Bounds on the speed of propagation of the KPP fronts in a cellular flow

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December 8, 2004

Abstract

We consider a reaction-diffusion-advection equation with a nonlinearity of the KPP type in a cellular flow. We show that the minimal pulsating traveling front speed $c_*(A)$ in a flow of amplitude A satisfies the upper and lower bounds $C_1 A^{1/4} \le c_*(A) \le C_2 A^{1/4}$ for $A \gg 1$. We also analyze a related eigenvalue problem and establish an "averaging along the streamlines" principle for the positive eigenfunction when $A \gg 1$.

1 Introduction

Recently there has been a lot of mathematical studies of the effects of a fluid flow on the propagation of the reaction-diffusion fronts: see [4, 23] for recent reviews. One of the simplest models of this phenomenon is a single reaction-diffusion advection equation

$$T_t + u \cdot \nabla T = \Delta T + f(T). \tag{1.1}$$

Here T is the temperature of the reactant, $0 \le T \le 1$, and u is a prescribed fluid flow. The nonlinearity $f(T)$ is usually taken to be a Lipschitz function either of the KPP type, that is,

$$f(0) = f(1) = 0, \quad f(T) > 0 \text{ for } 0 < T < 1, \quad f(T) \le f'(0)T, \tag{1.2}$$

or of the ignition class:

$$f(T) = 0 \text{ for } T \in [0, \theta_0] \cup \{1\},\ f(T) > 0 \text{ for } \theta_0 < T < 1. \tag{1.3}$$

This problem is considered in a two-dimensional strip $D = \mathbb{R}_x \times [0, L]_y$ with the periodic

$$T(x, y) = T(x, y + L),$$

or Neumann

$$\frac{\partial T(x, y)}{\partial y} = 0, \quad y = 0, L$$

boundary conditions in y. All our results below are formulated and proved with the periodic boundary conditions. However, the modifications required for the Neumann boundary conditions are invariably very minor, and the results still apply to that case as well. The boundary conditions at infinity are front-like:

$$T \to 0 \text{ as } x \to -\infty,\ T \to 1 \text{ as } x \to +\infty.$$

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The reaction-diffusion fronts propagate from the right to the left end of the strip with these boundary conditions. It has been shown in [5] and [22] that, when the flow u is spatially periodic and the nonlinearity $f(T)$ is either of the KPP or the ignition type, equation (1.1) admits pulsating traveling front solutions. They are of the form $T(t,x,y) = U(x-ct,x,y)$ with a function $U(s,x,y)$ that is monotonically increasing in the first variable and periodic in the last two. The speed c of the pulsating traveling front is unique in the ignition case, while in the KPP case such solutions exists for all $c \geq c_*$ where c_* is the minimal speed that is not a priori known explicitly. It has been further shown in [22] in the ignition case and in [6, 21] in the KPP case that the general solutions of the Cauchy problem with front-like initial conditions travel asymptotically with the speed of the pulsating traveling front (the minimal speed c_* in the KPP case).

One of the most physically interesting aspects of this problem is the speed-up of the propagation of the reaction-diffusion fronts by a strong advection. Mathematically, one is interested in the dependence of the asymptotic speed of propagation of front-like solutions of (1.1) on the amplitude and geometry of the flow u. The aforementioned results of [6, 21, 22] show that when the flow is periodic this question is reduced to the estimates on the speed of the pulsating traveling fronts of

$$T_t + Au\cdot\nabla T = \Delta T + f(T) \tag{1.4}$$

as the function of the flow amplitude A and geometry of the streamlines of the fixed flow u. In particular, it has been shown in [8] that if the streamlines of u are open and connect the left and right ends of the strip, then $c_*(A) \sim O(A)$, both in the ignition and KPP cases. In the special case of a shear flow of the form $u = (u(y), 0)$ an elementary proof of this result has been presented in [13]. Finally, it has been shown in [4] that in the KPP case the limit

$$\bar{c} = \lim_{A\to\infty} \frac{c_*(A)}{A}$$

exists for a shear flow.

The situation has been less clear when the flow has closed streamlines. We consider a class of cellular flows of the form $u = \nabla^\perp H = (H_y, -H_x)$ with the stream-function $H(x,y)$ that has period $L = 2$ both in x and y. Moreover, we assume that the level set $\{H = 0\}$ contains the union of the lines $\{x = 2N\}$ and $\{y = 2N\}$, $N \in \mathbb{N}$ and that the points of the form $(2n, 2m)$ are non-degenerate saddles of the function H (as depicted on Figure 1.1). Moreover, we assume that other

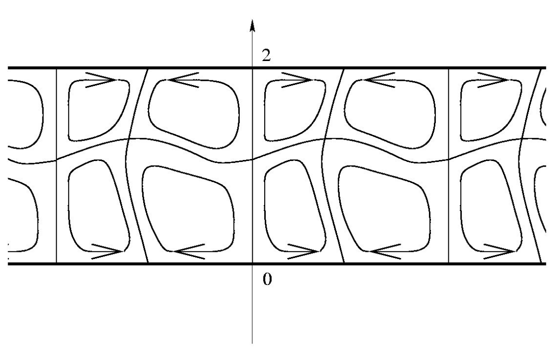

Figure 1.1: A sketch of the level sets of the stream-function $H(x,y)$.

critical points of the function H are also non-degenerate. A prototype example of such function is

$H(x, y) = \sin \pi x \sin \pi y$. It has been conjectured in [3] that

$$c_*(A) \sim A^{1/4} \tag{1.5}$$

both in the ignition and KPP cases. This prediction has been supported by a homogenization reasoning in [8] and [15], as well as by the physical arguments and numerical simulations in [1, 2, 20]. However, as far as the rigorous arguments are concerned, the only known estimates for the speed are the lower bound $c_*(A) \geq CA^{1/5}$ obtained in [16] and the upper bound $c_*(A) = o(A)$ shown in [4, 7]. They hold both for the ignition and KPP nonlinearities. The main result of this paper is the proof of (1.5) in the KPP case.

Theorem 1.1 *Let the nonlinearity $f(T)$ be of the KPP type (1.2) and the stream function $H(x, y)$ satisfy the above assumptions. Then there exist two constants $C_{1,2} > 0$ and $A_0 > 0$ so that the minimal pulsating traveling front speed $c_*(A)$ satisfies the upper and lower bounds*

$$C_1 A^{1/4} \leq c_*(A) \leq C_2 A^{1/4} \tag{1.6}$$

for all $A > A_0$.

As an immediate corollary of Theorem 1.1 and the maximum principle we deduce the upper bound $c(A) \leq CA^{1/4}$ for the unique speed of the pulsating traveling front in the ignition case. However, we do not prove the lower bound in the ignition case.

Theorem 1.1 is the main result of this paper. Its proof relies on the variational principle [5, 10] for the pulsating traveling front speed that reduces the problem to finding bounds for the principal eigenvalue of a certain advection-diffusion eigenvalue problem in a cellular flow. We also show that when the flow amplitude A is high the corresponding eigenfunction becomes approximately constant along the streamlines of the flow: see Theorem 4.1.

The paper is organized as follows. We first recall the variational principle for the minimal front speed and the relevant eigenvalue problem in Section 2. We also present some basic properties of the principle eigenvalue here. Most of the results in this section are not new but we adopt a slightly different point of view on the problem: the roles of the free parameter and the principle eigenvalue are reversed. The basic estimates on the principle eigenvalue that imply Theorem 1.1 in a straightforward fashion are presented in Theorem 3.1 in Section 3. Finally, Section 4 contains the proof of Theorem 4.1, "the oscillation along a streamline estimate" on the eigenfunction. The methods of [18] are used here. An additional twist in the present problem is that the eigenfunction satisfies only the maximum principle but not the minimum principle – this difficulty is circumvented by the introduction of its modification that obeys the minimum principle but not the maximum principle. A combination of the two functions allows us to control the oscillation over a streamline.

A word on notation: we denote by C all universal constants that do not depend on the flow amplitude A throughout the paper. The period cell is denoted $\mathcal{C} = [0, 2] \times [0, 2]$, while $\mathbf{x} = (x, y)$.

Acknowledgment. This work was started when both authors were visiting Stanford University. We thank G. Papanicolaou for his hospitality. The research was supported by NSF grant DMS-0203537 and ONR grant N00014-02-1-0089. LR was also supported by an Alfred P. Sloan Fellowship.

2 The variational principle for the speed

Let us recall the variational principle for the minimal front speed [5, 6, 7, 10]. It may be obtained in a quick formal way as follows. Consider the linearized KPP equation ahead of the front where T is small:

$$\frac{\partial T}{\partial t} + Au \cdot \nabla T = \Delta T + f'(0)T \tag{2.1}$$

and look for the pulsating traveling front solutions of the form $T(t, \mathbf{x}) = \phi(\mathbf{x}) \exp(\lambda(x + ct))$ with a positive periodic function $\phi(\mathbf{x})$. This leads to an eigenvalue problem for ϕ

$$\mathcal{M}(\lambda)\phi := \Delta\phi - Au \cdot \nabla\phi + 2\lambda\frac{\partial\phi}{\partial x} - \lambda A u_1 \phi = (c\lambda - \lambda^2 - f'(0))\phi. \tag{2.2}$$

The Krein-Rutman theorem implies that the operator $\mathcal{M}(\lambda)$ has a unique eigenvalue $h(\lambda)$ that corresponds to a positive periodic eigenfunction $\phi(\mathbf{x}; \lambda)$:

$$\Delta\phi - Au \cdot \nabla\phi + 2\lambda\frac{\partial\phi}{\partial x} - \lambda A u_1 \phi = h(\lambda)\phi, \quad \phi > 0 \text{ periodic.} \tag{2.3}$$

The front speed c then satisfies an equation

$$h(\lambda) = c\lambda - \lambda^2 - f'(0)$$

that may be re-written as

$$c = \frac{f'(0) + \lambda^2 + h(\lambda)}{\lambda}. \tag{2.4}$$

It has been shown in [5] that the minimal front speed for the full nonlinear KPP equation (1.4) is the same as that for the linearized equation (2.1). Moreover, it is given by infimum of the right side of (2.4).

Theorem 2.1 *[5] There exists a constant c_* so that a pulsating traveling front solution of (1.4) exists for all $c \geq c_*$. The minimal front speed is described by the variational principle*

$$c_* = \inf_{\lambda > 0} \frac{f'(0) + \lambda^2 + h(\lambda)}{\lambda}. \tag{2.5}$$

We note that the variational principle (2.5) as the characterization of the asymptotic propagation speed of the solutions of the Cauchy problem for (1.1) goes back to [10].

It is convenient to re-write the eigenvalue problem (2.3) in terms of the function $\psi(x, y) = \phi(x, y)e^{\lambda x}$. This function is not periodic but rather belongs to the set

$$E_\lambda^+ = \left\{\psi(x, y) \colon \phi(x, y) = \psi(x, y)e^{-\lambda x} \text{ is periodic and } \psi > 0\right\}.$$

The corresponding eigenvalue problem for $\psi(\mathbf{x})$ is

$$\mathcal{L}\psi := \Delta\psi - Au \cdot \nabla\psi = \mu(\lambda)\psi, \quad \psi > 0, \, \psi \in E_\lambda^+ \tag{2.6}$$

with $\mu(\lambda) = \lambda^2 + h(\lambda)$. Now, the variational principle (2.5) may be re-stated as

$$c_* = \inf_{\lambda > 0} \frac{f'(0) + \mu(\lambda)}{\lambda}, \tag{2.7}$$

where $\mu(\lambda)$ is the unique eigenvalue of (2.6). Expression (2.7) is the starting point of our analysis. Let us recall some basic properties of the function $\mu(\lambda)$.

Proposition 2.2 *The principal eigenvalue $\mu(\lambda)$ of the problem (2.6) is characterized as follows:*

$$\mu(\lambda) = \inf_{\phi \in E_\lambda^+} \sup_{\mathbf{x} \in \mathcal{C}} \frac{\mathcal{L}\phi}{\phi} = \sup_{\phi \in E_\lambda^+} \inf_{\mathbf{x} \in \mathcal{C}} \frac{\mathcal{L}\phi}{\phi}. \tag{2.8}$$

Here $\mathcal{C} = [0, 2] \times [0, 2]$ is the period cell and the operator $\mathcal{L}$ is defined by (2.6).

Proof. This proposition has also been proved in [5], we present the proof for the convenience of the reader. First, it is obvious that with ψ given by (2.6) we have

$$\inf_{\mathbf{x}\in\mathcal{C}} \frac{\mathcal{L}\psi}{\psi} = \sup_{\mathbf{x}\in\mathcal{C}} \frac{\mathcal{L}\psi}{\psi} = \mu. \tag{2.9}$$

Let us first assume that there exists a function $\eta \in E_\lambda^+$ so that

$$\inf_{\mathbf{x}\in\mathcal{C}} \frac{\mathcal{L}\eta}{\eta} = \mu + \delta, \quad \delta > 0.$$

Then we define $q_s(\mathbf{x}) = \psi(\mathbf{x}) - s\eta(\mathbf{x})$. The function $q_s(x,y) > 0$ for s sufficiently small while $q_s(x,y) < 0$ for s sufficiently large. Let τ be the first value of s so that $q_\tau(\mathbf{x}_0) = 0$ at some point $\mathbf{x}_0 \in \mathcal{C}$. We have $q_\tau(x,y) \geq 0$ – hence q_τ attains its minimum at $\mathbf{x}_0$. The same is true for the periodic function $w_\tau = e^{-\lambda x} q_\tau$. We recall that

$$\Delta\psi - Au\cdot\nabla\psi = \mu(\lambda)\psi$$

and

$$\Delta\eta - Au\cdot\nabla\eta > (\mu(\lambda)+\delta)\eta,$$

which implies that

$$\Delta q_\tau - Au\cdot\nabla q_\tau < \mu(\lambda)q_\tau - \delta\tau\eta.$$

Therefore, the function w_τ satisfies

$$\Delta w_\tau - Au\cdot\nabla w_\tau + 2\lambda\frac{\partial w_\tau}{\partial x} + \lambda^2 w_\tau - \lambda A u_1 w_\tau < \mu(\lambda)w_\tau - \delta\tau e^{-\lambda x}\eta. \tag{2.10}$$

However, w_τ is periodic and attains its minimum equal to zero at $\mathbf{x}_0$ so that both $w_\tau(\mathbf{x}_0) = 0$ and $\nabla w_\tau(\mathbf{x}_0) = 0$. Thus, it follows from (2.10) that at this point:

$$\Delta w(\mathbf{x}_0) < -\delta\tau e^{-\lambda x_0}\eta(\mathbf{x}_0) < 0.$$

This contradicts the fact that w attains its minimum at $\mathbf{x}_0$. Hence we have

$$\mu(\lambda) = \sup_{\phi\in E_\lambda^+} \inf_{x\in D} \frac{\mathcal{L}\phi}{\phi}.$$

The other equality in (2.8) is proved similarly. □

Proposition 2.3 *The function $\mu(\lambda)$ is convex.*

Proof. The proof is once again from [5]. We will show that

$$\mu(t\lambda_1 + (1-t)\lambda_2) \leq t\mu(\lambda_1) + (1-t)\mu(\lambda_2) \text{ for all } 0 \leq t \leq 1.$$

The min-max principle (2.8) implies that it suffices to show that given any pair of functions $f_1 \in E_{\lambda_1}^+$ and $f_2 \in E_{\lambda_2}^+$ there exists a function $\phi \in E_\lambda^+$, $\lambda = t\lambda_1 + (1-t)\lambda_2$, so that

$$\frac{\mathcal{L}\phi}{\phi} \leq t\frac{\mathcal{L}f_1}{f_1} + (1-t)\frac{\mathcal{L}f_2}{f_2}. \tag{2.11}$$

We claim that (2.11) holds with

$$\phi = f_1^t f_2^{1-t}. \tag{2.12}$$

Indeed, if $f_1 \in E^+_{\lambda_1}$, $f_2 \in E^+_{\lambda_2}$ it is straightforward to check that the function $e^{-\lambda x}\phi$ is periodic so that $\phi \in E^+_\lambda$. We verify that

$$\begin{aligned}\Delta\phi = \Delta(f_1^t f_2^{1-t}) &= t f_1^{t-1} f_2^{1-t}\Delta f_1 + (1-t) f_1^t f_2^{-t}\Delta f_2 - t(1-t) f_1^t f_2^{1-t}\left(\frac{|\nabla f_1|}{f_1} - \frac{|\nabla f_2|}{f_2}\right)^2 \\ &\le f_1^t f_2^{1-t}\left[t\frac{\Delta f_1}{f_1} + (1-t)\frac{\Delta f_2}{f_2}\right].\end{aligned}$$

Furthermore, using the above inequality and the function ϕ as in (2.12), we obtain

$$\begin{aligned}\frac{\mathcal{L}\phi}{\phi} &= \frac{1}{f_1^t f_2^{1-t}}\left[\Delta(f_1^t f_2^{1-t}) - Au\cdot\nabla(f_1^t f_2^{1-t})\right] \\ &\le t\frac{\Delta f_1}{f_1} + (1-t)\frac{\Delta f_2}{f_2} - A\left[t\frac{u\cdot\nabla f_1}{f_1} + (1-t)\frac{u\cdot\nabla f_2}{f_2}\right] = t\frac{\mathcal{L}f_1}{f_1} + (1-t)\frac{\mathcal{L}f_2}{f_2}.\end{aligned}$$

Thus (2.11) holds and therefore the function $\mu(\lambda)$ is convex. □

Proposition 2.4 *The function $\mu(\lambda) \ge 0$ is positive and monotonically increasing for $\lambda > 0$.*

Proof. Since $\mu(0) = 0$ and $\mu(\lambda)$ is a convex function, it suffices to check that $\mu'(0) = 0$. Let $w = e^{-\lambda x}\psi$ be the positive periodic function that satisfies

$$\Delta w - Au\cdot\nabla w + 2\lambda\frac{\partial w}{\partial x} - \lambda A u_1 w + \lambda^2 w = \mu(\lambda) w.$$

Integrating over the period cell $\mathcal{C}$ and using the incompressibility of the flow u we obtain

$$-\lambda A\int_{\mathcal{C}} u_1 w d\mathbf{x} + \lambda^2\int_{\mathcal{C}} w d\mathbf{x} = \mu(\lambda)\int_{\mathcal{C}} w d\mathbf{x}.$$

As $w(x;\lambda = 0) \equiv 1$, and u_1 has mean zero, it follows that

$$\mu'(0) = \lim_{\lambda\to 0}\frac{\mu(\lambda)}{\lambda} = -\frac{A}{|\mathcal{C}|}\int_{\mathcal{C}} u_1(\mathbf{x}) d\mathbf{x} = 0.$$

This finishes the proof of Proposition 2.4. □

Propositions 2.3 and 2.4 allow us to define the inverse function $\lambda = \lambda(\mu)$ that is increasing and concave. The eigenvalue problem (2.6) may re-formulated as follows: given $\mu \ge 0$ find the eigenvalue λ so that the problem

$$\Delta\psi - Au\cdot\nabla\psi = \mu\psi, \tag{2.13}$$

has a solution $\psi \in E^+_\lambda$. Existence and uniqueness of the eigenvalue $\lambda(\mu)$ follows from the previous arguments. We will adopt this point of view. The variational principle (2.7) for the minimal front speed now becomes

$$c_* = \inf_{\mu>0}\frac{f'(0)+\mu}{\lambda(\mu)}. \tag{2.14}$$

3 The eigenvalue enhancement estimate

The proof of Theorem 1.1 is based on the variational principle (2.14). Namely, we prove the following "eigenvalue enhancement" estimate.

Theorem 3.1 *There exist two constants $\mu_0 > 0$ and $A_0 > 0$ that are independent of the flow amplitude A and a pair of positive constants C_1 and C_2 so that*

$$\frac{C_1}{A^{1/4}}\sqrt{\mu} \le \lambda(\mu) \le \frac{C_2}{A^{1/4}}\sqrt{\mu} \tag{3.1}$$

for all $\mu < \mu_0$ and all $A > A_0$.

We first show that Theorem 1.1 follows immediately from Theorem 3.1.

Proof of Theorem 1.1. The variational principle (2.14) implies that

$$\min\left\{\inf_{0<\mu<\mu_0}\frac{f'(0)+\mu}{\lambda(\mu)}, \inf_{\mu_0\le\mu}\frac{\mu}{\lambda(\mu)}\right\} \le c_* \le \inf_{0<\mu<\mu_0}\frac{f'(0)+\mu}{\lambda(\mu)}. \tag{3.2}$$

Now, for the upper bound we have, using (3.1)

$$c_* \le \inf_{0<\mu<\mu_0}\frac{(f'(0)+\mu)A^{1/4}}{C_1\sqrt{\mu}} \le CA^{1/4}, \tag{3.3}$$

which is the upper bound in (1.6). In order to get the lower bound in (1.6) we observe that, first, it follows from (3.1) that

$$\inf_{0<\mu<\mu_0}\frac{f'(0)+\mu}{\lambda(\mu)} \ge \inf_{0<\mu<\mu_0}\frac{(f'(0)+\mu)A^{1/4}}{C_2\sqrt{\mu}} \ge C'A^{1/4}. \tag{3.4}$$

Second, as the function $\mu(\lambda)$ is concave and increasing,

$$\inf_{\mu_0\le\mu}\frac{\mu}{\lambda(\mu)} = \frac{\mu_0}{\lambda(\mu_0)} \ge \frac{A^{1/4}}{C_2}. \tag{3.5}$$

Using (3.4) and (3.5) in the left side of (3.2) we conclude that $c_* \ge CA^{1/4}$. This finishes the proof of Theorem 1.1. □

The proof of Theorem 3.1

Let us re-write the eigenvalue problem (2.13) in terms of the function $\zeta(\mathbf{x}) = \ln\psi(\mathbf{x})$. We obtain the following problem

$$\begin{aligned}
&\Delta\zeta - Au\cdot\nabla\zeta = \mu - |\nabla\zeta|^2, \quad (x,y)\in\mathcal{C} = [0,2]\times[0,2]\\
&\zeta(x+2,y) = \zeta(x,y) + 2\lambda(\mu),\\
&\zeta(x,y+2) = \zeta(x,y).
\end{aligned} \tag{3.6}$$

Note that $\nabla\zeta$ is a periodic function and $u\cdot n = 0$ on the boundary $\partial\mathcal{C}$ of the period cell. Therefore, integrating (3.6) over the period cell we obtain

$$\mu = \frac{1}{|\mathcal{C}|}\int_{\mathcal{C}}|\nabla\zeta|^2 dxdy. \tag{3.7}$$

Moreover, the log-eigenfunction ζ is defined up to an additive constant, and therefore can be chosen to be mean-zero. The Poincare inequality implies then

$$\|\zeta\|_{L^2(\mathcal{C})} \le C\sqrt{\mu}. \tag{3.8}$$

We decompose the function ζ into a linearly growing part and a periodic component as

$$\zeta = T_1 + S.$$

Here the function T_1 solves a homogeneous equation with the inhomogeneous boundary conditions

$$\Delta T_1 - Au \cdot \nabla T_1 = 0, \tag{3.9}$$
$$T_1(x+2,y) = T_1(x,y) + 2\lambda, \quad T_1(x,y+2) = T_1(x,y), \quad \int_{\mathcal{C}} T_1(x,y)dxdy = 0.$$

The mean-zero periodic function S solves

$$\Delta S - Au \cdot \nabla S = \mu - |\nabla(T_1 + S)|^2. \tag{3.10}$$

We first look at (3.9). This very problem arises as the cell problem in the computation of the effective diffusivity for the cellular flows [9, 14, 17, 18, 19]. We recall that the function T_1 has the form $T_1(x,y) = \lambda(x - 1 + \chi(x,y))$ with a mean-zero function χ that is periodic and satisfies the bounds [17]

$$C_1\sqrt{A} \leq \int_{\mathcal{C}} |\nabla\chi|^2 dxdy \leq C_2\sqrt{A}, \quad A > A_0. \tag{3.11}$$

It follows that

$$C_1\lambda^2\sqrt{A} \leq \int_{\mathcal{C}} |\nabla T_1|^2 dxdy \leq C_2\lambda^2\sqrt{A} \tag{3.12}$$

for $A > A_0$. Therefore, the Poincaré inequality implies that

$$\|T_1\|_{L^2(\mathcal{C})} \leq C\lambda A^{1/4}, \quad A > A_0. \tag{3.13}$$

Moreover, we will show that the function T_1 satisfies the following uniform upper bound.

Lemma 3.2 *There exist constants $C > 0$ and $A_0 > 0$ so that*

$$\|T_1\|_{L^\infty(\mathcal{C})} \leq C\lambda A^{1/4} \tag{3.14}$$

for all $A > A_0$

We postpone the proof of this lemma for the moment. Note, however, that when the stream-function $H(x,y)$ is symmetric

$$H(-x,y) = -H(x,y), \tag{3.15}$$

then the function T_1 satisfies a trivial estimate $-\lambda \leq T_1(x,y) \leq \lambda$: see [9].

Lemma 3.3 *There exists a constant $B_0 > 0$ so that*

$$\lambda(\mu) \leq B_0\sqrt{\mu}. \tag{3.16}$$

Proof. This inequality follows immediately from (3.7):

$$\mu = \int_{\mathcal{C}} |\nabla\zeta|^2 \frac{dxdy}{|\mathcal{C}|} \geq \int_{\mathcal{C}} |\zeta_x|^2 \frac{dxdy}{|\mathcal{C}|} \geq \left(\int_{\mathcal{C}} \zeta_x \frac{dxdy}{|\mathcal{C}|}\right)^2 = \frac{\lambda^2}{B_0^2} \tag{3.17}$$

with $B_0 = |\mathcal{C}|/2$. $\square$

A key ingredient in the proof of Theorem 3.1 is the following lemma.

Lemma 3.4 *There exist constants $C > 0$ and $A_0 > 0$ so that*

$$\zeta(x, y) \leq C\sqrt{\mu} \tag{3.18}$$

for all $(x, y) \in \mathcal{C}$ and $A > A_0$

The proof of Lemma 3.4 is also postponed for the moment. We first finish the proof of Theorem 3.1. It follows from (3.14) and (3.18) that

$$S(x, y) \leq C\left[\lambda A^{1/4} + \sqrt{\mu}\right]. \tag{3.19}$$

Furthermore, observe that it follows from (3.7) that

$$\mu = \int_{\mathcal{C}} |\nabla\zeta|^2 \frac{dxdy}{|\mathcal{C}|} = \int_{\mathcal{C}} |\nabla(T_1 + S)|^2 \frac{dxdy}{|\mathcal{C}|}.$$

Now, the triangle inequality implies that

$$|\sqrt{\mu} - \|\nabla T_1\|_{L^2}| \leq \|\nabla S\|_{L^2}. \tag{3.20}$$

Next, we multiply equation (3.10) by the periodic function S and integrate over the period $\mathcal{C}$. Using the fact that S has mean zero and (3.19), we obtain

$$\|\nabla S\|_{L_2}^2 = \int_{\mathcal{C}} S|\nabla(T_1 + S)|^2 dxdy \leq C\left[\lambda A^{1/4} + \sqrt{\mu}\right] \int_{\mathcal{C}} |\nabla\zeta|^2 dxdy \leq C\left[\lambda A^{1/4}\mu + \mu\sqrt{\mu}\right]. \tag{3.21}$$

Finally, (3.20) and the corrector bound (3.12) imply that we have

$$\sqrt{\mu} \leq C\left[\lambda A^{1/4} + \sqrt{\mu}\sqrt{\lambda}A^{1/8} + \mu^{3/4}\right] \leq C\left[\lambda A^{1/4} + \mu + \mu^{3/4}\right]$$

and

$$\sqrt{\mu} \geq C^{-1}\left[\lambda A^{1/4} - \sqrt{\mu}\sqrt{\lambda}A^{1/8} - \mu^{3/4}\right] \geq C^{-1}\left[\lambda A^{1/4} - \mu - \mu^{3/4}\right].$$

Hence, (3.1) follows for $\mu < \mu_0$ with μ_0 independent of A. This finishes the proof of Theorem 3.1. □

The proof of Lemmas 3.2 and 3.4

Both Lemma 3.2 and Lemma 3.4 follow from the following Proposition.

Proposition 3.5 *Let a mean-zero function $q(x, y)$ be periodic in y and satisfy*

$$\begin{aligned}
&\Delta q - Au \cdot \nabla q + |\nabla q|^2 \geq 0, \quad (x, y) \in \mathbb{R}^2, \\
&q(x + 2, y) = q(x, y) + 2\alpha, \quad q(x, y + 2) = q(x, y), \quad \int_{\mathcal{C}} q(x, y)dxdy = 0,
\end{aligned} \tag{3.22}$$

with a fixed number $\alpha > 0$. Then there exists a constant $C > 0$ so that

$$q(x, y) \leq C[\alpha + \|\nabla q\|_{L^2(\mathcal{C})}] \tag{3.23}$$

for all $(x, y) \in \mathcal{C} = [0, 2] \times [0, 2]$.

Proof. Let $\mathcal{C} = [0, 2] \times [0, 2]$ be the period cell. The function q satisfies the maximum principle and thus has to attain its maximum over $\mathcal{C}$ at the boundary $\partial\mathcal{C}$. Moreover, the periodicity of $q(x, y)$ in y implies that the maximum of the function q over the period cell is achieved at one of the vertical boundaries $x = 0$ or $x = 2$. However, the boundary condition

$$q(x + 2, y) = q(x, y) + 2\alpha$$

implies that the maximum is actually achieved along the line $x = 2$, say, at a point $\mathbf{x}_1 = (2, y_0)$. We denote $q(\mathbf{x}_1) = M := \sup_{\mathbf{x}\in\mathcal{C}} q(\mathbf{x})$. Consider the corresponding point $\mathbf{x}_0 = (0, y_0)$ on the left boundary of the cell, where $q(\mathbf{x}_0) = M - 2\alpha$. The function q attains its maximum over the previous period cell $\mathcal{C}_{-1} = [-2, 0] \times [0, 2]$ at this point. Therefore, according to the strong maximum principle, the vector $\nabla q(\mathbf{x}_0)$ points out of $\mathcal{C}_{-1}$ and into the cell $\mathcal{C}$. It follows that given any $\xi \in [0, 2]$ there exists $s(\xi) \in [0, 2]$ so that $q(\xi, s(\xi)) \geq M - 2\alpha$. Indeed, consider the function q in the rectangle $D(\xi) = [0, \xi] \times [0, 2]$. It has to attain its maximum over $D(\xi)$ on the boundary $\partial D(\xi)$. However, once again, the y-periodicity of q and the strong maximum principle imply that it cannot do so at the horizontal boundaries $\{y = 0\}$ or $\{y = 2\}$. Moreover, as ∇q points inside $D(\xi)$ at the point $\mathbf{x}_0$, where $q(x, y)$ attains its maximum over $\{x = 0\}$, it has to attain its maximum over $D(\xi)$ along the line $\{x = \xi\}$. In particular, the value of this maximum has to be larger than $M - 2\alpha$, which is its maximum over the line $\{x = 0\}$. We conclude that

$$\text{for any } \xi \in [0, 2] \text{ there exists } s(\xi) \in [0, 2] \text{ such that } q(\xi, s(\xi)) \geq \sup_{\mathbf{x}\in\mathcal{C}} q(\mathbf{x}) - 2\alpha. \tag{3.24}$$

On the other hand, observe that there exists a constant $B_1 > 0$ and a constant $B_2 > 0$ so that if we define the set $\mathcal{A} \subset [0, 2]$ as

$$\mathcal{A} = \left\{x \in [0, 2] : \ \exists r(x) \in [0, 2] \text{ such that } |q(x, r(x))| \leq B_1 \|q\|_{L^2(\mathcal{C})}\right\} \tag{3.25}$$

then

$$|\mathcal{A}| \geq B_2. \tag{3.26}$$

Now, (3.24) and (3.25)-(3.26) imply that

$$\int_0^2 \left|\frac{\partial q}{\partial y}\right|^2 dy \geq \frac{1}{2}\left(M - 2\alpha - B_1\|q\|_{L^2(\mathcal{C})}\right)^2$$

for all $x \in \mathcal{A}$. Integrating over $x \in \mathcal{A}$ we obtain

$$\int_{\mathcal{C}} |\nabla q|^2 \frac{dxdy}{|\mathcal{C}|} \geq \int_{\mathcal{A}} \left|\frac{\partial q}{\partial y}\right|^2 \frac{dxdy}{|\mathcal{C}|} \geq C B_2 \left(M - 2\alpha - B_1\|q\|_{L^2(\mathcal{C})}\right)^2.$$

It follows from the above and the Poincaré inequality that

$$M = \sup_{\mathbf{x}\in\mathcal{C}} q(\mathbf{x}) \leq C\left[\alpha + \|q\|_{L^2(\mathcal{C})} + \|\nabla q\|_{L^2(\mathcal{C})}\right] \leq C\left[\alpha + \|\nabla q\|_{L^2(\mathcal{C})}\right].$$

This completes the proof of Proposition 3.5. □

The proof of Lemma 3.2. Equation (3.9) implies that we may apply Proposition 3.5 both to the function $T_1(x, y)$ and $T_1'(x, y) = -T_1(-x, y)$ (the latter requires also reflecting the flow $u(x, y)$ but that is not important) with $\alpha = \lambda$. Then (3.14) follows from the gradient bound (3.12). □

The proof of Lemma 3.4. The proof of this lemma is another straightforward application of Proposition 3.5. Indeed, equations (3.6) and (3.23) imply that (3.18) follows immediately from (3.7) and (3.16). □

4 An oscillation bound on the log-eigenfunction

In this section we show that the log-eigenfunction $\zeta(\mathbf{x})$ (and hence the eigenfunction $\psi(\mathbf{x})$ itself) becomes nearly constant along each streamline of the flow away from the flow separatrices. This phenomenon is familiar from other problems that involve high amplitude cellular flows [11, 12, 17, 18, 19], where solutions of advection-diffusion problems become uniform over the flow streamlines. In this section we take a particular stream-function $H(x,y) = \sin \pi x \sin \pi y$ – this is not crucial for the proof but the fact that H is an eigenfunction of the Laplacian does simplify some of the arguments.

Theorem 4.1 *Let $\mathcal{L}(h) = \{H(\mathbf{x}) = h\}$ be a level set of the function $H(\mathbf{x})$ with $h \geq h_0 > 0$ and let $M(h)$ and $m(h)$ be the maximum and minimum of the function $\zeta(\mathbf{x})$ over $\mathcal{L}(h)$. Then there exist a constant $C > 0$ that depends on h_0 but not the flow amplitude A, and constants μ_0, A_0 so that if $0 < \mu < \mu_0$ and $A \geq A_0$, then*

$$M(h) - m(h) \leq C\left[\frac{\sqrt{\mu}}{A^{1/4}} + \frac{\mu}{\sqrt{A}}\right]. \tag{4.1}$$

Proof. We prove that ζ is almost constant along a level-set $H = h \geq h_0 > 0$, by using the techniques from [18]. The proof is by contradiction. We assume that

$$M(h) - m(h) \geq N\left[\frac{\sqrt{\mu}}{A^{1/4}} + \frac{\mu}{\sqrt{A}}\right] \tag{4.2}$$

and show that if the constant N is sufficiently large then we arrive at a contradiction.

The function ζ satisfies the maximum principle: given a domain Ω we have

$$\zeta(\mathbf{x}) \leq \max \zeta|_{\partial\Omega} \text{ for all } \mathbf{x} \in \Omega. \tag{4.3}$$

However, it might not satisfy the minimum principle. In order to overcome this issue we construct a new function ζ_0 that is close to ζ and satisfies the minimum principle. Here is the construction. Consider the flow cell $[0,1] \times [0,1]$, where $H > 0$ and define the function $\zeta_0 = \zeta + \eta g$, with $g = \ln H$. Here $H = \sin \pi x \sin \pi y$ is the stream-function for the cellular flow, and the constant η is to be determined. The function ζ_0 is defined away from the separatrices where $H = 0$. A direct computation shows that ζ_0 satisfies

$$\Delta\zeta_0 - Au \cdot \nabla\zeta_0 + 2\eta\nabla g \cdot \nabla\zeta_0 = -|\nabla\zeta_0|^2 - \eta^2|\nabla g|^2 + \mu + \eta\Delta g.$$

However, we have

$$\Delta g = \frac{\Delta H}{H} - \frac{|\nabla H|^2}{H^2} = -2\pi^2 - \frac{|\nabla H|^2}{H^2}$$

so that with the choice $\eta = \mu/(2\pi^2)$ we obtain

$$\Delta\zeta_0 - Au \cdot \nabla\zeta_0 + 2\mu\nabla g \cdot \nabla\zeta_0 = -|\nabla\zeta_0|^2 - \left(\frac{\mu^2}{4\pi^4} + \frac{\mu}{2\pi^2}\right)\frac{|\nabla H|^2}{H^2} \leq 0.$$

Therefore, the function ζ_0 satisfies the minimum principle – it may attain its minimum over a domain only on the boundary of the domain. Hence, we have inside a level set $\{H = h_0\}$

$$\zeta(x,y) + \frac{\mu}{2\pi^2}g(x,y) \geq \min \zeta|_{\partial\Omega_0} + \frac{\mu}{2\pi^2}g(h_0), \quad (x,y) \in \Omega_0,$$

where Ω_0 is the interior of that level set. Another way to write the minimum principle is

$$\zeta(x,y) \geq \min \zeta|_{\partial\Omega_0} + \frac{\mu}{2\pi^2}\ln\frac{h_0}{H(x,y)}, \quad (x,y) \in \Omega_0. \tag{4.4}$$

Let $\mathcal{L}(\alpha) = \{H = \alpha\}$ and $\mathcal{L}(\beta) = \{H = \beta\}$ with $\beta \geq \alpha$ be two level sets of the stream-function H, such that $\mathcal{L}(\beta)$ is contained in the interior of $\mathcal{L}(\alpha)$. We denote by $\mathcal{D}(h) = \{H \geq h\}$ the region bounded by $\mathcal{L}(h)$. Then (4.3) and (4.4) imply an oscillation inequality

$$M(\beta) - m(\beta) \leq M(\alpha) - m(\alpha) + \frac{\mu}{2\pi^2} \ln \frac{\beta}{\alpha}. \tag{4.5}$$

In order to show that the oscillation far away from the separatrices is small we start with a level set $\mathcal{L}(h) = \{H = h\}$ and consider a pair of gradient curves (see Figure 4.1)

$$\frac{d\gamma_m}{dt} = -\nabla\zeta_0(\gamma_m(t)), \quad \gamma_m(0) = \mathbf{x}_m(h), \tag{4.6}$$

and

$$\frac{d\gamma_M}{dt} = \nabla\zeta(\gamma_M(t)), \quad \gamma_M(0) = \mathbf{x}_M(h). \tag{4.7}$$

Here $\mathbf{x}_m$ and $\mathbf{x}_M$ are the points on the level set $\mathcal{L}(h)$ where the function ζ_0 attains its minimum and maximum, respectively. As the difference of the functions ζ_0 and ζ is constant on streamlines, these points are also the minimum and maximum of the function ζ_0 along $\mathcal{L}(h)$. Note that the gradient curve that exits from the maximum is determined by the function ζ, that satisfies the maximum principle, while the curve exiting from the minimum is determined by the function ζ_0, that obeys the minimum principle. As a consequence of the maximum and the minimum principles both of the curves $\gamma_m(t)$ and $\gamma_M(t)$ start in the direction of the outward normal to the region $\mathcal{D}(h)$.

In principle, one of these two curves may end at a critical point of the corresponding function, ζ or ζ_0, respectively. In order to avoid this situation we surround those critical points $\mathbf{x}_j$, $j = 1, \ldots, N_A$ by small circles $U_j = U(\mathbf{x}_j; r)$ of the radius $r \ll 1$ so small that the oscillations of the functions ζ and ζ_0 over each circle satisfy

$$\mathrm{osc}_{U_j}\zeta + \mathrm{osc}_{U_j}\zeta_0 \leq \frac{\delta(M(h) - m(h))}{1 + N_A}, \quad j = 1, \ldots, N_A \tag{4.8}$$

with a sufficiently small number $\delta > 0$. Then, if one of the gradient curves, say γ_M, hits one of the circles U_j, we continue it along U_j, with the speed equal to one, to the point $\mathbf{x}_j^M$, where the function ζ_M attains its maximum over U_j. The maximum principle implies that $\nabla\zeta$ points outward of U_j at this point, hence we may continue γ_M as a gradient curve that goes out of $\mathbf{x}_j^M$. Observe also that γ_m and γ_M may not hit the same circle U_j twice – the sequences of values of ζ_0 and ζ at the departure times from each consecutive circle are strictly decreasing and increasing, respectively.

Note that, as a consequence of their construction, $|\nabla\zeta|$ and $|\nabla\zeta_0|$ are bounded away from zero along γ_m and γ_M, hence, for instance,

$$\zeta(\gamma_M(t)) \geq \zeta(\mathbf{x}_M(h)) + C_A t - \frac{\delta(M(h) - m(h))}{1 + N_A}, \tag{4.9}$$

where $C_A > 0$. The last term above accounts for the fact that at time t the curve may be at one of the circles U_j. The lower bound (4.9) and the continuity of the function ζ imply that t is bounded above, so that given any level set $\mathcal{L}(h')$ that encloses $\mathcal{L}(h)$ both curves γ_m and γ_M have to intersect eventually $\mathcal{L}(h')$.

Next, we make sure that γ_m and γ_M do not intersect each other before they intersect a given level set $\mathcal{L}(h')$. Note that

$$\zeta(\gamma_M(t)) \geq M(h) - \frac{\delta(M(h) - m(h))}{1 + N_A}$$

for all $t \geq 0$, while

$$\zeta_0(\gamma_m(s)) \leq \zeta_0(\mathbf{x}_m) + \frac{\delta(M(h) - m(h))}{1 + N_A}.$$

The latter inequality implies that

$$\begin{aligned}\zeta(\gamma_m(s)) &\leq \zeta(\mathbf{x}_m) + \frac{\delta(M(h) - m(h))}{1 + N_A} + \frac{\mu}{2\pi^2} \ln\left[\frac{h}{H(\gamma_m(s))}\right] \\ &= m(h) + \frac{\mu}{2\pi^2} \ln\left[\frac{h}{H(\gamma_m(s))}\right] + \frac{\delta(M(h) - m(h))}{1 + N_A}.\end{aligned}$$

Therefore, if we take $\delta > 0$ sufficiently small, the curves γ_m and γ_M may not intersect, as long as they stay inside a level set $\mathcal{L}(h')$, with

$$h' = h - \frac{1}{\sqrt{A}},$$

provided that

$$M(h) - m(h) \geq C\mu \ln \frac{h}{h'} \geq \frac{C\mu}{\sqrt{A}}, \tag{4.10}$$

with a sufficiently large constant C independent of A. Note that (4.10) follows from (4.2) when the constant N is sufficiently large. Thus, under the assumption (4.2) with a sufficiently large N, the curves γ_m and γ_M do not intersect in the region $\mathcal{D}(h, h') = \{\mathbf{x} : \ h' < H(\mathbf{x}) < h\}$ between $\mathcal{L}(h)$ and $\mathcal{L}(h')$.

Observe that if (4.2) holds then not only the curves γ_m and γ_M do not intersect in $\mathcal{D}(h, h')$ but also

$$\zeta(\gamma_M(t)) - \zeta(\gamma_m(s)) \geq C(M(h) - m(h)) \tag{4.11}$$

for all t and s such that $\gamma_M(t)$ and $\gamma_m(s)$ are in $\mathcal{D}(h, h')$. As a consequence, neither of the curves γ_m or γ_M may wrap around the level set $\mathcal{L}(h)$. Indeed, if one of them did, the fact that they do not intersect would imply that the other curve would do the same. Then, as the width of $\mathcal{D}(h, h')$ is of the order $C/\sqrt{A}$, (4.2) and (4.11) would imply a lower bound

$$\mu = \int |\nabla\zeta|^2 \frac{dxdy}{|\mathcal{C}|} \geq C(M(h) - m(h))^2\sqrt{A} \geq CN^2\mu$$

which is impossible if N is sufficiently large. We conclude that it follows from (4.2) that neither γ_m nor γ_M wrap around the level set $\mathcal{L}(h)$.

We now consider two cases: whether one of γ_m or γ_M intersects a circle U_j before intersecting $\mathcal{L}(h')$ or not.

Case 1. Neither γ_m nor γ_M intersect any of U_j.

Let $\mathcal{R}$ be a domain bounded by the two gradient curves γ_M, γ_m, and parts of the level sets $\gamma = \mathcal{L}(h)$ and $\gamma' = \mathcal{L}(h')$ (see Figure 4.1). We integrate (3.6) over $\mathcal{R}$ and obtain

$$\int_{\gamma\cup\gamma'} n \cdot \nabla\zeta ds - \frac{\mu}{2\pi^2}\int_{\gamma_m} \frac{n \cdot \nabla H}{H} ds - A\int_{\gamma_m\cup\gamma_M} \zeta(n \cdot u)ds = \mu|\mathcal{R}| - \int_{\mathcal{R}} |\nabla\zeta|^2 dx. \tag{4.12}$$

For the last integral above we have a simple bound

$$\int_{\mathcal{R}} |\nabla\zeta|^2 dx \leq \mu. \tag{4.13}$$

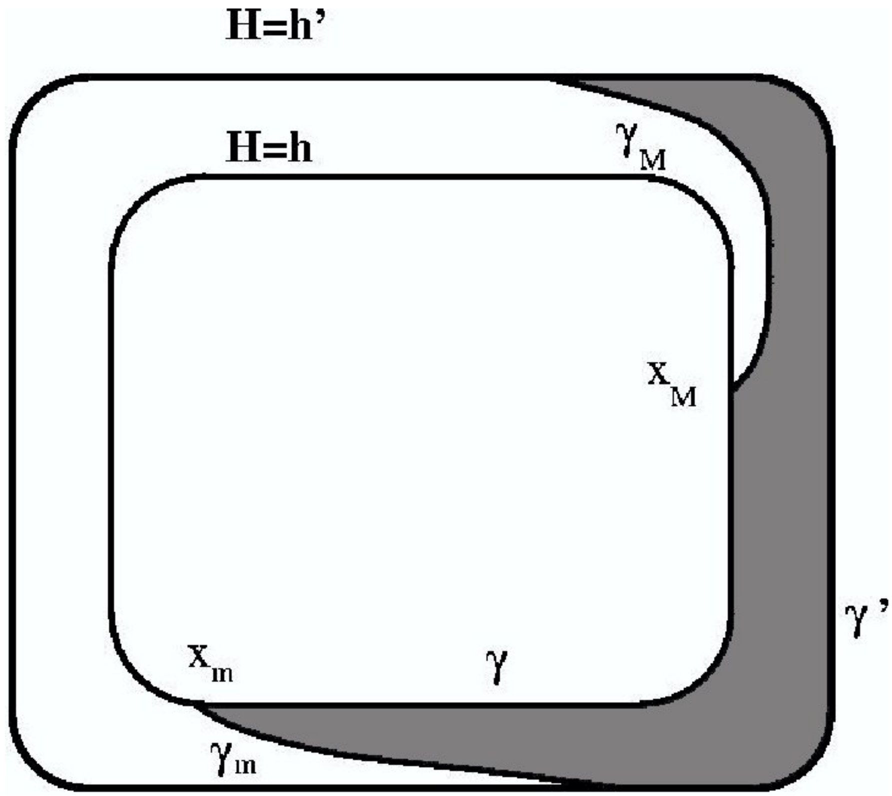

Figure 4.1: The domain $\mathcal{R}$ is shaded.

As the curve γ_m does not wrap around the level set $\mathcal{L}(h)$, the second term on the left-hand side may be bounded, using the fact that $h \geq h_0$, as follows:

$$\left|\int_{\gamma_m} \frac{n \cdot \nabla H}{H} ds\right| = \left|\int_{\gamma_m'} \frac{n \cdot \nabla H}{H} ds\right| + \left|\int_{\Omega_m} \left(2\pi^2 + \left|\frac{\nabla H}{H}\right|^2\right)\right| \leq C. \tag{4.14}$$

Here γ_m' is any other path connecting the endpoints of γ_m (that we may take to have the length of order one), while Ω_m is the domain bounded by γ_m and γ_m'.

We also recall the following fact (see (3.7) in [18]). Let $\gamma: [0,1] \to \mathcal{D}(\alpha, \beta)$ be any smooth curve of finite length that connects the level sets $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$: $\gamma(0) \in \mathcal{L}(\alpha)$, $\gamma(1) \in \mathcal{L}(\beta)$. Fix the normal n to γ so that $u \cdot n > 0$ for t sufficiently small, then $u \cdot n(\tau(\xi)) > 0$ for all ξ between α and β, with $\tau(\xi) = \sup\{t: \gamma(t) \in \mathcal{L}(\xi)\}$. That is, "$u \cdot n$ is positive when a streamline of u intersects γ for the last time." Let $f(\mathbf{x}) \geq 0$ be a continuous function monotonically increasing along γ. Then we have

$$F(\alpha, \beta) \inf_{\mathbf{x} \in \gamma} f \leq \int_\gamma (u \cdot n) f ds \leq F(\alpha, \beta) \sup_{\mathbf{x} \in \gamma} f, \tag{4.15}$$

where $F(\alpha, \beta)$ is the flux

$$F(\alpha, \beta) = \int_{\gamma_o} n \cdot u ds.$$

Here γ_o is any curve connecting the two level sets $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$.

We apply (4.15) to estimate the last term on the left side in (4.12). We have

$$F(h, h')M(h) \leq \int_{\gamma_M} \zeta(u \cdot n) ds \tag{4.16}$$

and

$$\int_{\gamma_m} \zeta_0(u\cdot n)ds \le F(h,h')\left[m(h)+\frac{\mu}{2\pi^2}\ln h\right].$$

The latter inequality implies that

$$\begin{aligned}\int_{\gamma_m} \zeta(u\cdot n)ds &\le F(h,h')\left[m(h)+\frac{\mu}{2\pi^2}\ln h\right]-\frac{\mu}{2\pi^2}\int_{\gamma_m}(u\cdot n)\ln(H(\gamma_m(s))ds\\ &= m(h)F(h,h')+\frac{\mu}{2\pi^2}\int_{\gamma_m}(u\cdot n)\ln\left(\frac{h}{H(\gamma_m(s))}\right)ds.\end{aligned}\tag{4.17}$$

As a consequence of the balance (4.12), the drop estimate (4.11) and the bounds (4.13), (4.14), the inequalities (4.16) and (4.17) imply that

$$\begin{aligned}(M(h)-m(h))\,F(h,h') &\le \frac{\mu}{A}(|\mathcal{R}|+1)+\frac{1}{A}\int\limits_{\gamma\cup\gamma'}|n\cdot\nabla\zeta|ds+\frac{C\mu}{A}\\ &+\frac{\mu}{2\pi^2}\int_{\gamma_o}(u\cdot n)\ln\left[\frac{h}{H(\gamma_o(s))}\right]ds.\end{aligned}\tag{4.18}$$

Here γ_o is any curve connecting the two level sets, and $F(h,h')$ is the flux between the level sets $\mathcal{L}(h)$ and $\mathcal{L}(h')$. We used above the fact that the integral

$$\int_\gamma f(H(\gamma(s))(u\cdot n)ds$$

is independent of the choice of the curve γ connecting two level sets.

Note that assumption (4.2) implies that

$$M(h)-m(h)\ge 10\mu\ln\frac{h}{h''}\tag{4.19}$$

for all $h'=h-\frac{1}{\sqrt{A}}<h''<h$, provided that N is sufficiently large. Therefore, in that case the left side in (4.18) dominates the last integral on the right and it follows that

$$(M(h)-m(h))\,F(h,h')\le\frac{C\mu}{A}+\frac{C}{A}\int_{\gamma\cup\gamma'}|n\cdot\nabla\zeta|ds.\tag{4.20}$$

We now integrate on a sub-strip strictly inside $\mathcal{D}(h,h')$. We use inequality (4.20) for a pair of level sets $\mathcal{L}((h+h')/2+H)$ and $\mathcal{L}(h'+H)$ with $0\le H\le\dfrac{h-h'}{2}=\dfrac{1}{2\sqrt{A}}$ to obtain

$$\begin{aligned}&\left(M\left(\frac{h+h'}{2}+H\right)-m\left(\frac{h+h'}{2}+H\right)\right)F\left(\frac{h+h'}{2}+H,h'+H\right)\\ &\le\frac{C}{A}\int\limits_{\mathcal{L}(\frac{h+h'}{2}+H)}\left|\frac{\partial\zeta}{\partial n}\right|ds+\frac{C}{A}\int\limits_{\mathcal{L}(h'+H)}\left|\frac{\partial\zeta}{\partial n}\right|ds+\frac{C\mu}{A}.\end{aligned}\tag{4.21}$$

However, we have

$$M(h)-m(h)\le M\left(\frac{h+h'}{2}+H\right)-m\left(\frac{h+h'}{2}+H\right)+\frac{\mu}{2\pi^2}\ln\frac{h}{\frac{h+h'}{2}+H}.$$

according to the oscillation inequality (4.5). Recalling (4.19) we obtain

$$M(h) - m(h) \leq C\left(M\left(\frac{h+h'}{2} + H\right) - m\left(\frac{h+h'}{2} + H\right)\right).$$

Therefore, we get from (4.21)

$$(M(h) - m(h))\, F\left(\frac{h+h'}{2} + H, \beta + H\right) \leq \frac{C}{A} \int\limits_{\mathcal{L}(\frac{h+h'}{2}+H)} \left|\frac{\partial\zeta}{\partial n}\right| ds + \frac{C}{A} \int\limits_{\mathcal{L}(h'+H)} \left|\frac{\partial\zeta}{\partial n}\right| ds + \frac{C\mu}{A}. \quad (4.22)$$

We integrate (4.22) with respect to $H \in (0, 1/(2\sqrt{A}))$ to obtain

$$(M(h) - m(h)) \int_0^{1/(2\sqrt{A})} F\left(\frac{h+h'}{2} + H, h' + H\right) dH \leq \frac{C}{A} \int_{h'}^{h} \int\limits_{\mathcal{L}(H)} \left|\frac{\partial\zeta}{\partial n}\right| ds\; dH + \frac{C\mu}{A^{3/2}}. \quad (4.23)$$

Furthermore, as $h > h_0$, the first integral on the right side of (4.23) may be estimated as

$$\int_{h'}^{h} \int\limits_{\mathcal{L}(H)} |n \cdot \nabla\zeta| ds dH \leq C \int_{\mathcal{D}(h,h')} |n \cdot \nabla\zeta| dx \quad (4.24)$$

$$\leq C|\mathcal{D}(h,h')|^{1/2} \sqrt{\int_{\mathcal{D}(h,h')} |\nabla\zeta|^2 dx} \leq C\sqrt{|h - h'|}\sqrt{\mu} = C\frac{\sqrt{\mu}}{A^{1/4}}$$

The left side of (4.23) satisfies

$$(M(h) - m(h)) \int\limits_0^{(h-h')/2} F\left(\frac{h+h'}{2} + H, h' + H\right) dH \geq C(M(h) - m(h))(h - h')^2 = \frac{C}{A}(M(h) - m(h)).$$

We arrive at

$$\frac{1}{A}(M(h) - m(h)) \leq C[A^{-5/4}\sqrt{\mu} + A^{-3/2}\mu]$$

so that

$$M(h) - m(h) \leq C\left[\frac{\sqrt{\mu}}{A^{1/4}} + \frac{\mu}{\sqrt{A}}\right]. \quad (4.25)$$

This contradicts (4.2) if N is sufficiently large, as the constant C in (4.25) does not depend on the choice of the constant N in (4.2). Therefore, (4.2) is impossible if N is large enough. Hence the proof of Theorem 4.1 in the case, when neither of the curves γ_m and γ_M intersects any of the circles U_j, is complete.

Case 2. One of the curves γ_m or γ_M intersects one of the circles U_j.

Let $\mathcal{R}$ be a domain bounded by the curves γ_M and γ_m (that now includes parts of some of the circles U_j) and parts of the level sets $\gamma = \mathcal{L}(h)$ and $\gamma' = \mathcal{L}(h')$. We integrate (3.6) over $\mathcal{R}$ and obtain

$$\int_{\gamma\cup\gamma'} \frac{\partial\zeta}{\partial n} ds + \sum_{k=1}^{N_A} \int\limits_{\gamma_M\cap\partial U_k} \frac{\partial\zeta}{\partial n} ds + \sum_{k=1}^{N_A} \int\limits_{\gamma_m\cap\partial U_k} \frac{\partial\zeta}{\partial n} ds - \frac{\mu}{2\pi^2} \sum_{k=1}^{N_A} \int\limits_{\gamma_M\cap\partial U_k} \frac{n\cdot\nabla H}{H} ds$$

$$-\frac{\mu}{2\pi^2} \int_{\gamma_m} \frac{n\cdot\nabla H}{H} ds + A \int_{\gamma_m\cup\gamma_M} \zeta(n \cdot u) ds = \mu|\mathcal{R}| - \int_{\mathcal{R}} |\nabla\zeta|^2 dxdy. \quad (4.26)$$

The first integral in the second line of (4.26) still satisfies the estimate (4.14) without any changes. The last term on the first line in (4.26) satisfies a simple estimate

$$\left|\sum_{k=1}^{N_A}\int\limits_{\gamma_M\cap\partial U_k}\frac{n\cdot\nabla H}{H}ds\right|\leq CN_Ar, \tag{4.27}$$

where r is the radius of each circle U_j. Hence it can be made arbitrarily small by taking a sufficiently small $r>0$.

Similarly, taking r sufficiently small we may ensure that $|\nabla\zeta|<\delta$ on all the circles U_j. Then we have

$$\left|\sum_{k=1}^{N_A}\int\limits_{\gamma_M\cap\partial U_k}\frac{\partial\zeta}{\partial n}ds\right|\leq CN_A\delta r, \tag{4.28}$$

with an identical estimate for the curve γ_m.

We may not apply the inequality (4.15) to the function ζ along γ_M, or ζ_0 along γ_m, as their monotonicity may be broken along the pieces of the circles U_j. However, we may modify the function ζ_M along these segments so as to make the modified function $\tilde{\zeta}$ be monotonic along the whole curve γ_M. The oscillation estimate (4.8) implies that the error thus created is bounded as

$$\left|\int_{\gamma_M}\tilde{\zeta}(u\cdot n)ds-\int_{\gamma_M}\zeta(u\cdot n)ds\right|\leq\frac{C\delta N_A}{1+N_A}\leq C\delta. \tag{4.29}$$

As δ and r are arbitrarily small, the above estimates allow us to recover (4.20). The proof of Theorem 4.1 then proceeds as in Case 1. □

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