Relaxation in reactive flows

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Abstract

We consider convective systems in a bounded domain, in which viscous fluids described by the Stokes system are coupled using the Boussinesq approximation to a reaction-advection-diffusion equation for the temperature. We show that the resulting flows possess relaxation-enhancing properties in the sense of [12]. In particular, we show that solutions of the nonlinear problems become small when the gravity is sufficiently strong due to the improved interaction with the cold boundary. As an application, we deduce that the explosion threshold for power-like nonlinearities tends to infinity in the large Rayleigh number limit. We also discuss the behavior of the principal eigenvalues of the corresponding advection-diffusion problem and the quenching phenomenon for reaction-diffusion equations.

1 Introduction

The presence of a strong incompressible flow in an advection-diffusion equation

$$\frac{\partial \phi}{\partial t} + Au \cdot \nabla \phi = \Delta \phi,$$

with a large parameter $A \gg 1$ improves the mixing properties of the pure diffusion process. This manifests itself as diffusivity enhancement (see [25] and references therein) in the whole space, or as accelerated convergence to an equilibrium in a smooth bounded domain Ω, in the strong flow limit $A \to +\infty$. As a measure of the latter effect the following definition has been proposed in [12]: a time-independent flow $u(x)$ is relaxation-enhancing if for any $\tau > 0$ and any $\delta > 0$ there exists $A_0(\tau, \delta)$ so that any solution of the initial value problem

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + Au \cdot \nabla \phi = \Delta \phi \text{ in } \Omega, \\ &\phi(0,x) = \phi_0(x), \\ &\phi = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

satisfies

$$\|\phi(\tau)\|_{L^\infty} \leq \delta \|\phi_0\|_{L^1} \tag{1.2}$$

for all $A \geq A_0(\tau, \delta)$. The $L^1 - L^\infty$ decay in (1.2) can be replaced by any $L^p - L^q$ decay with $1 \leq p, q \leq \infty$ – this does not change the class of relaxation enhancing flows [12]. It has been shown in [4, 12] that the flow $u(x)$ is relaxation-enhancing if and only if u has no first integrals in

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$H_0^1(\Omega)$. Equivalently, such flows lead to eigenvalue enhancement: the principal eigenvalue $\mu(A)$ of the Dirichlet problem

$$
\begin{aligned}
&-\Delta\phi + Au\cdot\nabla\phi = \mu(A)\phi \text{ in } \Omega,\\
&\phi = 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{1.3}
$$

satisfies $\lim_{A\to+\infty}\mu(A) = +\infty$ if and only if u is relaxation-enhancing [4, 12, 22].

Similar effects have been observed for reaction-advection-diffusion equations of the form

$$
\frac{\partial\phi}{\partial t} + Au\cdot\nabla\phi = \Delta\phi + f(\phi), \tag{1.4}
$$

with a non-negative nonlinearity $f(s)$ that vanishes at $s = 0$ and $s = 1$ – such equations appear in flame propagation as well as many other applied problems. It has been shown that a flow may speed-up a flame [9, 18, 23, 28, 30] or quench the propagation [11, 17, 24, 37, 38] as $A \to +\infty$.

Yet another problem where a prescribed flow has been shown to have a non-trivial effect is the explosion problem when the nonlinearity in (1.4) is of the form $f(s) = kg(s)$. Here $g(s)$ is a uniformly positive convex function that grows super-linearly as $s \to +\infty$, and the problem is posed in a bounded domain Ω with Dirichlet boundary conditions $\phi = 0$ on $\partial\Omega$. Solutions of the corresponding steady problem with $A = 0$ exist provided that $k < k_{cr}$, while no solutions exist for $k > k_{cr}$ [14, 19, 21]. Similarly, solutions of the parabolic problem with $A = 0$ blow-up for $k > k_{cr}$ [7] – see [8] for a recent review of related results. On the other hand, it has been shown in [6] that for $A \neq 0$ the flow may have a regularizing effect and increase the explosion threshold $k_{cr}(A)$. In particular, if u is relaxation enhancing then $k_{cr}(A) \to +\infty$ as $A \to \infty$.

In the present paper we consider similar mixing effects in nonlinear flows when the fluid is coupled to the temperature via a buoyancy force in the Stokes-Boussinesq approximation, in the infinite Prandtl number limit. The temperature $\theta(t, x)$ satisfies a semilinear advection-diffusion equation:

$$
\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta = \Delta\theta + kg(\theta)
$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with Dirichlet boundary conditions $\theta = 0$ on $\partial\Omega$. The flow is coupled to temperature by the Stokes-Boussinesq equation

$$
\frac{\partial u}{\partial t} - \Delta u + \nabla p = \rho\theta\hat{e}_z,
$$

with no-slip boundary conditions $u = 0$ on $\partial\Omega$. Here ρ is the Rayleigh number – the non-dimensional gravity strength – and we are interested in the behavior of solutions in the limit $\rho \to +\infty$, so that the flow is strong. The nonlinearity $g(s)$ is non-negative and the parameter $k > 0$ measures its strength.

The Stokes-Boussinesq reactive system has been actively studied recently when the nonlinearity $g(s)$ is either of the KPP or ignition type, mostly in unbounded domains. It has been shown that traveling fronts exist in various situations in infinite cylinders in two [2, 3, 13, 33, 34] and three [26] dimensions. Some a priori bounds in an infinite strip have been obtained in [10]. Numerical studies of the stability and qualitative behavior of such fronts have been performed in [15, 16, 35, 36]. The full convective Boussinesq explosion problem in a bounded domain has been studied numerically in [1] where a complex behavior of solutions has been observed. Existence and regularity of solutions have been established in [27]. However, much less of the qualitative properties of solutions is known in the Boussinesq case, apart from the aforementioned numerical simulations and lower bounds for the front speed obtained in [31].

In the present paper we investigate the effect of a strong convection with a large Rayleigh number. We prove several results that show the regularizing and mixing effects of convection. First, we show that for any parameter $k > 0$, steady solutions of the explosion problem do exist provided that the Rayleigh number is sufficiently large. This means that, unlike in the cellular flows, where the fluid flow may lower the critical threshold according to the numerical studies of [5], Boussinesq convection does not create hot spots, at least at high Rayleigh numbers. We also show that the fluid coupling has a regularizing effect: solutions become small as ρ tends to infinity. When the nonlinearity is of the ignition type we show that no steady solutions of the reactive Stokes-Boussinesq equation may exist when ρ is large. As a consequence we show that convection induces quenching in the time-dependent problem – the temperature goes to zero as $t \to +\infty$. This is an analogue of the corresponding results in [11, 17, 24, 37] for quenching in a prescribed flow.

We also extend the notion of relaxation-enhancing flows to families of flows. As in (1.1)-(1.2), a family of incompressible flows $u_\rho(x)$ is relaxation-enhancing if for any $\tau > 0$ and any $\delta > 0$ there exists $\rho_0(\tau, \delta)$ so that any solution of the initial value problem

$$\begin{aligned}&\frac{\partial \phi}{\partial t} + u_\rho \cdot \nabla \phi = \Delta \phi \text{ in } \Omega, \qquad (1.5)\\ &\phi(0, x) = \phi_0(x),\\ &\phi = 0 \text{ on } \partial\Omega,\end{aligned}$$

satisfies

$$\|\phi(\tau)\|_{L^\infty} \le \delta \|\phi_0\|_{L^1} \qquad (1.6)$$

for all $\rho \ge \rho_0(\tau, \delta)$. In particular, if $u(x)$ is a given relaxation enhancing flow, then $u_\rho(x) = \rho u(x)$ is a relaxation-enhancing family. Another typical class of relaxation-enhancing families comes from flows of the form $u_\rho(x) = \rho^{1+\alpha} v(\rho x)$, where $v(y)$ is a periodic cellular flow and $\alpha > 0$. In that case not only the flow rotates faster as ρ increases but also the invariant sets of u_ρ become smaller. We show that solutions of the steady reactive Stokes-Boussinesq form a relaxation enhancing family and the corresponding principal nonlinear eigenvalues tend to infinity. As a consequence, the exit time of the corresponding diffusion process tends to zero uniformly as $\rho \to +\infty$. Interestingly, while relaxation-enhancing flows are fairly difficult to construct explicitly, relaxation enhancing families arise naturally as solutions of nonlinear problems. Moreover, unlike in the case of prescribed flows [23, 29, 30], our proofs do not involve any delicate analysis of the behavior of the streamlines.

This paper is organized as follows. We present our main results in Section 2. Section 3 contains the analysis of the convective explosion problem. In Section 4 we establish the relaxation enhancement properties of the corresponding flows and discuss the nonlinear eigenvalue problem. Finally, in Section 5 we address quenching with the ignition nonlinearity.

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2 The main results

The steady explosion problem

We consider the nonlinear steady explosion problem

$$\begin{aligned}&-\Delta\theta + u \cdot \nabla\theta = kg(\theta), \qquad (2.1)\\ &-\Delta u + \nabla p = \rho\theta\hat{e}_z, \quad \nabla \cdot u = 0\end{aligned}$$

in a bounded two-dimensional domain $\Omega \subset \mathbb{R}^2$ with the Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0. \tag{2.2}$$

Here $k \geq 0$ is a parameter measuring the strength of nonlinearity. We assume that $g(s)$ is positive for all $s \in \mathbb{R}$: $g(s) \geq g_0 > 0$ and grows at most polynomially at infinity: there exists $C > 0$ and $m \geq 0$ so that

$$0 < g_0 \leq g(s) \leq C(1 + s^m) \text{ for } s \geq 0. \tag{2.3}$$

If, in addition, we assume that $g(s)$ is convex and grows super-linearly at infinity, then there exists a critical threshold k_* so that a solution of (2.1) with $u = 0$, or, equivalently, $\rho = 0$, exists for all $k \in (0, k_*)$ while (2.1) has no classical (or weak) solutions for $k > k_*$ [14, 19, 21]. It turns out that a strong convection has the following regularizing effect.

Theorem 2.1 *Let $g(s)$ be a Lipschitz nonlinearity satisfying (2.3) and let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any $k > 0$ there exists $\rho_0(k)$ so that (2.1)-(2.2) has a classical solution for all $\rho > \rho_0(k)$. Moreover, for any $\varepsilon > 0$ there exists $\rho_1(\varepsilon, k)$ so that for all $\rho > \rho_1(\varepsilon, k)$ the system (2.1)-(2.2) has a solution which, in addition, satisfies $\|\theta\|_{H_0^1(\Omega)} < \varepsilon$.*

If we consider a nonlinearity $g(s)$ which is a priori bounded: $0 \leq g(s) \leq M$ for all $s \in \mathbb{R}$ then we have a stronger result which holds both in dimensions two and three.

Theorem 2.2 *Let $g(s)$ be a Lipschitz nonlinearity satisfying $0 \leq g(s) \leq M$ for all $s \in \mathbb{R}$, and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a smooth bounded domain. Then for any $k > 0$ there exists $\rho_0(k)$ so that (2.1)-(2.2) has a classical solution for all $\rho > \rho_0(k)$. Moreover, for any $\varepsilon > 0$ there exists $\rho_1(\varepsilon, k)$ so that for all $\rho > \rho_1(\varepsilon, k)$ the system (2.1)-(2.2) has a solution which, in addition, satisfies $\|\theta\|_{H_0^1(\Omega)} < \varepsilon$.*

The small solutions constructed in Theorems 2.1 and 2.2 are analogous to the minimal solutions which exist for $\rho = 0$. However, for $\rho > 0$ different solutions $\theta(x)$ correspond to different flows $u(x)$ so the maximum principle is not available to compare them and it is not clear if solutions are ordered. We also note that if $g(0) = 0$ (and only in this case) then solutions constructed in Theorem 2.2 may be equal identically to zero.

The relaxation enhancing properties of Boussinesq flows

The families of flows constructed in Theorems 2.1 turn out to be relaxation enhancing.

Theorem 2.3 *For any $k > 0$ the system (2.1) with the nonlinearity $g(s)$ as in (2.3) has a relaxation-enhancing family of solutions $u_\rho(x)$, $\rho > \rho_0(k)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. The same is true for nonlinearities $g(s)$ as in Theorem 2.2 with an additional assumption $g(s) \geq g_0 > 0$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$.*

This theorem provides a large class of relaxation-enhancing families in two and three dimensions.

As we have mentioned, a single flow $u(x)$ is relaxation-enhancing if and only if the principal eigenvalue $\mu(A)$ of the operator $-\Delta + Au \cdot \nabla$ with the Dirichlet boundary conditions on $\partial\Omega$ tends to infinity [4, 12, 22]. In our situation a family of flows which have the principal eigenvalue tending to infinity comes from the nonlinear eigenvalue problems of the form

$$\begin{aligned} &-\Delta\theta + u \cdot \nabla\theta = \mu\theta, \\ &-\Delta u + \nabla p = \rho\theta\hat{e}_z, \quad \nabla \cdot u = 0, \\ &\theta \geq 0 \text{ in } \Omega \subset \mathbb{R}^3, \ u = 0 \text{ and } \theta = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.4}$$

Here $\Omega \subset \mathbb{R}^3$ is a smooth domain and μ is a nonlinear eigenvalue which is to be determined as part of the problem. The next theorem provides an analogue of the principal eigenvalue for the problem with a prescribed flow.

Theorem 2.4 *For each $\rho > 0$ and each $M > 0$ there exists an eigenvalue $\mu(\rho, M) > 0$ so that (2.4) has a solution $\theta_{\rho,M} \geq 0$ with $\|\theta_{\rho,M}\|_{L^2} = M$. Moreover, for each fixed $M > 0$ we have $\mu(\rho, M) \to +\infty$.*

Note that we have a simple scaling identity $\lambda\theta_{\rho,M} = \theta_{\lambda\rho,\lambda M}$ for any $\lambda > 0$ and therefore $\mu(\lambda\rho, \lambda M) = \mu(\rho, M)$. As a consequence, the fact that $\mu(\rho, M) \to +\infty$ as $\rho \to +\infty$ for a fixed M is equivalent to $\mu(\rho, M) \to +\infty$ as $M \to 0$ at a fixed $\rho > 0$. It is not clear to us at the moment whether the fact that the principal eigenvalue tends to infinity as $\rho \to +\infty$ is sufficient to ensure that the flows $u_\rho(x)$ form a relaxation enhancing flow.

The Boussinesq problem with a combustion nonlinearity

Here we consider a nonlinearity $g(s)$ of the form $g(s) = \beta(s)(1 - s)$. The Lipschitz function $\beta(s)$ is non-decreasing and has an ignition cut-off:

$$\beta(s) = 0 \text{ for all } s \in [0, \theta_0] \text{ and } \beta(s) > 0 \text{ for all } s > \theta_0. \tag{2.5}$$

Such nonlinearities are commonly used in flame propagation problems. The steady problem with no advection

$$-\Delta\theta = k\beta(\theta)(1 - \theta), \tag{2.6}$$

in a smooth bounded three-dimensional domain $\Omega \subset \mathbb{R}^3$ with the Dirichlet boundary conditions $\theta|_{\partial\Omega} = 0$ has a non-trivial ($\theta \not\equiv 0$) solution provided that the ignition cut-off $\theta_0 < 2/3$ and $k \geq k_0$ is sufficiently large. The corresponding steady Boussinesq convective problem is

$$\begin{aligned} -\Delta\theta + u \cdot \nabla\theta &= k\beta(\theta)(1 - \theta), \\ -\Delta u + \nabla p &= \rho\theta\hat{e}_z, \quad \nabla \cdot u = 0 \end{aligned} \tag{2.7}$$

with the Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0. \tag{2.8}$$

It turns out that (2.7)-(2.8) has no non-trivial solutions if convection is strong enough.

Theorem 2.5 *For any $k > 0$ there exists $\rho_0(k)$ so that the only classical solution of (2.7)-(2.8) with $\rho > \rho_0(k)$ is $\theta \equiv 0$.*

The Cauchy problem for the convective problem is

$$\begin{aligned} &\frac{\partial\theta}{\partial t} - \Delta\theta + u \cdot \nabla\theta = k\beta(\theta)(1 - \theta), \\ &\frac{\partial u}{\partial t} - \Delta u = \rho\theta\hat{e}_z - \nabla p, \quad \nabla \cdot u = 0, \end{aligned} \tag{2.9}$$

supplemented by the boundary conditions (2.8) and the Cauchy data $\theta(0, x) = \theta_0(x) \in C^\infty(\Omega)$, $0 \leq \theta_0(x) \leq 1$, and $u(0, x) = u_0(x) \in C^\infty(\Omega)$. The time-dependent version of Theorem 2.5 is the following.

Theorem 2.6 *For any smooth bounded domain $\Omega \subset \mathbb{R}^3$ and any $k > 0$ there exists ρ_0 so that for any $\rho > \rho_0$ and any initial data $0 \le \theta_0(x) \le 1$ the solution of (2.9)-(2.8) satisfies*

$$\lim_{t \to +\infty} \|\theta(t)\|_{L^\infty(\Omega)} = 0. \tag{2.10}$$

The time-dependent problem with $u = 0$ on the real line has been studied in the pioneering works of Kanel [20] who has shown that if $\|\theta_0\|_{L^1}$ is smaller than a critical value l_{cr} then (2.10) holds. These results have been recently sharpened in [38]. The Cauchy problem in prescribed flows has been studied in [10, 17, 37] where it has been shown that the critical mass $l_{cr}(A)$ may tend to infinity as $A \to +\infty$ which comes from the additional mixing by the flow. Theorem 2.6 may be seen as the Boussinesq version of the quenching results of [11, 17, 37] with the size of the domain serving as the scale l_{cr} – no matter how large the domain Ω is, solutions decay (the flame is extinguished) provided that gravity is sufficiently strong.

3 Existence of solutions with strong convection

We prove Theorems 2.1 and 2.2 in this section.

A uniform bound for a linear advection-diffusion equation

We first recall a uniform bound for solutions of advection-diffusion equations with an incompressible drift, which holds uniformly in the advecting flow (the only reference for this fact we are aware of is [6] so we present the proof for the convenience of the reader).

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $\phi(x)$ be the solution of the linear elliptic problem*

$$-\Delta\phi + u \cdot \nabla\phi = f(x), \quad x \in \Omega, \tag{3.1}$$

with the Dirichlet boundary conditions $\phi = 0$ on $\partial\Omega$. Assume that $f \in L^p(\Omega)$, $p > n/2$ and u is incompressible: $\nabla \cdot u = 0$. There exists a constant $C > 0$ which depends on Ω and p but not on the flow u so that $\|\phi\|_{L^\infty(\Omega)} \le C\|f\|_{L^p}$.

Proof. The function $\phi(x)$ can be represented as

$$\phi(x) = \int_0^\infty \psi(t, x)dt. \tag{3.2}$$

Here the function $\psi(t, x)$ solves the Cauchy problem

$$\begin{aligned} &\frac{\partial\psi}{\partial t} - \Delta\psi + u \cdot \nabla\psi = 0, \quad x \in \Omega, \\ &\psi(0, x) = f(x) \text{ for } x \in \Omega, \end{aligned} \tag{3.3}$$

with the boundary condition $\psi(t, x) = 0$ for $x \in \partial\Omega$. As in the proof of Lemma 5.6 in [12], we have the following bound for any $\varepsilon > 0$, $t \ge 0$ and $\tau > 0$:

$$\|\psi(t + \tau)\|_{L^2} \le \frac{C_\varepsilon}{\tau^{n/4+\varepsilon}}\|\psi(t)\|_{L^1},$$

with the constant C_ε which depends only on the domain Ω but not on the incompressible flow $u(x)$. In addition, due to incompressibility of $u(x)$, there exists a constant $\alpha > 0$ so that

$$\|\psi(t + \tau)\|_{L^2} \le e^{-\alpha\tau}\|\psi(t)\|_{L^2}.$$

It follows that

$$\|\psi(t+\tau)\|_{L^2} \le \frac{C_\varepsilon e^{-\alpha\tau}}{\tau^{n/4+\varepsilon}}\|\psi(t)\|_{L^1},$$

where here and below C_ε and α are some new universal constants. Let $\mathcal{P}_\tau$ be the evolution operator $\mathcal{P}_\tau[\psi(t)] = \psi(t+\tau)$. Since u is divergence-free, the adjoint operator $\mathcal{P}_\tau^*$ is the evolution operator for (3.3) with the flow $u(x)$ replaced by $(-u(x))$. Therefore, $\mathcal{P}_\tau^*$ obeys the same bound

$$\|\mathcal{P}_\tau^*\|_{L^1\to L^2} \le \frac{C_\varepsilon e^{-\alpha\tau}}{\tau^{n/4+\varepsilon}}.$$

As a consequence, we have

$$\|\mathcal{P}_\tau\|_{L^2\to L^\infty} \le \frac{C_\varepsilon e^{-\alpha\tau}}{\tau^{n/4+\varepsilon}}.$$

Using the semi-group property $\mathcal{P}_\tau = \mathcal{P}_{\tau/2}\circ\mathcal{P}_{\tau/2}$, we obtain

$$\|\mathcal{P}_\tau\|_{L^1\to L^\infty} \le \frac{C_\varepsilon e^{-\alpha\tau}}{\tau^{n/2+\varepsilon}}.$$

The maximum principle implies the trivial $L^\infty \to L^\infty$ bound $\|\mathcal{P}_\tau\|_{L^\infty\to L^\infty} \le 1$. Using the Riesz-Thorin interpolation theorem, we conclude that for any $1 < p < \infty$ we have

$$\|\mathcal{P}_\tau\|_{L^p\to L^\infty} \le \frac{C_\varepsilon e^{-\alpha\tau/p}}{\tau^{n/(2p)+\varepsilon}},$$

so that for ψ from (3.3) we have

$$|\psi(t,x)| \le \frac{C_\varepsilon e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}}\|f\|_{L^p}, \tag{3.4}$$

for all $x \in \Omega$ and $t > 0$. Using this bound in (3.2) we deduce that for $p > n/2$ we may choose $\varepsilon > 0$ sufficiently small so that we have the required estimate

$$|\phi(x)| \le \int_0^\infty \|\psi(t)\|_{L^\infty}dt \le \int_0^\infty \frac{C_\varepsilon e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}}\|f\|_{L^p}dt = C\|f\|_{L^p}.$$

This finishes the proof of Lemma 3.1. □

The semigroup property in the proof above is not really necessary and the result can be extended to smooth time-dependent velocities $u = u(x,t)$: consider the Cauchy problem

$$\begin{aligned}&\frac{\partial\phi}{\partial t} + u\cdot\nabla\phi = \Delta\phi + f(t,x),\\ &\phi(0,x) = g(x),\\ &\phi = 0 \text{ on } \partial\Omega.\end{aligned} \tag{3.5}$$

Then we can write, using the Duhamel formula

$$\phi(t,x) = \int_0^t \psi(t,x;s)ds + \eta(t,x).$$

Here $\eta(t,x)$ is the solution of the Cauchy problem (3.5) with $f = 0$, while the function $\psi(t,x;s)$ satisfies for $t \ge s$

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + u\cdot\nabla\psi = \Delta\psi,\\ &\psi(s,x;s) = f(s,x),\\ &\psi = 0 \text{ on } \partial\Omega.\end{aligned} \tag{3.6}$$

We can prove that we have, similarly to Lemma 3.1,

$$\|\psi(t,\cdot;s)\|_{L^\infty(\Omega)} \le \frac{C_\varepsilon e^{-\alpha_p(t-s)}}{(t-s)^{n/(2p)+\varepsilon}}\|f(s,\cdot)\|_{L^p(\Omega)}.$$

It follows from this estimate and (3.4) that for any $p > n/2$ we have

$$\|\phi(t,\cdot)\|_{L^\infty(\Omega)} \le C_p\|f\|_{L^\infty([0,t];L^p(\Omega))} + \frac{C_\varepsilon e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}}\|g\|_{L^p(\Omega)},$$

with constants that are independent of the flow $u(t,x)$.

The almost linear problem: existence and smallness

We first consider an almost linear problem with a prescribed heating $f_\rho(x)$ (which may depend on the parameter ρ as well)

$$\begin{aligned} &-\Delta\phi + u\cdot\nabla\phi = f_\rho(x), \\ &-\Delta u + \nabla p = \rho\phi\hat{e}_z, \quad \nabla\cdot u = 0, \end{aligned} \tag{3.7}$$

in a smooth domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$ with the Dirichlet boundary conditions (2.2): $u|_{\partial\Omega} = 0$, $\phi|_{\partial\Omega} = 0$.

Lemma 3.2 *Suppose $\Omega \subset \mathbb{R}^n$, $n = 2,3$, is a bounded domain, and $f_\rho \in L^p(\Omega)$, $n/2 < p \le \infty$. Then there exists a solution $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ to (3.7).*

Proof. Consider the map $F:\ L^2(\Omega) \to H_0^1(\Omega)$, where $\Theta = F(\Theta_0)$ is given by the solution of the quasi-linear elliptic problem

$$\begin{cases} -\Delta\Theta + u\cdot\nabla\Theta = f_\rho(x), \\ -\Delta u + \nabla p = \rho\Theta_0\hat{e}_z, \quad \nabla\cdot u = 0, \\ u|_{\partial\Omega} = 0, \quad \Theta|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

As the flow u is incompressible, for any $\Theta_0 \in L^2(\Omega)$, we multiply the first equation in (3.7) by Θ and integrate over Ω. Let us choose $q < 2n/(n-2)$ (and hence $\|\Theta\|_{L^q} \le C\|\Theta\|_{H_0^1}$) so that $q' = q/(q-1) < p$ – this is possible for $n \ge 2$ provided that $p > n/2$. As Θ vanishes on the boundary, this leads to a uniform bound:

$$\|\nabla\Theta\|_{L^2}^2 \le C\|f_\rho\|_{L^{q'}}\|\Theta\|_{L^q} \le C\|f_\rho\|_{L^p}\|\nabla\Theta\|_{L^2}, \quad 1/q' + 1/q = 1,$$

with the constant $C > 0$ which is independent of the function Θ_0. Therefore, F maps $L^2(\Omega)$ into a fixed ball in $H_0^1(\Omega)$. In particular, for a ball $B_K = \{||\Theta_0||_{L_2} \le K\} \subset L^2(\Omega)$ with a sufficiently large radius K

$$F(B_K) \subset \{||\Theta||_{H_0^1} \le C\} \subset\subset B_K.$$

Hence, F is a compact map from B_K into itself so that by the Schauder's fixed point theorem it has a fixed point, which is a solution to (3.7) in $H_0^1(\Omega)$. The L^∞-estimate for the solution $\phi(x)$ follows from Lemma 3.1. □

Lemma 3.3 *Let $n/2 < p < \infty$ and $K > 0$. Then for any $\varepsilon > 0$, there exists $\rho_{cr} = \rho_{cr}(K,\varepsilon)$ such that if $\|f_\rho\|_{L^p(\Omega)} \le K$ and $\rho > \rho_{cr}(K,\varepsilon)$ then any solution $\phi(x)$ to (3.7) satisfies*

$$\|\phi\|_{H_0^1} \le \varepsilon. \tag{3.9}$$

Proof. As in the proof of Lemma 3.2, we chose $q < 2n/(n-2)$ so that $p > q' = q/(q-1)$. Integrating by parts and using incompressibility of $u(x)$ we obtain the following a priori estimates:

$$\|\phi\|_{H_0^1}^2 \leq C\|f_\rho\|_{L^{q'}}\|\phi\|_{L^q} \leq C\|f_\rho\|_{L^p}\|\phi\|_{L^q}, \quad 1/q' + 1/q = 1, \tag{3.10}$$

as $p > q'$. In addition, we have $\|\phi\|_{L^q} \leq C\|\phi\|_{H_0^1}$ for $q < (2n)/(n-2)$, and thus

$$\|u\|_{H_0^1} \leq \rho C\|\phi\|_{L^2}, \quad \|\phi\|_{H_0^1} \leq \|f_\rho\|_{L^p}. \tag{3.11}$$

The first bound in (3.11) comes from the Stokes equation for $u(x)$.

The proof of Lemma 3.3 is by contradiction. Suppose there exists a sequence (u_n, ϕ_n, ρ_n) of solutions to (3.7) with $\rho = \rho_n$, such that $\rho_n \to \infty$ and $\|f_{\rho_n}\|_{L^p} \leq K$ but

$$\|\phi_n\|_{H_0^1} \geq \varepsilon_0 > 0. \tag{3.12}$$

Then for $\|u_n\|_{L^2}$ we have two possibilities: either there exist $C > 0$ and a subsequence, still denoted u_n, such that

$$\|u_n\|_{L^2} \geq C\rho_n, \tag{3.13}$$

or

$$\lim_{n\to\infty}\left(\frac{\|u_n\|_{L^2}}{\rho_n}\right) = 0. \tag{3.14}$$

Suppose first that (3.13) is true. Then using (3.11) and rescaling $\bar{u}_n = u_n/\rho_n$ we find a subsequence $\bar{u}_n$ that converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to $\bar{u}_0 \in H_0^1(\Omega)$. Moreover, it follows from Proposition 2.2 on p. 33 in [32] that

$$\|\bar{u}_n\|_{H^2} \leq C\|\phi_n\|_2$$

and thus $\bar{u}_0 \in H^2(\Omega)$ and convergence of $\bar{u}_n$ is strong in $H_0^1(\Omega)$. The uniform lower bound (3.13) implies that

$$\|\bar{u}_0\|_{L^2} \geq C \tag{3.15}$$

which, in turn, implies that $\bar{u}_0 \not\equiv 0$. The sequence ϕ_n is uniformly bounded in $H_0^1(\Omega)$ and hence (up to extraction of a subsequence) ϕ_n converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function $\phi_0 \in H_0^1(\Omega)$. Let $h(x)$ be a smooth test function with support contained inside Ω. We multiply the first equation in (3.7) by $h(x)$ and integrate over Ω:

$$\frac{1}{\rho_n}\int \nabla h \cdot \nabla\phi_n dx + \int h(\bar{u}_n \cdot \nabla\phi_n)dx = \frac{1}{\rho_n}\int h(x)f_{\rho_n}(x)dx. \tag{3.16}$$

The uniform H^1-bound for ϕ_n implies that the first term on the left vanishes as $n \to +\infty$ and so does the right side of (3.16) as well. As for the second term on the left, the gradient $\nabla\phi_n$ converges weakly in $L^2(\Omega)$ to $\nabla\phi_0$, while $v_n = h\bar{u}_n$ converge strongly in $H_0^1(\Omega)$ to $v_0 = h\bar{u}_0$. We conclude that for any smooth function $h(x)$ with the support inside Ω we have

$$\int h(\bar{u}_n \cdot \nabla\phi_n) \to \int h(\bar{u}_0 \cdot \nabla\phi_0).$$

Using this in (3.16) we obtain

$$\int h(\bar{u}_0 \cdot \nabla\phi_0) = 0, \tag{3.17}$$

and thus, in particular,

$$\bar{u}_0 \cdot \nabla \phi_0 = 0 \text{ a.e.} \tag{3.18}$$

The function $\bar{u}_0 \cdot \nabla \phi_0$ lies in $L^q(\Omega)$ for some $q > 1$, as, using the Hölder inequality, we get

$$\int |\bar{u}_0|^q |\nabla \phi_0|^q dx \le \left(\int |\nabla \phi_0|^2 \right)^{q/2} \left(\int |\bar{u}_0|^{2q/(2-q)} \right)^{(2-q)/2} \le C \|\phi_0\|_{H_0^1}^q \|\bar{u}_0\|_{H_0^1}^q \le C$$

if q is sufficiently close to 1. Therefore, as compactly supported functions are dense in $L^{q'}(\Omega)$, identity (3.17) holds for all $h \in L^{q'}(\Omega)$, $1/q + 1/q' = 1$.

In addition, $\bar{u}_0$ and ϕ_0 satisfy the Stokes equations

$$-\Delta \bar{u}_0 + \nabla p = \phi_0 \hat{e}_z, \quad \nabla \cdot \bar{u}_0 = 0.$$

Multiplying the Stokes equation by $\bar{u}_0$ and using the fact that $\bar{u}_0 \in H_0^1(\Omega)$ we deduce that

$$\|\nabla \bar{u}_0\|_{L^2}^2 = \int_\Omega \bar{u}_{0,z} \phi_0 dx, \tag{3.19}$$

where $\bar{u}_{0,z}$ is the third component of $\bar{u}_0$. On the other hand, using the test function $\eta(x) = z$ in (3.17), we obtain

$$0 = \int_\Omega (z \bar{u}_0 \cdot \nabla \phi_0) = \int_\Omega \bar{u}_{0,z} \phi_0 dx.$$

It follows from (3.19) that $\bar{u}_0 = 0$, which contradicts (3.15). Therefore, (3.13) is impossible.

However, if (3.14) holds then we may divide the Stokes equation by ρ_n and pass to the limit $n \to \infty$. As the sequence ϕ_n is still bounded in H_0^1, there exists a weakly converging subsequence ϕ_n in $H_0^1(\Omega)$ (which converges strongly in any $L^q(\Omega)$, $q < 2n/(n-2)$) with a limit $\phi_0 \in H_0^1(\Omega)$. As $\bar{u}_n = u_n/\rho_n$ converges strongly to zero in $L^2(\Omega)$, we obtain that

$$0 = \phi_0 \hat{e}_z + \nabla p$$

holds weakly. Therefore $\phi_0 \hat{e}_z$ is a gradient and $\phi_0 = h(z)$. Since $\phi_n \to \phi_0$ strongly in any $L^q(\Omega)$, $q < 2n/(n-2)$, using (3.12) and (3.10) we have that $\|\phi_0\|_{L^q} > 0$ and $\|\phi_0\|_{H_0^1} > 0$. But this is impossible if $\phi_0 = h(z) \in H_0^1(\Omega)$ is a function of z only and $\phi_0|_{\partial\Omega} = 0$. □

Proof of Theorem 2.2. The "smallness" part of Theorem 2.2 is an immediate consequence of Lemma 3.3. The existence part in this theorem follows from an argument identical to that in the proof of Lemma 3.2. □

Proof of Theorem 2.1

Define a truncated nonlinearity

$$g_M(s) = \begin{cases} M, & g(s) \ge M \\ g(s), & -M \le f(x) \le M \\ -M, & g(x) \le -M. \end{cases}$$

Since $g_M(\Theta(x)) \in L^\infty(\Omega)$, it follows from Theorem 2.2 that there exists a function $\theta_M(x)$ and the corresponding flow $u_M(x)$ that satisfy the system

$$\begin{aligned} -\Delta \theta_M + u_M \cdot \nabla \theta_M &= k g_M(\theta_M), \\ -\Delta u_M + \nabla p &= \rho \theta_M \hat{e}_z, \quad \nabla \cdot u_M = 0, \end{aligned} \tag{3.20}$$

with the boundary conditions (2.2).

By Lemma 3.3, given any $\varepsilon > 0$, we can choose $\rho_{cr}(\varepsilon, M)$ so that for any $\rho > \rho_{cr}(\varepsilon, M)$ the solution we have found satisfies the bound

$$\|\nabla\theta_M\|_{H_0^1} \leq \varepsilon.$$

By the Sobolev embedding we have

$$\|g_M(\theta)\|_2^2 \leq C\left(1 + (\|\theta\|_{L^{2m}})^{2m}\right) \leq C\left(1 + \left(\|\nabla\theta\|_{H_0^1}\right)^{2m}\right) \leq C(1 + \varepsilon^{2m}).$$

If we take ε so small that $C(1 + \varepsilon^{2m}) < 2C$ and use Lemma 3.1 we conclude that

$$\|\theta_M\|_{L^\infty} \leq C\|g_M\|_{L^2} \leq K,$$

with the constant K independent of M. It follows that if we take $M > 5K$ then there is no truncation and $g_M(\theta_M) = g(\theta_M)$. Hence, we have found a solution to (2.1), that satisfies $\|\theta\|_{H_0^1(\Omega)} < \varepsilon$. □

4 The mixing properties of the reactive convective flows

Proof of Theorem 2.3

We will prove only the first statement which is a simple consequence of Theorem 2.1. The second is proved identically using the smallness of the solutions constructed in Theorem 2.2. Let us set $k = 1$ in (2.1) for convenience. Take a function $f \in L^\infty(\Omega)$ with $\|f\|_{L^\infty} = 1$, assume without loss of generality that $f \geq 0$, and let the function $\zeta_\rho(t, x)$ be the solution of the Cauchy problem

$$\frac{\partial\zeta_\rho}{\partial t} + u_\rho(x) \cdot \nabla\zeta_\rho = \Delta\zeta_\rho \tag{4.1}$$

with the Dirichlet boundary conditions $\zeta_\rho(t, x) = 0$ for all $x \in \partial\Omega$, and the initial data $\zeta_\rho(0, x) = f(x)$. Given any $\delta > 0$ we can find $\rho_{cr}(\delta)$ and a family of solutions (θ_ρ, u_ρ) of (2.1) such that $\|\theta_\rho\|_{H_0^1} < \delta$ for all $\rho > \rho_{cr}(\delta)$. Consider now the functions ψ_ρ which solve

$$-\Delta\psi_\rho + u_\rho \cdot \nabla\psi_\rho = f(x) \text{ for } x \in \Omega$$
$$\psi_\rho = 0 \text{ for } x \in \partial\Omega.$$

The maximum principle implies that $0 \leq \psi_\rho(x) \leq \theta_\rho(x)/g_0$. Hence, it follows from Theorem 2.1 that $\|\psi_\rho\|_{L^2} \leq \delta/g_0$ for all $\rho \geq \rho_{cr}(\delta)$. However, the function $\psi_\rho(x)$ may be written as

$$\psi_\rho(x) = \int_0^\infty \zeta_\rho(t, x)dt.$$

Therefore, we have

$$\int_0^\infty \|\zeta_\rho(t)\|_{L^2}dt \leq \delta/g_0.$$

As the function $s(t) = \|\zeta_\rho(t)\|_{L^2}$ is monotonically decreasing in time and δ is arbitrary, it follows that for any $\tau > 0$ any any $\varepsilon > 0$ we can find $\rho_{cr}(\tau, \varepsilon)$ so that $\|\zeta_\rho(\tau/2)\|_{L^2} < \varepsilon\tau^{3/4+1}$ for all $\rho > \rho_{cr}(\tau, \varepsilon)$. Then Lemma 5.6 in [12] (uniform in the flow $L^2 - L^\infty$ decay for solutions of advection-diffusion equations in an incompressible flows) implies that $\|\zeta_\rho(\tau)\|_{L^\infty} < \varepsilon$. Thus, the flows $u_\rho(x)$ indeed form a relaxation enhancing family. □

An alternative definition of the relaxation enhancing families is in terms of the exit times. The following proposition generalizes the corresponding results from [22] for a multiple of a single flow: $u_\rho(x) = \rho u(x)$.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^n$, $n = 2,3$ be a smooth bounded domain and let τ_ρ be the exit time, solution of*

$$-\Delta\tau_\rho + u_\rho \cdot \nabla\tau_\rho = 1 \text{ for } x \in \Omega. \qquad (4.2)$$
$$\tau_\rho = 0 \text{ for } x \in \partial\Omega.$$

The family $u_\rho(x)$ of incompressible flows is relaxation enhancing if and only if

$$\lim_{\rho\to+\infty} \|\tau_\rho\|_{L^\infty} = 0. \qquad (4.3)$$

Proof. Once again, we write

$$\tau_\rho(x) = \int_0^\infty \psi_\rho(t,x)dt, \qquad (4.4)$$

with the function $\psi_\rho(t,x)$ which solves

$$\frac{\partial\psi_\rho}{\partial t} + u_\rho(x)\cdot\nabla\psi_\rho = \Delta\psi_\rho \qquad (4.5)$$

with the Dirichlet boundary conditions $\psi_\rho(t,x) = 0$ for $x \in \partial\Omega$, and the Cauchy data $\psi_\rho(0,x) = 1$. It follows from (3.4) with $n = p$ that for any $\varepsilon > 0$ there exist $C > 0$ and $\alpha > 0$ that do not depend on the flow u_ρ so that

$$\|\psi_\rho(t+s)\|_{L^\infty} \le \frac{Ce^{-\alpha t}}{t^{1/2+\varepsilon}}\|\psi_\rho(s)\|_{L^p} \le \frac{Ce^{-\alpha t}}{t^{1/2+\varepsilon}}\|\psi_\rho(s)\|_{L^\infty}.$$

Assume first that u_ρ is relaxation enhancing. Then for any $t > 0$ there exists ρ_{cr} so that for all $\rho > \rho_{cr}$ we have $\|\psi_\rho(t)\|_{L^\infty} \le \varepsilon$. The exit time can be then estimated as

$$\|\tau_\rho\|_{L^\infty} \le \int_0^t \|\psi_\rho(s)\|_{L^\infty}ds + \varepsilon\int_t^\infty \frac{Ce^{-\alpha(s-t)}}{(s-t)^{1/2+\varepsilon}}ds \le t + C\varepsilon \le \delta$$

if we choose $t > 0$ and $\varepsilon > 0$ sufficiently small. Therefore, (4.3) holds.

On the other hand, as $\|\psi_\rho(t)\|_{L^\infty}$ is decreasing in time, it follows from (4.3) and (4.4) that for any $\varepsilon > 0$ and any $t > 0$ we have for a sufficiently large $\rho > \rho_0(\varepsilon,t)$ and $\delta = t\varepsilon$:

$$t\|\psi_\rho(t)\|_{L^\infty} \le \|\tau_\rho\|_{L^\infty} \le \delta = t\varepsilon,$$

so that $\|\psi_\rho(t)\|_{L^\infty} \le \varepsilon$. Therefore, u_ρ is a relaxation enhancing family. □

As a consequence of Theorem 2.3 and Proposition 4.1 we conclude that the exit times for the flows generated by Stokes-Boussinesq systems (2.1) tend to zero in the limit of the large Rayleigh number. This is one way to formalize the fact that any point in the domain Ω is nearly connected to the boundary by such flows.

The nonlinear eigenvalue problem

Next, we prove Theorem 2.4.

Proof of Theorem 2.4. Let us fix $M > 0$ and $\rho \ge 0$ and consider the map $\mathcal{K}: L^2(\Omega) \to L^2(\Omega)$ defined as follows: given a function $\theta \in L^2(\Omega)$ construct the flow $u(x) \in H^2(\Omega)$ by solving the Stokes-Boussinesq problem

$$-\Delta u + \nabla p = \rho\theta\hat{e}_z, \quad \nabla\cdot u = 0 \text{ in } \Omega,$$

with the no-slip boundary conditions $u = 0$ on $\partial\Omega$. Then $\psi = \mathcal{K}\theta$ is the positive in Ω eigenfunction of the operator

$$-\Delta\psi + u \cdot \nabla\psi = \mu\psi, \quad \psi > 0 \text{ in } \Omega,$$

with the Dirichlet boundary conditions $\psi = 0$ on $\partial\Omega$, normalized so that $\|\psi\|_{L^2} = M$. The operator $\mathcal{K}$ is compact. This is seen as follows: take any ball $B_K = \{\theta \in L^2(\Omega) : \|\theta\|_{L^2} \le K\}$, then $\|u(\theta)\|_{H^2} \le CK$ for any $\theta \in B_K$ [32]. Therefore, we have $|\mu| \le C(K)$ and thus there exists a constant $C_0(K)$ such that $\|\psi\|_{H_0^1} \le C_0(K)$ for any $\theta \in B_K$ – therefore, $\mathcal{K}$ is compact. The Schauder fixed point theorem implies that the operator $\mathcal{K} : B_K \to B_K$ has a fixed point θ for any $K \ge M$. However, a fixed point of $\mathcal{K}$ is a solution to the nonlinear eigenvalue problem (2.4).

Now, we show that $\mu(\rho, M) \to +\infty$ as $\rho \to +\infty$ for any fixed $M > 0$. Let us assume that this is not the case. Then there exists a sequence $\rho_n \to +\infty$ such that $\mu(\rho_n) < K_0$ for some fixed $K_0 > 0$. It follows from Lemma 3.1 that $\|\theta_{\rho_n}\|_{L^\infty} \le CK_0M$ for all $n \in \mathbb{N}$ since

$$\|\theta_{\rho_n}\|_{L^2} = M, \tag{4.6}$$

and dimension $n \le 3$. Lemma 3.3 implies then that there exists N_0 so that $\|\theta_{\rho_n}\|_{H_0^1} \le M/10$ for $n \ge N_0$. This is a contradiction to (4.6), therefore we have $\mu(\rho, M) \to +\infty$ as $\rho \to +\infty$. □

5 Reactive convection with an ignition nonlinearity

In this section we consider the reaction-diffusion-convection problems with an ignition nonlinearity satisfying (2.5). Throughout this section Ω is a smooth bounded domain in $\mathbb{R}^3$.

The non-convective steady problem

First, we recall the following result for the steady problem without convection.

Lemma 5.1 *Let the function $\beta(s)$ satisfy (2.5) and assume that the ignition cut-off $\theta_0 < 2/3$. There exists $\kappa_0 > 0$ so that the semi-linear elliptic problem*

$$\begin{cases} -\Delta\Theta = \kappa\beta(\Theta)(1-\Theta), \\ \Theta|_{\partial\Omega} = 0 \end{cases} \tag{5.1}$$

has a non-negative solution $\Theta(x)$, which is not identically equal to zero, for all $\kappa \ge \kappa_0$.

Proof. Let us first build a nontrivial positive sub-solution of (5.1). Choose Θ_1 and Θ_2 so that $\Theta_1 < 2\Theta_2/3$ and $\theta_0 < \Theta_1 < \Theta_2 < 1$ and then find $\alpha > 0$ such that

$$g(s) := \beta(s)(1-s) \ge \alpha, \text{ for } \Theta_1 \le s \le \Theta_2. \tag{5.2}$$

Now, set

$$\tilde{g}(s) = \begin{cases} 0 & s \le \Theta_1, \\ \alpha & s \ge \Theta_1. \end{cases}$$

We construct a sub-solution, which is radially symmetric with respect to a point x_0 inside the domain Ω – we set $x_0 = 0$ without loss of generality. Let us define the function $\phi(r)$ as

$$\phi(r) = \Theta_2 - \frac{\kappa\alpha r^2}{2}$$

for $0 \le r \le r_0$, where $r_0 = (2(\Theta_2 - \Theta_1)/(\alpha\kappa))^{1/2}$, that is, $\phi(r_0) = \Theta_1$, and we also set

$$\phi(r) = \frac{\kappa\alpha r_0^3}{r} - (2\Theta_2 - 3\Theta_1)$$

for $r \ge r_0$. Then both ϕ and ϕ' are continuous at r_0 and, in addition, ϕ satisfies

$$-\Delta\phi = \kappa\tilde{g}(\phi) \le \kappa g(\phi).$$

Moreover, we have $\phi(R_0) = 0$, where

$$R_0 = \frac{\kappa\alpha r_0^3}{(2\Theta_2 - 3\Theta_1)} = \frac{2\sqrt{2}(\Theta_2 - \Theta_1)^{3/2}}{(\alpha\kappa)^{1/2}(\Theta_2 - \Theta_1)}.$$

Now, with the sub-solution $\phi(x)$ in hand, we let $\psi(t,x)$ be the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial\psi}{\partial t} - \Delta\psi = \kappa g(\psi), \\ \psi|_{\partial\Omega} = 0 \\ \psi(0,x) = \phi_0(x) = \max\{\phi(x), 0\}. \end{cases} \tag{5.3}$$

The maximum principle implies that the solution $\psi(t,x)$ satisfies $0 \le \psi(x) \le 1$ and, in addition, as $\phi(x)$ is a sub-solution for the elliptic problem, $\psi(t,x)$ is increasing in t, point-wise in x. Therefore, as $t \to +\infty$ the function $\psi(t,x)$ converges to a steady solution for (5.1). □

The convective steady problem

We may now prove Theorem 2.5.

Proof of Theorem 2.5. Let $\theta(x)$ and $u(x)$ be a solution of

$$\begin{aligned} -\Delta\theta + u \cdot \nabla\theta &= k\beta(\theta)(1-\theta), \\ -\Delta u + \nabla p &= \rho\theta\hat{e}_z, \quad \nabla \cdot u = 0 \end{aligned} \tag{5.4}$$

with the Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0, \tag{5.5}$$

and assume that $\theta \not\equiv 0$. The maximum principle implies that $0 \le \theta \le 1$ and hence $\|g(\theta)\|_{L^\infty} \le K$, $g = k\beta(\theta)(1-\theta)$ with a constant K which depends only on the nonlinearity g and the domain Ω. Lemma 3.3 implies that for any $\varepsilon > 0$, there exists ρ_{cr} such that $\|\theta\|_{H_0^1} \le \varepsilon$ for all $\rho \ge \rho_{cr}$. We know, however, from Lemma 3.1 that in three dimensions

$$\|\theta\|_{L^\infty} \le C\|g(\theta)\|_{L^2} \le C\|\theta\|_{H_0^1} \le C\varepsilon.$$

Hence, choosing ε so that $C\varepsilon < \theta_0$ we have that $\|\theta\|_{L^\infty} < \theta_0$. It means that the right-hand side of the first equation in (5.4) is identically equal to zero. The only solution then must be equal identically to zero. □

Quenching in reactive convection

Here we prove Theorem 2.6. Let us recall the system (2.9):

$$\begin{aligned} &\frac{\partial \theta}{\partial t} - \Delta\theta + u\cdot\nabla\theta = kg(\theta), \\ &\frac{\partial u}{\partial t} - \Delta u + \nabla p = \rho\theta\hat{e}_z, \quad \nabla\cdot u = 0, \end{aligned} \tag{5.6}$$

with the Dirichlet and no-slip boundary conditions:

$$u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0, \tag{5.7}$$

and the Cauchy data $\theta(0,x) = \theta_0(x) \in C^\infty(\Omega)$, $0 \le \theta_0(x) \le 1$, and $u(0,x) = u_0(x) \in C^\infty(\Omega)$. The maximum principle implies an a priori bound:

$$0 \le \theta(t,x) \le 1 \tag{5.8}$$

The first step is to prove the following lemma.

Lemma 5.2 *For any $\varepsilon > 0$ there exists ρ_{cr} such that for any $\rho \ge \rho_{cr}$ we have*

$$\inf_{\tau\in[0,1]} \|\theta(\tau)\|_{L^2} \le \varepsilon.$$

Proof. Consider any time $0 < t \le 1$. First, multiplying the Stokes equation by u and integrating by parts, we obtain the usual bound

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int |\nabla u|^2 dx = \rho\int_\Omega (u\cdot\hat{e}_z)\theta dx,$$

so that, as u vanishes on the boundary and $0 \le \Theta \le 1$,

$$\int_0^t \int |\nabla u(s,x)|^2 dxds \le C(\rho^2+1).$$

Next, we multiply the Stokes equation by u_t and integrate by parts:

$$\begin{aligned} \int_0^t \int_\Omega |u_s|^2 dxds &= \rho\int_0^t \int_\Omega (u_s\cdot\hat{e}_z)\theta dxds + \int_0^t \int_\Omega (\Delta u\cdot u_s) dxds \\ &\le \frac{1}{2}\int_0^t \int_\Omega |u_s|^2 dxds + \frac{\rho^2}{2}\int_0^t \int_\Omega \theta^2 dxds - \frac{1}{2}\int_0^t \frac{d}{ds}\left(\int_\Omega |\nabla u(s,x)|^2 dx\right) ds. \end{aligned}$$

Hence, we have

$$\int_0^t \int_\Omega |u_s|^2(s,x) dxds + \int_\Omega |\nabla u(t,x)|^2 dx \le C\rho^2 \int_0^t \int_\Omega \theta^2(s,x) dxds + C_0 \le C(1+\rho^2)$$

and thus

$$\int_0^t \int_\Omega \left(u_t^2 + |\nabla u|^2\right) dxdt \le C(1+\rho^2). \tag{5.9}$$

We also know that $\theta \in L^2(\Omega\times[0,1])$. Therefore, we can use the argument presented in Lemma 3.3. Let us show that for any ε_0 there exists ρ_{cr} such that for any $\rho \ge \rho_{cr}$ we can find a time $\tau(\rho) \in [0,1]$

such that $\|\theta(\tau)\|_{H_0^1} \le \varepsilon_0$. Suppose, by contradiction, that for some $\varepsilon_0 > 0$ and a sequence (u_n, θ_n, ρ_n) of solutions to (5.6) with $\rho = \rho_n$, such that $\rho_n \to \infty$, we have, for all $t \in [0, 1]$,

$$\|\theta_n(t)\|_{L^2} \ge \varepsilon_0 > 0. \tag{5.10}$$

Then, as in the proof of Lemma 3.3, we have two possibilities: either there exist $C > 0$ and a subsequence, which we still denote u_n, such that

$$\int_0^1 \|u_n(t)\|_{L^2}^2 dt \ge C\rho_n^2, \tag{5.11}$$

or

$$\lim_{n\to\infty} \int_0^1 \|u_n(t)\|_{L^2}^2 dt/\rho_n^2 = 0. \tag{5.12}$$

Assume first that (5.11) holds. As a consequence of (5.9) we have

$$\|u_n\|_{H^1(\Omega\times[0,1])} \le C(1 + \rho^2).$$

Then, rescaling $\bar{u}_n = u_n/\rho_n$ we find a subsequence $\bar{u}_n$ that converges weakly in $H^1(\Omega \times [0, 1])$ and strongly in $L^2(\Omega \times [0, 1])$ to a limit $\bar{u}_0 \in H^1(\Omega \times [0, 1])$. Moreover, convergence of $\bar{u}_n$ is strong in $L^p(\Omega \times [0, 1])$, for some $p > 2$. From the uniform bound (5.11) we have that

$$\|\bar{u}_0\|_{L^2(\Omega\times[0,1])} \ge C \tag{5.13}$$

which implies that $\bar{u}_0 \not\equiv 0$. Next, the sequence θ_n converges weakly (after extraction of a subsequence) in $L^2(\Omega \times [0, 1])$ to θ_0 and $\nabla\theta_n$ converges weakly to $\nabla\theta_0 \in L^2(\Omega \times [0, 1])$. It follows that the product $\bar{u}_n \cdot \nabla\theta_n$ converges weakly to the corresponding product in the sense that for any test function $h \in C^\infty(\Omega \times [0, 1])$ we have

$$\int_0^1 \int_\Omega h(x, t)(\bar{u}_n \cdot \nabla\theta_n)dxdt \to \int_0^1 \int_\Omega h(x, t)(\bar{u}_0 \cdot \nabla\theta_0)dxdt.$$

From (5.6) we have that for any $h \in C_0^\infty(\Omega \times [0, 1])$

$$\frac{1}{\rho_n}\int_0^1 \int_\Omega \nabla h(x, t) \cdot \nabla\theta_n dxdt + \int_0^1 \int_\Omega h(x, t)(\bar{u}_n \cdot \nabla\theta_n)dxdt = \frac{k}{\rho_n}\int_0^1 \int_\Omega h(x, t)g(\theta_n)dxdt.$$

Passing to the limit $\rho_n \to +\infty$ we obtain

$$\int_0^1 \int_\Omega h(x, t)(\bar{u}_0 \cdot \nabla\phi_0)dxdt = 0, \text{ for any } h \in C_0^\infty(\Omega \times [0, 1]).$$

Since $C_0^\infty(\Omega \times [0, 1])$ is dense in $L^{q'}(\Omega \times [0, 1])$ for $1 \le q' < \infty$ we have

$$\bar{u}_0 \cdot \nabla\theta_0 = 0 \text{ a.e.}, \tag{5.14}$$

or,

$$\int_0^1 \int_\Omega h\bar{u}_0 \cdot \nabla\theta_0 dxdt = 0 \tag{5.15}$$

for any test function $h \in L^{q'}(\Omega \times [0, 1])$. In addition, $\bar{u}_0$ and θ_0 satisfy the Stokes equation with the unit Rayleigh number:

$$\frac{\partial\bar{u}_0}{\partial t} - \Delta\bar{u}_0 + \nabla p = \phi_0\hat{e}_z, \quad \nabla \cdot \bar{u}_0 = 0.$$

Multiplying the Stokes equation by $\bar u_0$ and using the fact that $\bar u_0 = 0$ on $\partial\Omega$ we deduce that

$$\int_0^1 \|\nabla \bar u_0(t)\|_{L^2}^2 dt + \|\bar u_0(t=1)\|_{L^2}^2 dt = \int_0^1 \int_\Omega \bar u_{0,z}\theta_0 dxdt.$$

On the other hand, as before, using the test function $\eta(x) = z$ in (5.15), we obtain

$$0 = \int_0^1 \int_\Omega (z\bar u_0 \cdot \nabla\theta_0) = \int_0^1 \int_\Omega \bar u_{0,z}\theta_0 dx.$$

It follows that $\bar u_0 = 0$, which contradicts (5.13). Therefore, (5.11) is impossible.

Assume now that (5.12) holds. Then we may divide the Stokes equation by ρ_n and pass to the limit $n \to \infty$. As the sequence θ_n is positive and is bounded by 1 in $L^\infty(\Omega \times [0,1])$, there exists a weakly converging subsequence θ_n in $L^2(\Omega \times [0,1]))$ with a limit $\theta_0 \not\equiv 0$ because of (5.10) and since $0 \le \theta_n \le 1$. In particular, using (5.10) we obtain

$$\int_0^1 \int_\Omega \theta_n dxdt \to \int_0^1 \int_\Omega \theta_0 dxdt = C > 0. \tag{5.16}$$

Let us set

$$\psi_n(x) = \int_0^1 \theta_n(t,x)dt.$$

Then we have

$$\|\nabla\psi_n(x)\|_{L_2(\Omega)} \le \int_0^1 \|\nabla\theta_n\|_{L^2} dtdx \le C\left(\int_0^1 \int_\Omega |\nabla\theta_n|^2 dxdt\right)^{1/2} \le C.$$

Therefore, the sequence $\psi_n(x)$ is uniformly bounded in $H_0^1(\Omega)$, and thus has a subsequence that converges strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to

$$\psi_0(x) = \int_0^1 \theta_0(t,x)dt.$$

As $\bar u_n = u_n/\rho_n$ converges strongly to zero in $L^2(\Omega \times [0,1])$ because of (5.12), we obtain that

$$0 = \theta_0 \hat e_z + \nabla P$$

holds in the weak sense. Therefore, ψ_0 satisfies

$$0 = \psi_0 \hat e_z + \nabla\bar P, \quad \bar P(x) = \int_0^1 P(t,x)dt$$

and hence $\psi_0 \hat e_z$ is a gradient: $\psi_0 = h(z)$. It follows from (5.16) that $\|\psi_0\|_{L^2} > 0$. But this is impossible if $\psi_0 = h(z)$, $\psi_0 \in H_0^1(\Omega)$ and $\psi_0|_{\partial\Omega} = 0$. This finishes the proof of Lemma 5.2. □

Proof of Theorem 2.5. Let us use Lemma 5.2 to choose ρ_{cr} so that for any $\rho > \rho_{cr}$ we can find some time $\tau \in [0,1]$ such that $\|\theta(\tau)\|_{L^2} \le c_0\theta_0$, where c_0 is an appropriately small constant. The maximum principle implies that for $t \ge \tau$ we have $\theta(t,x) \le e^{M(t-\tau)}\zeta(t,x)$. The constant M is chosen so that $g(s) \le Ms$ for all $s \ge 0$, and the function $\zeta(t,x)$ is the solution of the advection-diffusion problem with the flow $u(t,x)$ which solves (5.6):

$$\begin{aligned} &\frac{\partial\zeta}{\partial t} - \Delta\zeta + u(t,x)\cdot\nabla\zeta = 0, \quad t \ge \tau, \\ &\zeta|_{\partial\Omega} = 0, \quad \zeta(\tau,x) = \theta(\tau,x). \end{aligned} \tag{5.17}$$

Using Lemma 5.6 of [12] (which applies also to time-dependent incompressible flows as well since it uses only integration by parts) we obtain that

$$\|\zeta(t=\tau+1)\|_{L^\infty} \le C\|\zeta(\tau)\|_{L^2},$$

with the constant C independent of the flow $u(t,x)$. Therefore, if we choose c_0 so that $c_0 Ce^M < 1/2$ then $\|\theta(\tau+1)\|_{L^\infty} < \theta_0/2$. It follows from the maximum principle that $\theta(t,x) \le \theta_0/2$ for all $t \ge \tau+1$ and all $x \in \Omega$. Thus, in particular, $g(\theta) \equiv 0$ for $t \ge \tau+1$. As a consequence, θ and u solve the Boussinesq system without any reaction for $t > \tau+1$ and hence (2.10) holds. □

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