Math 63CM Homework # 6

Due in section on Friday, February 21.

1. Let
$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$
and consider the initial value problem
$$x'(t) = Ax(t), \quad x(0) = x_0.$$
Find all values of $a, b \in \mathbb{R}$ such that
(i) 0 is a stable equilibrium,
(ii) 0 is an asymptotically stable equilibrium,
(iii) 0 is an unstable equilibrium.
Justify your answers. Finally, choose some values of a and b such that (c) above holds and sketch the corresponding phase portrait.
2. Let A be a (4×4) complex matrix with eigenvalues $-1 \pm i$, $\pm i$.
(i) Show that there exists a non-zero periodic solution $x(t)$ to $x'(t) = Ax(t)$.
(ii) Show that there exists a non-zero solution $x(t)$ to $x'(t) = Ax(t)$ such that $\|x(t)\| \to 0$ as $t \to \infty$.
(iii) Show that there is an open and dense subset $U \subset \mathbb{C}^4$ such that if $x_0 \in U$, then the unique solution to
$$x'(t) = Ax(t), \quad x(0) = x_0$$
is <u>not</u> periodic and does <u>not</u> satisfy $\|x(t)\| \to 0$ as $t \to \infty$.
3. Consider the system
$$\begin{aligned} y_1''(t) &= -\omega_1^2 y_1(t), \\ y_2''(t) &= -\omega_2^2 y_2(t), \end{aligned} \tag{1}$$
for real-valued functions $y_1(t)$ and $y_2(t)$. Assume that $\omega_1,\ \omega_2 \in \mathbb{R} \setminus \{0\}$ are such that ω_1/ω_2 is an irrational number. This system describes two particles attached to two different springs. The purpose of the problem is to show that for ω_1, ω_2 as above, the solution is in general not periodic, but is always "almost" periodic (in a sense to be made precise below).
(i) Show that $E_1(t) = (y_1(t))^2 + \omega_1^{-2}(y_1'(t))^2$ and $E_2(t) = (y_2(t))^2 + \omega_2^{-2}(y_2'(t))^2$ are independent of t, that is $E_1(t) = E_1(0)$ and $E_2(t) = E_2(0)$ for all $t \geq 0$. For the rest of this problem, we fix the initial conditions $y_1(0)$, $y_1'(0)$, $y_2(0)$, $y_2'(0)$ so that $E_1(0) \neq 0$ and $E_2(0) \neq 0$.
(ii) Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if $x - y \in 2\pi\mathbb{Z} = \{2\pi n : n \in \mathbb{Z}\}$, and the distance on $\mathbb{R}/\sim$ by $d(\theta, \phi) = \min_{\substack{\theta' \sim \theta \\ \phi' \sim \phi}} |\theta' - \phi'|$. Show that this distance depends only on the equivalence classes of θ and ϕ.
(iii) Let $(\theta_1, \theta_2) : \mathbb{R} \to \mathbb{R}/\sim$ be defined by
$$\cos\theta_i(t) = \frac{y_i(t)}{\sqrt{(y_i(t))^2 + \omega_i^{-2}(y_i'(t))^2}}, \ \sin\theta_i(t) = \frac{\omega_i^{-1} y_i'(t)}{\sqrt{(y_i(t))^2 + \omega_i^{-2}(y_i'(t))^2}},$$
for $i = 1, 2$. By deriving equations for θ_1 and θ_2, or otherwise, show that for each $i = 1, 2$, the corresponding component of the solution $(y_i, y_i') : \mathbb{R} \to \mathbb{R}^2$ is periodic of period $\frac{2\pi}{\omega_i}$, but the full

solution $(y_1, y_1', y_2, y_2') : \mathbb{R} \to \mathbb{R}^4$ is not periodic.
(iii) Show that for every $\delta > 0$, there exists $T \in \mathbb{R}$ such that

$$d(\theta_1(t+T), \theta_1(t)) < \delta \text{ and } d(\theta_2(t+T), \theta_2(t)) < \delta \text{ for every } t \in \mathbb{R}.$$

Hint: First show by a pigeon hole principle argument that there exists $m \in \mathbb{N}$ such that

$$d(0, \frac{2\pi m \omega_1}{\omega_2}) < \delta.$$

Now take $T = \frac{2\pi m}{\omega_2}$.
(iv) We say that a function $f : \mathbb{R} \to \mathbb{R}^n$ is *almost periodic* if for every $\epsilon > 0$, there exists $T > 0$ such that $\|f(t+T) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$. Conclude, using part (c) or otherwise, that the solution $(y_1, y_1', y_2, y_2') : \mathbb{R} \to \mathbb{R}^4$ is almost periodic.
4. Problem 2.5 in Brendle.
5. Problem 2.6 in Brendle.