed, follows from Proposition 3.7.

Proof of Proposition 3.7

To simplify the notation, we assume that $\ell = 1$. Using the symmetry considerations, we conclude that expression in (3.34) equals

$$\sum_{k_1,\ldots,k_n \in \mathbb{Z}} \varepsilon^n n! \mathbb{E}\left\{\int_{\Delta_n(t/\varepsilon^2)} \exp\left\{i\varepsilon^{2H}\xi\Delta B_{s_{n+1}}\right\} \prod_{p=1}^n \left[\hat{v}(k_p)\exp\left\{\mathrm{i}(k_{p,n} + \varepsilon^{2H}\xi)\Delta B_{s_p}\right\}\right] d\mathbf{s}_{1n}\right\}, \tag{3.38}$$

with the convention $s_{n+1} := t/\varepsilon^2$. We can repeat the arguments in the proof of Lemmas 3.4 and 3.5 to conclude that the limit of the expression in (3.38) can be nonzero only if $n = 2m$ for some non-negative integer m. In addition, the limit, as $\varepsilon \to 0+$, is the same as

$$(2m)! \sum_{k_1,\ldots,k_{2m-1} \in \mathbb{Z}_*} \left[\prod_{p=1}^m |\hat{v}(k_{2p-1})|^2\right] \lim_{\varepsilon \to +0} \varepsilon^{2m} \mathbb{E}\tilde{\mathcal{D}}^{(1)}_{2m}\left(\frac{t}{\varepsilon^2}, 0\right). \tag{3.39}$$

with $\tilde{\mathcal{D}}^{(1)}_{2m}(T,t)$ defined as the integral over $\Delta^{(1)}_{2m,\varepsilon}(t/\varepsilon^2, 0)$ (see (3.16)) of

$$\tilde{\mathcal{E}}_{2m}(\mathbf{s}_{1,2m}) := \exp\left\{\mathrm{i}\varepsilon^{2H}\xi \sum_{p=1}^{m+1} \Delta B_{s_{2p-1}}\right\} \exp\left\{-\mathrm{i}\sum_{p=1}^m (k_{2p-1} - \varepsilon^{2H}\xi)\Delta B_{s_{2p}}\right\}.$$

Here $s_{2m+1} = t/\varepsilon^2$ and $s_0 := 0$. Since $|s_{2p} - s_{2p-1}| \le \log^r \varepsilon^{-1}$, we conclude that limit in (3.39) is the same when $\tilde{\mathcal{E}}_{2m}(\mathbf{s}_{1,2m})$ is replaced by

$$\tilde{\mathcal{E}}^{(1)}_{2m}(\mathbf{s}_{1,2m}) := \exp\left\{ \mathrm{i}\varepsilon^{2H}\xi \sum_{p=1}^{m+1} (B_{s_{2p+1}} - B_{s_{2p-1}}) \right\} \exp\left\{ -\mathrm{i}\sum_{p=1}^{m} k_{2p-1}\Delta B_{s_{2p}} \right\}$$
$$= \exp\left\{ \mathrm{i}\varepsilon^{2H}\xi B_{t/\varepsilon^2} \right\} \exp\left\{ -\mathrm{i}\sum_{p=1}^{m} k_{2p-1}\Delta B_{s_{2p}} \right\},$$

here $s_{-1} := 0$. Rewriting, modified in such a way (3.39), as in (3.18) and using Lemma 3.6 we arrive at

$$\lim_{\varepsilon\to+0} \left| \varepsilon^{2m}\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right) - \varepsilon^{2m}\mathbb{E}\exp\left\{ \mathrm{i}\varepsilon^{2H}\xi B_{t/\varepsilon^2} \right\} \int\limits_{\Delta_m(t/\varepsilon^2)} \prod_{p=1}^{m} \bar{\mathcal{J}}_\varepsilon(s_{2p-2}, s_{2p}, k_{2p-1}) d\tilde{\mathbf{s}}_{2,2m} \right| = 0, \tag{3.40}$$

where $d\tilde{\mathbf{s}}_{2,2m} := ds_2 \dots ds_{2m}$, $s_0 := 0$,

$$\bar{\mathcal{J}}_\varepsilon(s_1, s_2, k) := \mathbb{E}\left[\int_{(s_2 - \log^r \varepsilon^{-1})\vee s_1}^{s_2} \exp\left\{ -\mathrm{i}(k + \varepsilon^{2H}\xi)[B_{s_2} - B_s] \right\} ds \right].$$

From (3.40) we get

$$\lim_{\varepsilon\to+0} \left| \varepsilon^{2m}\mathbb{E}\mathcal{D}^{(1)}_{2m,\varepsilon}\left(\frac{t}{\varepsilon^2}, 0\right) - \exp\left\{ -(1/2)\xi^2 t^{2H} \right\} \frac{(R_* t)^m}{m!} \right| = 0. \tag{3.41}$$

Combining (3.41) with (3.39) we obtain the statement of Proposition 3.7 for $\ell = 1$. The argument can be easily generalized to an arbitrary $\ell \ge 1$ and the conclusion of the theorem follows. □

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