A sharp bound on the L^2 norm of the solution of a random elliptic difference equation

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Abstract

We consider a stationary solution of the Poisson equation $(\lambda - L^\omega)\phi_\lambda(x;\omega) = -\partial^* b(x;\omega)$, where $\lambda > 0$ and L^ω is a random, discrete, elliptic operator given by $L^\omega u(x) := \partial^*[a(x;\omega)\partial u(x)]$, $x \in \mathbb{Z}$. Here $\partial f(x) := f(x+1) - f(x)$ and $\partial^* f(x) := f(x-1) - f(x)$ for an arbitrary function $f : \mathbb{Z} \to \mathbb{R}$. The coefficients $\{(a(x;\omega), b(x;\omega)),\, x \in \mathbb{Z}\}$ form a stationary random field over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We prove that if the field of coefficients is sufficiently strongly mixing then $\|\phi_\lambda(0)\|_{\mathbb{P}}$ - the L^2 norm of w.r.t. the probability measure $\mathbb{P}$ - behaves as $\hat{C}\lambda^{-1/4}$, as $\lambda \ll 1$ for some constant $\hat{C} > 0$. In addition $\|\partial\phi_\lambda(0) - \partial\phi_0(0)\|_{\mathbb{P}} \le C\lambda^{1/4}$ for $\lambda \in (0,1]$ and some constant $C > 0$. These results complement those of [6] and [8] that hold for an analogous problem in the multidimensional setting.

1 Introduction

Suppose that $\{(a(x;\omega), b(x;\omega))\, x \in \mathbb{Z}\}$ is a stationary random field over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall be concerned with the stationary solutions of the equation

$$(\lambda + L^\omega)\phi_\lambda(x;\omega) = -\partial^* b(x;\omega), \tag{1.1}$$

where $\lambda > 0$ is small,

$$L^\omega u(x) := \partial^*[a(x;\omega)\partial u(x)], \quad x \in \mathbb{Z}$$

where $u : \mathbb{Z} \to \mathbb{R}$, $\partial u(x) := u(x+1) - u(x)$ is the discrete difference operator and $\partial^* u(x) := u(x-1) - u(x)$ is its adjoint. We assume that there exist constants $0 < a_* < a^* < +\infty$ and $b^* < +\infty$, so that

$$a(x;\omega) \in [a_*, a^*], \quad |b(x;\omega)| \le b^*, \quad \forall x \in \mathbb{Z},\, \mathbb{P} \text{ a.s. in } \omega. \tag{1.2}$$

Note that the operator L^ω is positive-definite and is the discrete version of $(-\nabla \cdot (a(x)\nabla))$ in the continuous case, thus all $\lambda > 0$ belong to its resolvent set. This observation allows to find a (unique) stationary solution of (1.1) for any $\lambda > 0$, see e.g. [7] for a details. On the other hand since $L^\omega 1 = 0$, $\lambda_0 = 0$ belongs to the spectrum of the operator. We shall be concerned with the limiting behavior of $\phi_\lambda(x)$, as $\lambda \downarrow 0$.

It has been shown recently (somewhat surprisingly) in [6] (see also [8] for another, more probabilistic, argument) that when $d \ge 3$ ($d \ge 9$ in [8]), and the coefficients $a(x) = b(x)$ (in [8] $a(x)$ and $b(x)$ are allowed to be different) are i.i.d., $\|\phi_\lambda(0)\|_{\mathbb{P}}$ stays bounded, as $\lambda \downarrow 0$. We denote here by $\|\cdot\|_{\mathbb{P}}$ the L^2 norm with respect to the probability measure $\mathbb{P}$:

$$\|f\|_{\mathbb{P}} = \left[\int f^2(\omega)d\mathbb{P}\right]^{1/2}.$$

When $d=2$ one can prove, see ibid., a logarithmic bound $\|\phi_\lambda(0)\|_{\mathbb{P}} \le C\log^\gamma \lambda^{-1}$, for $\lambda\in(0,1]$. In the present note we complete the picture by proving that in one dimension $\|\phi_\lambda(0)\|_{\mathbb{P}} \sim \hat{C}\lambda^{-1/4}$, with an explicit constant $\hat{C}>0$, as $\lambda\downarrow 0$, see Theorem 1.1 below, provided that the field $a(x)$ is sufficiently strongly mixing. The case when $a(x)=b(x)$ is of particular interest in the homogenization theory as the respective field $\phi_\lambda(x)$, called *the corrector*, can be used to show the convergence of solutions of equations with fast varying coefficients. A somewhat related question of determining the convergence rate for homogenization in one dimension has been considered in [1].

Our second result concerns the rate of convergence of the gradient of the λ-corrector in one dimension. It has been shown in [13] (see also [2] for the discrete setting) that in the continuum case when $d\ge 3$ and the coefficients are sufficiently strongly mixing there exist constants $C,\gamma>0$ such that $\|\nabla\phi_\lambda(0)-\nabla\phi_0(0)\|_{\mathbb{P}} \le C\lambda^\gamma$, $\lambda\in(0,1]$. In fact, in the discrete setting, for an i.i.d. field $a(x)$ one can show that γ can be chosen arbitrarily in the interval $(0,(d-2)/(d+8))$, see [2]. When $d=2$ the corresponding result is slightly weaker, see [10], Lemma 7.1 – it asserts that $\|\nabla\phi_\lambda(0)-\nabla\phi_0(0)\|_{\mathbb{P}} \le C\lambda^{\gamma/\log\log(\lambda^{-1})}$, $\lambda\in(0,1]$ for some $C,\gamma>0$. We prove that in the case $d=1$, under the aforementioned mixing assumption, $\|\partial\phi_\lambda(0)-\partial\phi_0(0)\|_{\mathbb{P}} \le C\lambda^{1/4}$ for all $\lambda\in(0,1]$, where $C>0$ is a constant.

Finally, we use our approach to obtain estimates of the convergence rate of solutions of parabolic equations with random coefficients and random initial data towards the expected value of the initial data, see Theorem 3.1. This property is known as stabilization of solutions of the heat equation and has been introduced by Zhikov in [14]. Our contribution is to establish the rate of convergence to equilibrium.

The method of the proof relies on a Feynman-Kac type of representation of the gradient of the corrector given by formula (2.4) below. This representation in turn allows us to write the corrector itself in terms of the Green's function of the symmetric, simple random walk, which is given explicitly. These formulas together allow us to describe the precise asymptotics of both $\phi_\lambda(0)$ and $\phi'_\lambda(0)$, as $\lambda\downarrow 0$, see Theorem 1.1.

The main result

We assume that the field $\{(a(x),b(x)), x\in\mathbb{Z}\}$ satisfies (1.2), and the following:

(1) Stationarity: for any $N\ge 1$, $x_1,\dots,x_N$ and $x\in\mathbb{Z}$ the laws of $(a(x_1),b(x_1),\dots,a(x_N),b(x_N))$ and $(a(x_1+x),b(x_1+x),\dots,a(x_N+x),b(x_N+x))$ are identical. Under this hypothesis there exists a unique stationary solution to (1.1) for each $\lambda>0$, see [7].

(2) Mixing: denote by $\int_{\mathbb{Z}}$ the summation over all integers, $B(x):=b(x)/a(x)$ and

$$\alpha(x) := \frac{1}{a(x)} - \frac{1}{\hat{a}}, \quad \beta(x) = B(x) - \hat{b},$$

where $\hat{a} := \langle a^{-1}(0)\rangle_{\mathbb{P}}^{-1}$, and $\hat{b} = \langle B(0)\rangle_{\mathbb{P}}$, so that $\langle\alpha(0)\rangle_{\mathbb{P}} = \langle\beta(0)\rangle_{\mathbb{P}} = 0$. We require that the two point statistics satisfy

$$\int_{\mathbb{Z}} x^2[|\langle\alpha(x)\alpha(0)\rangle_{\mathbb{P}}| + |\langle\beta(x)\beta(0)\rangle_{\mathbb{P}}|]dx < +\infty, \tag{1.3}$$

and, in addition, the higher moments satisfy

$$\mathcal{I}_N := \sup_{x_2,\dots,x_{2n}} \int_{[x_1\ge x_2\ge\dots\ge x_{2N-1}\ge x_{2N}]} \left| \left\langle \prod_{i=1}^{2N} \gamma_k(x_i) \right\rangle_{\mathbb{P}} \right| dx_1 dx_3 \dots dx_{2N-1} < +\infty \tag{1.4}$$

for $N=1,\dots,5$, $k=0,1$, and $\gamma_0(x)=\alpha(x)$, $\gamma_1(x)=\beta(x)$.

The main result of this note is the following.

Theorem 1.1 *Under the foregoing hypotheses we have*

$$\|\phi_\lambda(0)\|_{\mathbb{P}} = \mathcal{C}_*\lambda^{-1/4} + O(1) \quad \text{as } \lambda \downarrow 0, \quad \mathcal{C}_* = \hat{a}^{1/4} G_0^{1/2}/2 \tag{1.5}$$

where

$$G_0 := \int_{\mathbb{Z}} \langle \Gamma(x)\Gamma(0)\rangle_{\mathbb{P}} dx$$

and

$$\Gamma(x) := \hat{a}\hat{b}\alpha(x) + \beta(x). \tag{1.6}$$

In addition, there exists $\hat{C} > 0$ such that

$$\|\partial\phi_\lambda(0) - \partial\phi_0(0)\|_{\mathbb{P}} \le \hat{C}\lambda^{1/4} \quad \text{for all } \lambda \in (0,1]. \tag{1.7}$$

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2 The proof of Theorem 1.1

2.1 The proof of (1.5)

In order to obtain a precise asymptotics in (1.5) we will split the field ϕ_λ into several terms (see decomposition (2.7) below), and estimate each of them separately. Denote $\psi_\lambda(x) := a(x)\partial\phi_\lambda(x)$. Using equation (1.1) we obtain

$$\phi_\lambda(x) = -\frac{1}{\lambda}\partial^* f_\lambda(x), \tag{2.1}$$

where

$$f_\lambda(x) := \psi_\lambda(x) - \psi_0(x). \tag{2.2}$$

Here $\psi_0(x) := \hat{a}\hat{b} - b(x)$. Note that the field $\psi_\lambda(x)$ converges, as $\lambda \to 0+$, in $L^2(\mathbb{P})$ to $\psi_0(x)$. This can been seen as follows. Using Theorem 2.4 of [11] one can deduce that $\partial\phi_\lambda(x)$ converges to some stationary, zero mean, field $\Phi_*(x)$ in $L^2(\mathbb{P})$. From (1.1) we get $\partial^*[a(x)\Phi_*(x)] = -\partial^* b(x)$ hence $\Phi_*(x) = -B(x) + Ca^{-1}(x)$ for some deterministic constant C. Since $\langle\Phi_*(x)\rangle_{\mathbb{P}} = 0$ we conclude that $C = \hat{a}\hat{b}$ and the assertion follows due to the fact that $\psi_0(x) = a(x)\Phi_*(x)$.

Observe that $\psi_\lambda(x)$ satisfies

$$(\lambda/2)a^{-1}(x)\psi_\lambda(x) + (1/2)\partial^*\partial\psi_\lambda(x) = -(1/2)\partial^*\partial b(x), \quad \forall\, \lambda > 0. \tag{2.3}$$

Therefore, it can be written as

$$\psi_\lambda(x) = -\frac{1}{2}\int_0^{+\infty} \mathbf{E}\left[e_\lambda(t,x)\partial^*\partial b(X_t^x)\right] dt, \tag{2.4}$$

where

$$e_\lambda(t,x) := \exp\left\{-(\lambda/2)\int_0^t a^{-1}(X_s^x) ds\right\}.$$

Here $\{X_t,\, t\ge 0\}$ is a symmetric, simple random walk on $\mathbb{Z}$ with continuous time starting at x, given over another probability space $(\Sigma,\mathcal{A},\mathbb{Q})$, and $\mathbf{E}$ denotes the expectation with respect to $\mathbb{Q}$. We shall drop the superscipt x in the case wen the walk starts at the origin. Using the fact that

$$M_t = b(X_t^x) - b(x) + \frac{1}{2}\int_0^t \partial^*\partial b(X_s^x)ds$$

is a mean zero martingale, we conclude that (recall $B(x) = b(x)a^{-1}(x)$),

$$\begin{aligned}\psi_\lambda(x) &= \int_0^{+\infty} \mathbf{E}\left[e_\lambda(t,x)db(X_t^x)\right] = \frac{\lambda}{2}\int_0^{+\infty}\mathbf{E}[e_\lambda(t,x)B(X_t)]dt - b(x) \\ &= \psi_0(x) - \hat{a}\hat{b} + \sum_{i=0}^{n} D_\lambda^{(i)}(x) + R_\lambda^{(n)}(x),\end{aligned} \tag{2.5}$$

where

$$D_\lambda^{(i)}(x) := \frac{1}{i!}\left(\frac{\lambda}{2}\right)^{i+1}\int_0^{+\infty}\mathbf{E}\left\{B(X_t)\left[\int_0^t \alpha(X_s)ds\right]^i\right\}\exp\left\{-t\hat{a}^{-1}\lambda/2\right\}dt,$$

and

$$R_\lambda^{(n)}(x) := \frac{\lambda}{2}\int_0^{+\infty}\mathbf{E}\left\{B(X_t)\left[e_\lambda(t,x) - \exp\left\{-t\hat{a}^{-1}\lambda/2\right\}\sum_{i=0}^{n}\frac{1}{i!}\left\{\frac{\lambda}{2}\int_0^t\alpha(X_s)ds\right\}^i\right]\right\}dt. \tag{2.6}$$

Substituting (2.5) into the right hand side of (2.1) we obtain

$$\phi_\lambda(x) = \sum_{i=0}^{n}\phi_\lambda^{(i)}(x) + r_\lambda^{(n)}(x), \tag{2.7}$$

where

$$\begin{aligned}\phi_\lambda^{(0)}(x) &= \frac{1}{\lambda}\partial^*\left[\hat{a}\hat{b} - D_\lambda^{(0)}(x)\right], \\ \phi_\lambda^{(i)}(x) &= -\frac{1}{\lambda}\partial^* D_\lambda^{(i)}(x), \quad \text{for } i = 1,\ldots,n\end{aligned} \tag{2.8}$$

and

$$r_\lambda^{(n)}(x) = -\frac{1}{\lambda}\partial^* R_\lambda^{(n)}(x). \tag{2.9}$$

As we will see, the main contribution to ϕ_λ comes from $\phi_\lambda^{(0)}(x)+\phi_\lambda^{(1)}(x)$ that is of the order $O(\lambda^{-1/4})$, while the other terms are of the size at most $O(1)$, provided that $n\ge 3$.

Before we proceed to the estimates, note that simple symmetry considerations give for $i\ge 1$

$$\begin{aligned}D_\lambda^{(i)}(x) &= \left(\frac{\lambda}{2}\right)^{i+1}\int_0^{+\infty}\exp\left\{-t\hat{a}^{-1}\lambda/2\right\}dt\int_{\Delta_i(t)}\mathbf{E}\left\{B(X_t)\left[\prod_{k=1}^{i}\alpha(X_{s_k})\right]\right\}ds_1\ldots ds_i \\ &= \left(\frac{\lambda}{2}\right)^{i+1}\int_0^{+\infty}ds_1\int_{s_1}^{+\infty}ds_2\ldots\int_{s_{i-1}}^{+\infty}ds_i\int_{s_i}^{+\infty}ds_{i+1}\exp\left\{-s_{i+1}\hat{a}^{-1}\lambda/2\right\} \\ &\times\int_{\mathbb{Z}^{i+1}}B(x_{i+1})p(s_{i+1}-s_i,x_{i+1}-x_i)\left[\prod_{k=1}^{i}\alpha(x_k)p(s_k-s_{k-1},x_k-x_{k-1})\right]dx_1\ldots dx_i,\end{aligned} \tag{2.10}$$

where $\Delta_i(t) := [(s_1,\ldots,s_i) : 0 \le s_1 \le \ldots \le s_i]$, $s_0 := 0$, and $x_0 := x$. Recall that the Green's function corresponding to the operator $\mu + (1/2)\partial^*\partial$ is

$$G_\mu(x) := \int_0^{+\infty} e^{-\mu t} p(t,x)dt,$$

where $p(t,x) := \mathbb{Q}[X_t = x]$ for $t > 0$, $x \in \mathbb{Z}$. It is explicitly given by (see, e.g. (3.134) p. 141 of [4])

$$G_\mu(x) = \xi(1-\xi^2)^{-1/2} q_\xi^{|x|}, \quad x \in \mathbb{Z} \tag{2.11}$$

with $\xi := (1+\mu)^{-1}$ and $q_\xi := (1-\sqrt{1-\xi^2})\xi^{-1}$. Observe that for small μ we have

$$\xi_1 = 1 - \mu + o(\mu), \tag{2.12}$$

and

$$q_{\xi_1} = \frac{1-\sqrt{1-\xi_1^2}}{\xi_1} = 1 - \sqrt{2\mu} + o(\sqrt{\mu}). \tag{2.13}$$

Integrating out the s_{i+1}-variable in (2.10) and using the definition of the Green's function we can write

$$D_\lambda^{(i)}(x) = \left(\frac{\lambda}{2}\right)^{i+1} \int_{\mathbb{Z}^{i+1}} \prod_{k=1}^{i} \left[\alpha(x_k) G_{\lambda/(2\hat{a})}(x_{k-1}-x_k)\right] \tag{2.14}$$

$$\times B(x_{i+1}) G_{\lambda/(2\hat{a})}(x_i - x_{i+1}) dx_1 \ldots dx_{i+1},\ i \ge 1. \tag{2.15}$$

When $i = 0$ we can write

$$D_\lambda^{(0)}(x) - \hat{a}\hat{b} = \frac{\lambda}{2} \int_{\mathbb{Z}} G_{\lambda/(2\hat{a})}(x - x_1)\beta(x_1)dx_1, \tag{2.16}$$

where, as we recall, $\beta(x) = B(x) - \hat{b}$.

Asymptotics of $r_\lambda^{(n)}$

The begin the proof of (1.5) with the estimate of $r_\lambda^{(n)}$ since some elements of the proof of this bound will be used later in estimating the other terms.

Lemma 2.1 *Suppose that $n \ge 3$. Then, there exists a constant C_r such that*

$$\|r_\lambda^{(n)}\|_{\mathbb{P}} \le C_r \lambda^{(n+1)/4-1}, \quad \forall\, \lambda \in (0,1]. \tag{2.17}$$

Proof. It suffices to prove that here exists a constant $C > 0$ so that

$$\|R_\lambda^{(n)}\|_{\mathbb{P}} \le C\lambda^{(n+1)/4}, \quad \forall\, \lambda \in (0,1], \tag{2.18}$$

with $R_\lambda^{(n)}(x)$ given by (2.6), and $R_\lambda^{(n)} := R_\lambda^{(n)}(0)$. We use an elementary inequality

$$\left| e^{-a} - \sum_{i=0}^{n} e^{-b}\frac{(b-a)^i}{i!} \right| \le \frac{1}{(n+1)!} \max\{e^{-a}, e^{-b}\}|b-a|^{n+1}$$

valid for any $a, b > 0$. This inequality and the ellipticity assumption (1.2) together imply that

$$|R_\lambda^{(n)}| \le C_1 \lambda^{n+2} \int_0^{+\infty} V(t) \exp\left\{-(\lambda/2)(a^*)^{-1} t\right\} dt \tag{2.19}$$

where

$$V(t) := \mathbf{E}\left|\int_0^t \alpha(X_s)ds\right|^{n+1},$$

with a deterministic constant $C_1 > 0$. Calculations similar to those leading to (2.14) yield

$$|R_\lambda| \le C_2\lambda^{n+1}\int_{\mathbb{Z}^{n+1}}\prod_{k=1}^{n+1}\left[\alpha(x_i)G_{\lambda_1}(x_i - x_{i-1})\right]dx_1\dots dx_{n+1}, \tag{2.20}$$

where $\lambda_1 := a^*\lambda/2$, and thus

$$\langle R_\lambda^2\rangle_{\mathbb{P}} \le C_2^2\lambda^{2n+2}\int_{\mathbb{Z}^{2n+2}}\left|\left\langle\prod_{k=1}^{2n+2}\alpha(x_i)\right\rangle_{\mathbb{P}}\right| \tag{2.21}$$
$$\times\prod_{i=1}^{n+1}\left[G_{\lambda_1}(x_i - x_{i-1})G_{\lambda_1}(x_{i+n+2} - x_{i+n+1})\right]dx_1\dots dx_{2n+2},$$

where $x_0 = x_{2n+3} = 0$. Using (2.11), we conclude that

$$\langle R_\lambda^2\rangle_{\mathbb{P}} \le C_3\lambda^{n+1}\int_{\mathbb{Z}^{2n+2}}\left|\left\langle\prod_{k=1}^{2n+2}\alpha(x_i)\right\rangle_{\mathbb{P}}\right|\prod_{i=1}^{n+1}\left[q_{\xi_1}^{|x_i-x_{i-1}|}q_{\xi_1}^{|x_{i+n+2}-x_{i+n+1}|}\right]dx_1\dots dx_{2n+2}, \tag{2.22}$$

and $\xi_1 := (1+\lambda_1)^{-1}$.

We divide $\mathbb{Z}^{2n+2}$ into simplicies $\Delta_\sigma := [x_{\sigma(2n+2)} \ge \dots \ge x_{\sigma(1)}]$, where σ is a permutation of the set $\{1,\dots,2n+2\}$. Each simplex is further split as $\Delta_\sigma = \Delta_\sigma^{(1)} \cup \Delta_\sigma^{(2)}$. Here $(x_1,\dots,x_{2n+2})$ is in $\Delta_\sigma^{(1)}$ if $0 \in [x_{\sigma(2)}, x_{\sigma(2n+2)})$, and in $\Delta_\sigma^{(2)}$ if $0 \notin [x_{\sigma(2)}, x_{\sigma(2n+2)})$.

Lemma 2.2 *We have*

$$\prod_{i=1}^{n+1}\left[q_{\xi_1}^{|x_i-x_{i-1}|}q_{\xi_1}^{|x_{i+n+2}-x_{i+n+1}|}\right] \le q_{\xi_1}^{x_{\sigma(2n+2)}+|x_{\sigma(2)}|} \qquad \textit{on } \Delta_\sigma^{(1)}, \tag{2.23}$$

$$\prod_{i=1}^{n+1}\left[q_{\xi_1}^{|x_i-x_{i-1}|}q_{\xi_1}^{|x_{i+n+2}-x_{i+n+1}|}\right] \le q_{\xi_1}^{x_{\sigma(2n+2)}} \qquad \textit{on } \Delta_\sigma^{(2)'} := \Delta_\sigma^{(2)} \cap [x_{\sigma(2n+2)} > 0], \tag{2.24}$$

and

$$\prod_{i=1}^{n+1}\left[q_{\xi_1}^{|x_i-x_{i-1}|}q_{\xi_1}^{|x_{i+n+2}-x_{i+n+1}|}\right] \le q_{\xi_1}^{|x_{\sigma(2)}|} \qquad \textit{on } \Delta_\sigma^{(2)''} := \Delta_\sigma^{(2)} \cap [x_{\sigma(2n+2)} < 0]. \tag{2.25}$$

Proof of Lemma 2.2. In order to show (2.23), suppose that $x_{\sigma(2)} = x_j$ and $x_{\sigma(2n+2)} = x_k$. If $j \le n+1$ and $k \ge n+2$, as $x_{2n+3} = x_0 = 0$, and

$$x_{\sigma(2)} \le 0 \le x_{\sigma(2n+2)} \text{ on } \Delta_\sigma^{(1)}, \tag{2.26}$$

it is clear that

$$x_{\sigma(2n+2)} + |x_{\sigma(2)}| \le |x_k - x_{k+1}| + \dots + |x_{2n+2} - x_{2n+3}| + |x_0 - x_1| + \dots + |x_{j-1} - x_j| \tag{2.27}$$

and (2.23) holds since $q_{\xi_1} \in (0,1)$. When $j,k \le n+1$ we can write, using (2.26),

$$x_{\sigma(2n+2)} + |x_{\sigma(2)}| = |x_{\sigma(2n+2)} - x_{\sigma(2)}| \le |x_0 - x_1| + \dots + |x_n - x_{n+1}|, \tag{2.28}$$

whence (2.23) holds. The case $j, k \geq n+2$ can be verified analogously.

In order to verify that (2.24) and (2.25) hold, we simply note that, say, for (2.24) if $\sigma(2n+2) \leq n+1$ then we would use the fact that

$$x_{\sigma(2n+2)} = |x_{\sigma(2n+2)} - x_0| \leq |x_1 - x_0| + \cdots + |x_{n+1} - x_n|,$$

and the other cases are very similar. $\square$

We now finish the proof of Lemma 2.1. The integral in (2.22) can be written as

$$\int_{\Delta_\sigma} \left| \left\langle \prod_{k=1}^{2n+2} \alpha(x_i) \right\rangle_{\mathbb{P}} \right| \prod_{i=1}^{n+1} \left[q_{\xi_1}^{|x_i - x_{i-1}|} q_{\xi_1}^{|x_{i+5} - x_{i+4}|} \right] dx_1 \dots dx_{2n+2} = I_1 + I_2,$$

where I_ℓ correspond to the integration over domains $\Delta_\sigma^{(\ell)}$, $\ell = 1, 2$. Using the mixing condition (1.4) for $N = n+1$ and (2.23) we conclude that, with

$$A_\sigma^{(1)} := \left\{ [x_{\sigma(2n+2)} \geq x_{\sigma(2n)} \geq \ldots \geq x_{\sigma(2)}], 0 \in [x_{\sigma(2)}, x_{\sigma(2n+2)}] \right\},$$

we have

$$\begin{aligned} I_1 &\leq \int\limits_{A_\sigma^{(1)}} q_{\xi_1}^{x_{\sigma(2n+2)} + |x_{\sigma(2)}|} dx_{\sigma(2)} \dots dx_{\sigma(2n+2)} \sup_{x_{\sigma(2n+2)}, \dots, x_{\sigma(2)}} \left[\int\limits_{\mathbb{Z}^{n+1}} \left| \left\langle \prod_{k=1}^{2n+2} \alpha(x_i) \right\rangle_{\mathbb{P}} \right| dx_{\sigma(1)} \dots dx_{\sigma(2n+1)} \right] \\ &\leq \mathcal{I}_{n+1} \int_{A_\sigma^{(1)}} q_{\xi_1}^{x_{\sigma(2n+2)} + |x_{\sigma(2)}|} dx_{\sigma(2)} \dots dx_{\sigma(2n+2)} \leq \mathcal{I}_{n+1} \int_{\mathbb{Z}^{n+1}} q_{\xi_1}^{(\sum_{i=1}^{n+1} |x_i|)/n+1} dx_1 \dots dx_{n+1} \\ &\leq \mathcal{I}_{n+1} \left(1 - q_{\xi_1}^{1/(n+1)}\right)^{-(n+1)} \leq \frac{C}{\lambda^{(n+1)/2}} \end{aligned}$$

for some constant $C > 0$. We have used (2.13) in the last step. On the other hand the mixing condition (1.4) and (2.24), (2.25) yield

$$I_2 \leq C \int_{[x_{\sigma(2n+2)} \geq x_{\sigma(2n)} \geq \ldots \geq x_{\sigma(2)} \geq 0]} q_{\xi_1}^{x_{\sigma(2n+2)}} dx_{\sigma(2)} \dots dx_{\sigma(2n+2)} \leq \frac{C}{\lambda^{(n+1)/2}}$$

Coming back to (2.22) we conclude that

$$\langle [R_\lambda^{(n)}]^2 \rangle_{\mathbb{P}} \leq C_n \lambda^{(n+1)/2}, \tag{2.29}$$

which in turn implies (2.18). This finishes the proof of Lemma 2.1. $\square$

Asymptotics of $\phi_\lambda^{(0)}(0) + \phi_\lambda^{(1)}(0)$

Here, we identify the leading order contribution in (1.5).

Lemma 2.3 *We have*

$$\|\phi_\lambda^{(0)}(0) + \phi_\lambda^{(1)}(0)\|_{\mathbb{P}} = \mathcal{C}_* \lambda^{-1/4} + O(1) \quad \text{as } \lambda \downarrow 0, \tag{2.30}$$

with the constant $\mathcal{C}_*$ as in (1.5).

Proof. From (2.8) and (2.16) we conclude that

$$\phi_\lambda^{(0)}(0) = -\frac{1}{2}\int_{\mathbb{Z}} \partial^* G_{\lambda/(2\hat{a})}(x_1)\beta(x_1)dx_1 = \frac{1}{2}\int_{\mathbb{Z}} g(x_1;\xi_1)q_{\xi_1}^{|x_1|}\beta(x_1)dx_1, \tag{2.31}$$

where $\xi_1 := [1+\lambda/(2\hat{a})]^{-1}$, and

$$g(x;\xi) := \begin{cases} 1+\sqrt{\dfrac{1-\xi}{1+\xi}}, & \text{when } x \ge 1,\ |\xi| \le 1, \\ \\ -\left(1-\sqrt{\dfrac{1-\xi}{1+\xi}}\right), & \text{when } x \le 0,\ |\xi| \le 1. \end{cases}$$

There exists a constant $C > 0$ such that

$$|g(x;\xi) - \mathrm{sgn}(x)| \le C\sqrt{\lambda}, \quad \forall\, x \in \mathbb{Z},\ \lambda \in (0,1],\ |\xi| \in [1/2,1], \tag{2.32}$$

with the convention $\mathrm{sgn}(x) := 1$ for $x \ge 1$ and $\mathrm{sgn}\ x := -1$ for $x \le 0$. Likewise, using (2.8) and (2.10) we obtain that

$$\begin{aligned} \phi_\lambda^{(1)}(0) &= -\frac{\lambda}{4}\int_{\mathbb{Z}^2} \partial^* G_{\lambda/(2\hat{a})}(x_1)G_{\lambda/(2\hat{a})}(x_2-x_1)\alpha(x_1)B(x_2)dx_1dx_2 \\ &= \frac{\lambda\xi_1}{4\sqrt{1-\xi_1^2}}\int_{\mathbb{Z}^2} g(x_1;\xi_1)q_{\xi_1}^{|x_1|}q_{\xi_1}^{|x_1-x_2|}\alpha(x_1)B(x_2)dx_1dx_2, \end{aligned} \tag{2.33}$$

Using decomposition $B(x) = \hat{b}+\beta(x)$, we obtain from (2.31) and (2.33) that $\phi_\lambda^{(0)}(0)+\phi_\lambda^{(1)}(0) = J_1+J_2$, with

$$J_1 = \frac{1}{2}\int_{\mathbb{Z}} g(x_1;\xi_1)q_{\xi_1}^{|x_1|}\Gamma(x_1)dx_1dx_2,$$

and

$$J_2 = \frac{(\hat{a}\lambda)^{1/2}}{2\sqrt{2}}\left(\frac{\xi_1}{1+\xi_1}\right)^{1/2}\int_{\mathbb{Z}^2} g(x_1;\xi_1)q_{\xi_1}^{|x_1|}q_{\xi_1}^{|x_1-x_2|}\alpha(x_1)\beta(x_2)dx_1dx_2.$$

Here $\Gamma(x)$ is given by (1.6).

Asymptotics of J_1

By virtue of (1.3) and (2.32), we deduce that, as $\lambda \downarrow 0$,

$$\begin{aligned} \|J_1\|_{\mathbb{P}}^2 &= \frac{1}{4}\int_{\mathbb{Z}^2} g(x;\xi_1)g(x';\xi_1)q_{\xi_1}^{|x|+|x'|}\langle\Gamma(x-x')\Gamma(0)\rangle_{\mathbb{P}}dxdx' \\ &= \frac{1}{4}\int_{\mathbb{Z}^2} \mathrm{sgn}\ x\,\mathrm{sgn}\ x'\ q_{\xi_1}^{|x|+|x'|}\langle\Gamma(x-x')\Gamma(0)\rangle_{\mathbb{P}}dxdx' + O(1) \\ &= \frac{1}{8\pi}\int_0^{2\pi} |F(q_{\xi_1}e^{i\zeta})|^2 G(\zeta)d\zeta + O(1), \end{aligned} \tag{2.34}$$

where

$$F(z) = -1+2i\mathrm{Im}\left[\int_{x\ge 1} z^x\right] = 2i(\mathrm{Im}z)|1-z|^{-2}-1,$$

and

$$G(\zeta) := \int_{\mathbb{Z}} e^{i\zeta x} \langle \Gamma(x)\Gamma(0)\rangle_{\mathbb{P}} dx. \tag{2.35}$$

Bochner's theorem implies that

$$0 \le G(\zeta) \le G_* := \int_{\mathbb{Z}} |\langle \Gamma(x)\Gamma(0)\rangle_{\mathbb{P}}| dx < +\infty,$$

due to (1.3). In order to pass to the limit $\lambda \downarrow 0$ we use (2.12) and (2.13), and obtain that

$$\xi_1 = 1 - \frac{\lambda}{2\hat{a}} + o(\lambda),$$

and

$$q_{\xi_1} = \frac{1 - \sqrt{1-\xi_1^2}}{\xi_1} = 1 - \sqrt{\frac{\lambda}{\hat{a}}} + o(\sqrt{\lambda}).$$

Thanks to (1.3) we have $|G(\zeta) - G(0)| \sim \zeta^2$ for $\zeta \ll 1$. One can conclude that

$$\begin{aligned} \mathcal{C}_*^2 &:= \lim_{\lambda\downarrow 0} \sqrt{\lambda}\|J_1\|_{\mathbb{P}}^2 = \frac{1}{4}\lim_{\lambda\downarrow 0}\sqrt{\lambda}\int_0^{2\pi} |F(q_{\xi_1}e^{i\zeta})|^2 G(\zeta)\frac{d\zeta}{2\pi} \\ &= \frac{G(0)}{4}\lim_{\lambda\downarrow 0}\sqrt{\lambda}\int_0^{2\pi} |F(q_{\xi_1}e^{i\zeta})|^2 \frac{d\zeta}{2\pi}. \end{aligned} \tag{2.36}$$

However, we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} |F(q_{\xi_1}e^{i\zeta})|^2 d\zeta = \frac{1}{2\pi}\int_{\mathbb{Z}^2}\int_0^{2\pi} \operatorname{sgn} x \operatorname{sgn} x'\, q_{\xi_1}^{|x|+|x'|} e^{i\zeta x - i\zeta x'} dx dx' d\zeta \\ &= \int_{\mathbb{Z}} q_{\xi_1}^{2|x|} dx = \frac{1+q_{\xi_1}^2}{1-q_{\xi_1}^2}, \end{aligned}$$

whence

$$\mathcal{C}_*^2 = \frac{G(0)}{4}\lim_{\lambda\downarrow 0}\lambda^{1/2}\frac{1+q_{\xi_1}^2}{1-q_{\xi_1}^2} = \frac{\hat{a}^{1/2}G(0)}{4}, \tag{2.37}$$

which is the constant appearing in (1.5) in Theorem 1.1.

Asymptotics of J_2

The L^2-norm of J_2 satisfies

$$\|J_2\|_{\mathbb{P}}^2 \le C\lambda \int_{\mathbb{Z}^4} q_{\xi_1}^{|x_1|} q_{\xi_1}^{|x_3|} q_{\xi_1}^{|x_1-x_2|} q_{\xi_1}^{|x_3-x_4|}\, |\langle \alpha(x_1)\alpha(x_3)\beta(x_2)\beta(x_4)\rangle_{\mathbb{P}}|\, dx_1 dx_2 dx_3 dx_4,$$

with some constant $C > 0$. To estimate the right side we use the mixing condition (1.4) in the same way as in the proof of Lemma 2.1. We divide the domain of integration $\mathbb{Z}^4$ into subdomains of the form $\Delta_\sigma := [x_{\sigma(1)} \ge x_{\sigma(2)} \ge x_{\sigma(3)} \ge x_{\sigma(4)}]$ where σ is a permutation of $(1, 2, 3, 4)$. In case the permutation equals identity we can estimate it by

$$C'\lambda \int_{x_2,x_4} q_{\xi_1}^{|x_2|} q_{\xi_1}^{|x_4|} dx_2 dx_4 \left\{ \sup_{x_2,x_4} \int_{[x_1\ge x_2\ge x_3\ge x_4]} |\langle \alpha(x_1)\alpha(x_3)\beta(x_2)\beta(x_4)\rangle_{\mathbb{P}}|\, dx_1 dx_3 \right\}$$

This expression can be further estimated by

$$C''\lambda(1-q_{\xi_1})^{-2} \le C_1, \quad \forall \in \lambda \in (0,1]$$

with some C'', C_1. The cases corresponding to other domains can be dealt with similarly. This completes the proof of Lemma 2.3. □

Asymptotics of $\phi_\lambda^{(i)}$ for $i \ge 2$

Next, we show that the contribution of both $\phi_\lambda^{(2)}$ and $\phi_\lambda^{(3)}$ in ϕ_λ is small.

Lemma 2.4 *There exist constants $\mathcal{C}_*^{(i)}$, $i = 2, 3$ such that*

$$\|\phi_\lambda^{(i)}(0)\|_{\mathbb{P}} \le \mathcal{C}_*^{(i)} \lambda^{i/2-1} \quad \text{for } \lambda \in (0,1]. \tag{2.38}$$

Proof. We start with the argument for $i = 2$. A simple calculation, using (2.1) and (2.14) shows that

$$\begin{aligned}\phi_\lambda^{(2)}(0) = &-\frac{\lambda^2}{8}\int_{\mathbb{Z}} \left[\partial^* G_{\lambda/(2\hat{a})}(x_1)\right] G_{\lambda/(2\hat{a})}(x_2 - x_1) G_{\lambda/(2\hat{a})}(x_3 - x_2) \\ &\times \alpha(x_1)\alpha(x_2)B(x_3)dx_1dx_2dx_3 = K_1 + K_2,\end{aligned} \tag{2.39}$$

where

$$K_1 := 2^{3/2}\xi_1^{1/2}(1+\xi_1)^{-1/2}\frac{\lambda^{1/2}\hat{a}^{3/2}\hat{b}}{8}\int_{\mathbb{Z}^2} g(x_1;\xi_1) q_{\xi_1}^{|x_1|} q_{\xi_1}^{|x_1-x_2|}\alpha(x_1)\alpha(x_2)dx_1dx_2,$$

$$K_2 := \frac{\hat{a}\lambda\xi_1}{4(1+\xi_1)}\int_{\mathbb{Z}^3} g(x_1;\xi_1) q_{\xi_1}^{|x_1|} q_{\xi_1}^{|x_1-x_2|} q_{\xi_1}^{|x_2-x_3|}\alpha(x_1)\alpha(x_2)\beta(x_3)dx_1dx_2dx_3.$$

The L^2 norm of K_1 satisfies

$$\begin{aligned}\|K_1\|_{\mathbb{P}}^2 &\le C\lambda \int_{\mathbb{Z}^4} g(x_1;\xi_1)g(x_3;\xi_1) q_{\xi_1}^{|x_1|} q_{\xi_1}^{|x_1-x_2|} q_{\xi_1}^{|x_3|} q_{\xi_1}^{|x_3-x_4|}\left|\left\langle \prod_{i=1}^4 \alpha(x_i)\right\rangle_{\mathbb{P}}\right| dx_1dx_2dx_3dx_4 \\ &\le C'\lambda \int_{\mathbb{Z}^4} q_{\xi_1}^{|x_1|} q_{\xi_1}^{|x_3|} q_{\xi_1}^{|x_1-x_2|} q_{\xi_1}^{|x_3-x_4|}\left|\left\langle \prod_{i=1}^4 \alpha(x_i)\right\rangle_{\mathbb{P}}\right| dx_1dx_2dx_3dx_4,\end{aligned} \tag{2.40}$$

with some constants $C, C' > 0$. To estimate the utmost right side of (2.40) we use the mixing condition (1.4) with $N = 2$. We divide the domain of the integration $\mathbb{Z}^4$ into the subdomains of the form $\Delta_\sigma := [x_{\sigma(1)} \ge x_{\sigma(2)} \ge x_{\sigma(3)} \ge x_{\sigma(4)}]$, where σ is a permutation of $(1,2,3,4)$ and use an argument detailed in the proof of Lemma 2.2 below. When the permutation equals identity we can estimate this term by

$$C'\lambda \int_{\mathbb{Z}^2} q_{\xi_1}^{|x_2|} q_{\xi_1}^{|x_4|} dx_2dx_4 \left\{\sup_{x_2,x_4}\int_{[x_1\ge x_2\ge x_3\ge x_4]}\left|\left\langle \prod_{i=1}^4 \alpha(x_i)\right\rangle_{\mathbb{P}}\right| dx_1dx_3\right\}.$$

The last expression can be further estimated by

$$C''\lambda(1-q_{\xi_1})^{-2} \le C_1, \quad \forall \in \lambda \in (0,1]$$

for some C'', C_1. The other domains of integration can be dealt with similarly. The considerations for $\|K_2\|_{\mathbb{P}}^2$ are similar. Finally, to estimate $\|\phi_\lambda^{(i)}(0)\|_{\mathbb{P}}^2$, for $i \ge 3$ we can easily generalize the above argument applying the mixing condition (1.4) for $N = i$. □

To finish the proof of Theorem 1.1 we use expansion (2.7) for $n = 3$. The result is a direct consequence of Lemmas 2.1, 2.3 and 2.4.

2.2 The gradient estimate

We now prove (1.7). It suffices to show that

$$\|\psi_\lambda(0) - \psi_0(0)\|_{\mathbb{P}} \le C\lambda^{1/4}, \quad \forall\, \lambda \in (0,1] \tag{2.41}$$

for some constant $C > 0$. Using (2.5) it is enough to estimate

$$\|D_\lambda^{(0)} + D_\lambda^{(1)} - \hat{a}\hat{b}\|_{\mathbb{P}},$$

$\|D_\lambda^{(i)}\|_{\mathbb{P}}$ for $i = 2, 3$ and $\|R_\lambda\|_{\mathbb{P}}$. We have used a shorthand notation $D_\lambda^{(i)} := D_\lambda^{(i)}(0)$. From (2.14) we obtain after elementary calculations the decomposition $D_\lambda^{(1)} = L_1 + L_2$, where

$$L_1 := \frac{\lambda}{2}\int_{\mathbb{Z}} \Gamma(x_1) G_{\lambda/(2\hat{a})}(x_1)dx_1$$
$$L_2 := \frac{\lambda\hat{a}\xi_1^2}{4(1+\xi_1)}\int_{\mathbb{Z}^2} \alpha(x_1)\beta(x_2) q_{\xi_1}^{|x_1-x_2|} q_{\xi_1}^{|x_1|} dx_1 dx_2.$$

Thus,

$$\|L_1\|_{\mathbb{P}}^2 = \frac{\xi_1^2\lambda^2}{4(1-\xi_1^2)}\int_{\mathbb{Z}^2} q_{\xi_1}^{|x|+|x'|}\langle\Gamma(x-x')\Gamma(0)\rangle_{\mathbb{P}} dx dx' = \frac{\lambda\hat{a}}{2^4\pi}\int_0^{2\pi} |F_1(q_{\xi_1}e^{i\zeta})|^2 G(\zeta)d\zeta + O(\lambda),$$

where $G(\zeta)$ is given by (2.35), $F_1(z) := (1-|z|^2)|1-z|^{-2}$ is the Poisson kernel in dimension $d = 2$. Since $|G(\zeta) - G(0)| \sim \zeta^2$ for $\zeta \ll 1$ one can easily deduce that

$$\|L_1\|_{\mathbb{P}}^2 = \frac{G(0)\lambda\hat{a}}{2^4\pi}\int_0^{2\pi} |F_1(q_{\xi_1}e^{i\zeta})|^2 d\zeta + O(\lambda).$$

We have

$$\int_0^{2\pi} |F_1(q_{\xi_1}e^{i\zeta})|^2 d\zeta \le C_1 \int_{\mathbb{R}} \frac{d\zeta}{1 - q_{\xi_1} + \zeta^2}$$

for $\lambda \in (0,1]$ and some constant $C_1 > 0$ and $1 - q_{\xi_1} \sim \lambda^{1/2}$. Hence, after elementary computations, we get

$$\|L_1\|_{\mathbb{P}}^2 \le C_2\lambda^{1/2}$$

for $\lambda \in (0,1]$ and some constant $C_2 > 0$.

To estimate $\|L_2\|_{\mathbb{P}}^2$ we repeat essentially the estimates of $\|J_2\|_{\mathbb{P}}^2$ and obtain

$$\|L_2\|_{\mathbb{P}}^2 \le C\lambda$$

for $\lambda \in (0,1]$ and some constant $C > 0$.

The computation that $\|D_\lambda^{(i)}(0)\|_{\mathbb{P}}^2 \le C_i\lambda^{1/2}$ for $i = 2, 3$ (in fact both these quantities are of order $o(\lambda^{1/2})$) is quite routine taking into account the arguments contained in the proofs of Lemmas 2.1 and 2.4. This ends the proof of (1.7) and that of Theorem 1.1. □

3 Asymptotics of transition semigroup of the environment process.

Expansion (2.8) can be used to describe the asymptotics of the solution of the initial value problem

$$(\partial_t + L^\omega)\Phi(t,x;\omega) = 0 \qquad (3.1)$$
$$\Phi(0,x;\omega) = c(x;\omega),$$

as $t \to +\infty$, where $\{(a(x;\omega), c(x;\omega)),\, x \in \mathbb{Z}\}$ is a stationary field satisfying assumptions (1) and (2) from Section 1. In addition, we assume $\langle c(0)\rangle_{\mathbb{P}} = 0$.

We obtain, in the one dimensional situation, estimate of the rate of convergence in the stabilization problem. Namely, the following result holds.

Theorem 3.1 *Under the above assumptions there exists a constant $C > 0$ such that*

$$\frac{1}{T}\int_0^T \|\Phi(t,x)\|_{\mathbb{P}}^2\, dt \le \frac{C}{T^{1/2}}, \quad \forall\, x \in \mathbb{Z},\, T > 1. \qquad (3.2)$$

Remark. Property expressed in (3.2) is known as *the stabilization (in the mean) of solutions of the heat conduction equation*, see [14], and has been considered in various versions in a number of papers, see e.g. [15, 16, 3] and the references therein.

Proof of Theorem 3.1. The proof of this result shall be done in a number of steps.

Step 1: representation of $\Phi(t,x)$

Suppose that $\{Y_t^{x,\omega},\, t \ge 0\}$ is a random walk, starting at x and corresponding to the generator $-L^\omega$. We have

$$\Phi(t,x;\omega) = \mathbf{E}[c(Y_t^{x,\omega})] = c(x;\omega) - L_t(x;\omega), \qquad (3.3)$$

where

$$L_t(x;\omega) := \int_0^t \mathbf{E} L^\omega c(Y_s^{x,\omega}) ds.$$

Let

$$\varphi(t) := \int_0^t \|\Phi(s,0)\|_{\mathbb{P}}^2 ds.$$

Since

$$\|\Phi(t,x)\|_{\mathbb{P}}^2 = \|\Phi(t,0)\|_{\mathbb{P}}^2 = \|c(0)\|_{\mathbb{P}}^2 - 2\langle c(0), L_t(0)\rangle_{\mathbb{P}} + \|L_t(0)\|_{\mathbb{P}}^2$$

we obtain,

$$\hat{\varphi}(\lambda) := \int_0^{+\infty} e^{-\lambda t}\varphi(t) dt = \frac{1}{\lambda^2}\left[\|c(0)\|_{\mathbb{P}}^2 - 2\langle \phi_\lambda(0), c(0)\rangle_{\mathbb{P}} + \langle \phi_\lambda(0), \phi_{\lambda/2}(0)\rangle_{\mathbb{P}}\right], \quad \lambda > 0 \qquad (3.4)$$

with $\phi_\lambda(x)$ the solution of (1.1) corresponding to $b(x) := a(x)\partial c(x)$. Indeed, denote $F(t,x;\omega) := -\mathbf{E} L^\omega c(Y_t^{x,\omega};\omega)$ and $F(t) := F(t,0)$. Then,

$$\phi_\lambda(x;\omega) = \int_0^{+\infty} e^{-\lambda t} F(t,x;\omega) dt.$$

A direct application of the integration by parts formula gives

$$-2\int_0^{+\infty} e^{-\lambda t} dt \int_0^t \langle c(0), L_s(0)\rangle_{\mathbb{P}} ds = -\frac{2}{\lambda^2}\langle c(0), \phi_\lambda(0)\rangle_{\mathbb{P}} ds. \qquad (3.5)$$

For any $t > t' \geq 0$ we have

$$\langle F(t,x), F(t',x)\rangle_{\mathbb{P}} = \langle F(t), F(t')\rangle_{\mathbb{P}} = \langle F(t-t'), F(2t')\rangle_{\mathbb{P}}. \tag{3.6}$$

We prove this identity momentarily but first use it to verify (3.4). We have

$$\int_0^{+\infty} e^{-\lambda t}dt \int_0^t \|L_s(0)\|_{\mathbb{P}}^2 ds = 2\int_{[t\geq s\geq s_1\geq s_2\geq 0]} e^{-\lambda t}\langle F(s_2), F(s_1)\rangle_{\mathbb{P}} dt ds ds_1 ds_2$$
$$\overset{(3.6)}{=} \frac{2}{\lambda^2}\int_{[s_1\geq s_2\geq 0]} e^{-\lambda s_1}\langle F(s_1-s_2), F(2s_2)\rangle_{\mathbb{P}} ds_1 ds_2$$
$$= \frac{2}{\lambda^2}\int_{[s_2\geq 0]} e^{-\lambda s_2}\langle \phi_\lambda(0), F(2s_2)\rangle_{\mathbb{P}} ds_2 = \frac{1}{\lambda^2}\langle \phi_\lambda(0), \phi_{\lambda/2}(0)\rangle_{\mathbb{P}}$$

and the second equality in (3.4) follows.

The proof of (3.6)

To show (3.6) we use the notation $p^\omega(t,x,y)$ to denote transition probabilities corresponding to $Y_t^{x,\omega}$. The first equality follows easily from stationarity of the environment so we only need to use the second one. Because the generator $-L^\omega$ is in a divergence form and counting measure is invariant and reversible we have $p^\omega(t,x,y) = p^\omega(t,y,x)$ for all $x,y \in \mathbb{Z}$. The middle term in (3.6) equals

$$\int_{\mathbb{Z}^2} L^\omega c(y) L^\omega c(y') p^\omega(t,0,y) p^\omega(t',0,y') dy dy'$$
$$= \int_{\mathbb{Z}^2} L^\omega c(y) L^\omega c(y') p^\omega(t,0,y) p^\omega(t,0,z) p^\omega(t'-t,z,y') dy dy'$$
$$= \int_{\mathbb{Z}^3} L^\omega c(y) L^\omega c(y') p^\omega(t,0,y) p^\omega(t,0,z) p^\omega(t'-t,z,y') dy dy' dz.$$

Using stationarity of the environment we can rewrite the right hand side as being equal to

$$\int_{\mathbb{Z}^3} L^\omega c(y-z) L^\omega c(y'-z) p^\omega(t,-z,y-z) p^\omega(t,-z,0) p^\omega(t'-t,0,y'-z) dy dy' dz.$$

Changing variables $y := y - z$, $y' := y' - z$, $z := -z$ and using symmetry of $p^\omega(t,z,0)$ we obtain that the above expression equals

$$\int_{\mathbb{Z}^3} L^\omega c(y) L^\omega c(y') p^\omega(t,z,y) p^\omega(t,0,z) p^\omega(t'-t,0,y') dy dy' dz$$
$$= \int_{\mathbb{Z}^2} L^\omega c(y) L^\omega c(y') p^\omega(2t,0,y) p^\omega(t'-t,0,y') dy dy'.$$

and the last equality in (3.6) follows.

Step 2: estimates of the resolvent

We make use of computations made in Section 2.1 with $b(x) = a(x)\partial c(x)$. Notice that $B(x) = \partial c(x)$ and $\hat{b} = \langle B(0)\rangle_{\mathbb{P}} = 0$. We prove the following.

Proposition 3.2 *Under the above assumptions there exist $C_1, C_2 > 0$ such that*

$$\|\phi_\lambda(0) - \phi_\lambda^{(0)}(0)\|_{\mathbb{P}} \leq C_1\lambda^{1/2}, \tag{3.7}$$

and

$$\|c(0) + \phi_\lambda^{(0)}(0)\|_{\mathbb{P}} \leq C_2\lambda^{1/2}, \quad \lambda \in (0,1] \tag{3.8}$$

Proof. The argument is very similar to what has been done in Section 2.1. This time however we use the expansion (2.5) with $n = 6$. From Lemma 2.1 we can estimate $\|r_\lambda^{(6)}\|_{\mathbb{P}} \le C_r \lambda^{1/2}$. To estimate $\|\phi_\lambda^{(1)}(0)\|_{\mathbb{P}}$ we use representation (2.33). Because $\hat{b} = 0$ we get (recall that $B(x) = \partial c(x)$)

$$\begin{aligned}\phi_\lambda^{(1)}(0) &= -\frac{\lambda}{4}\int_{\mathbb{Z}^2} \partial^* G_{\lambda/(2\hat{a})}(x_1)\partial^* G_{\lambda/(2\hat{a})}(x_2 - x_1)\alpha(x_1)c(x_2)dx_1dx_2 \\ &= -\frac{\lambda}{4}\int_{\mathbb{Z}^2} g(x_1;\xi_1)g(x_2;\xi_1)q_{\xi_1}^{|x_1|}q_{\xi_1}^{|x_2-x_1|}\alpha(x_1)c(x_2)dx_1dx_2.\end{aligned} \tag{3.9}$$

Using mixing assumption in the same way as in the proof of Lemma 2.3 we conclude that

$$\|\phi_\lambda^{(1)}(0)\|_{\mathbb{P}} \le C_1\lambda^{1/2}, \quad \lambda \in (0,1]. \tag{3.10}$$

A slight modification of the proof of estimates of $\phi_\lambda^{(i)}$ for $i \ge 2$ is also possible due to the fact that $B(x)$ is a gradient of a zero mean field $c(x)$. In that case we can write

$$\begin{aligned}\phi_\lambda^{(i)}(0) &= -\frac{\lambda^i}{2^{i+1}}\int_{\mathbb{Z}^{i+1}} \partial^* G_{\lambda/(2\hat{a})}(x_1)\prod_{k=1}^{i-1} G_{\lambda/(2\hat{a})}(x_{k+1}-x_k)\partial^* G_{\lambda/(2\hat{a})}(x_{i+1}-x_i) \\ &\times \prod_{k=1}^{i}\alpha(x_k)c(x_{i+1})dx_1\dots dx_{i+1} \\ &= -\frac{\lambda^{(i+1)/2}}{2^{i+1}}\int_{\mathbb{Z}^{i+1}} g(x_1;\xi_1)g(x_{i+1};\xi_1)q_{\xi_1}^{|x_1|}\prod_{k=1}^{i}\left[q_{\xi_1}^{|x_{k+1}-x_k|}\alpha(x_k)\right]c(x_{i+1})dx_1\dots dx_{i+1}.\end{aligned} \tag{3.11}$$

Using the mixing lemma for $N = i + 1$ we arrive at the estimate

$$\|\phi_\lambda^{(i)}(0)\|_{\mathbb{P}} \le C_1\lambda^{i/2}, \quad \lambda \in (0,1]. \tag{3.12}$$

This, and expansion (2.33) implies (3.7). To show (3.8) observe, see (2.31), that

$$\begin{aligned}\phi_\lambda^{(0)}(x) &= -\frac{1}{2}\int_{\mathbb{Z}} \partial^* G_{\lambda/(2\hat{a})}(x-x_1)\partial c(x_1)dx_1 = -\frac{1}{2}\int_{\mathbb{Z}} \partial^*\partial G_{\lambda/(2\hat{a})}(x-x_1)c(x_1)dx_1 \\ &= \frac{\lambda}{2\hat{a}}\int_{\mathbb{Z}} G_{\lambda/(2\hat{a})}(x-x_1)c(x_1)dx_1 - c(x) = \frac{\lambda\xi_1}{2\hat{a}(1-\xi_1^2)^{1/2}}\int_{\mathbb{Z}} q_{\xi_1}^{|x-x_1|}c(x_1)dx_1 - c(x).\end{aligned}$$

Hence,

$$\|\phi_\lambda^{(0)}(0) + c(0)\|_{\mathbb{P}}^2 \le C\lambda\left\|\int_{\mathbb{Z}} q_{\xi_1}^{|x_1|}c(x_1)dx_1\right\|_{\mathbb{P}}^2.$$

The L^2 norm on the right hand side is of order of magnitude $\lambda^{-1/2}$, which can be seen analogously to the estimates of J_1 done previously, see (2.34) and following estimates. □

Step 3: the end of the proof of Theorem 3.1

Note also that, directly from the definition in (3.4), it follows that $\lambda^{-1}\varphi\left(\lambda^{-1}\right) \le \hat{\varphi}(\lambda)$ hence

$$\lambda\varphi\left(\lambda^{-1}\right) \le \lambda^2\hat{\varphi}(\lambda), \quad \forall\, \lambda \in (0,1]. \tag{3.13}$$

This in turn implies that, with $\lambda = T^{-1}$,

$$\frac{1}{T}\int_0^T \|\Phi(t,x)\|_{\mathbb{P}}^2 dt \le T^{-2}\hat{\varphi}(T^{-1}). \tag{3.14}$$

By virtue of (2.34) and Theorem 3.2 we conclude that the right hand side of (3.14) can be estimated by $CT^{-1/2}$, which implies (3.2). □

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