Math 63CM Homework # 5

Due in section on Friday, February 14.

1. Let $k \geq 0$ be an integer, and A be an $n \times n$ complex matrix.
(i) Show that $A^k, A^{k+1}, \ldots, A^{k+n}$ are linearly dependent.
(ii) For every $n \geq 2$, give an example of an $n \times n$ matrix A such that $I, A, \ldots, A^{n-1}$ are linearly independent.
2. (i) Let A be an $n \times n$ complex matrix, and λ be an eigenvalue of A of multiplicity k. That is, λ is a root of $\chi_A(\lambda) = \det(\lambda I - A)$ of multiplicity k. Show that for any $\ell \geq k$, we have

$$\ker(\lambda I - A)^k = \ker(\lambda I - A)^\ell.$$

(ii) Let A and B be two $n \times n$ complex matrices that commute: $AB = BA$. Show that B leaves $\ker A$ invariant: if $w \in \ker A$ then $Bw \in \ker A$, and A leaves $\ker B$ invariant: if $w \in \ker B$ then $Aw \in \ker B$. Is it true that we necessarily have $\ker A = \ker B$?
3. Let A be an $n \times n$ complex matrix. Suppose $\lambda_1, \ldots, \lambda_m$ are (all) the distinct (complex) eigenvalues of A. Show that A is diagonalizable if and only if $p(A) = 0$ where p is the polynomial $p(\lambda) = \Pi_{i=1}^m (\lambda - \lambda_i)$.
4. Let A be an $(n \times n)$ complex matrix such that $\det(\lambda I - A) = \Pi_{i=1}^m (\lambda - \lambda_i)^{\nu_i}$, where λ_i are distinct. Find $\det(\lambda I - e^A)$. Justify your answer.
5. (i) Problem 2.2 in Brendle.
(ii) Consider the matrix A as in Problem 2.2 in Brendle. Find a matrix S such that $S^{-1}AS$ is in the Jordan canonical form.
6. A real $n \times n$ matrix A is positive-definite if it is symmetric and for any $x \in \mathbb{R}^n$ such that $x \neq 0$ we have $(Ax \cdot x) > 0$, where $(x \cdot y)$ is the standard dot product on $\mathbb{R}^n$.
(i) Show that if A is positive-definite, then there exists $c > 0$ so that $(Ax \cdot x) > c\|x\|^2$ for all $x \in \mathbb{R}^n$.
(ii) Show that if A is positive-definite, then all eigenvalues of A are positive.
(iii) Show that if A is positive definite then the limit

$$\int_0^\infty e^{-tA} dt = \lim_{T \to +\infty} \int_0^T e^{-tA} dt$$

exists.
(iv) Show that A is positive definite then

$$A^{-1} = \int_0^\infty e^{-tA} dt.$$

7. Recall that the inner product in $\mathbb{C}^n$ is defined as

$$\langle v, w \rangle = v_1 \bar{w}_1 + v_2 \bar{w}_2 + \cdots + v_n \bar{w}_n,$$

for $v, w \in \mathbb{C}^n$, and the norm of a vector $v \in \mathbb{C}^n$ is

$$\|v\| = \langle v, v \rangle = (v_1 \bar{v}_1 + \cdots + v_n \bar{v}_n)^{1/2}.$$

Here, for a complex number $z = a + ib$, its complex conjugate is $\bar{z} = x - iy$. Note that for any complex number $z \in \mathbb{C}^n$ we have $z\bar{z} = x^2 + y^2 \geq 0$. We define $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$.

Also, recall that a complex $n \times n$ matrix A is self-adjoint if $a_{ij} = \bar{a}_{ji}$. We assume below that A is an $n \times n$ complex self-adjoint matrix.
(i) Show that $\langle Av, w \rangle = \langle v, Aw \rangle$, for all $v, w \in \mathbb{C}^n$.

(ii) Show that all eigenvalues of A are real.
(iii) Show that if v_1 and v_2 are two eigenvectors of A that correspond to eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$ then $\langle v_1, v_2 \rangle = 0$.
(iv) Let $v(t)$ be the solution to a system of linear ODE

$$i\frac{dv}{dt} = Av, \;\; v(0) = z \in \mathbb{C}^n.$$

Show that $\|v(t)\| = \|z\|$ for all $t \geq 0$. Hint: differentiate $\|v(t)\|^2$ in time. Be careful with complex conjugates.
(v) Let $v(t)$ be the solution to a system of non-linear ODE for $v(t) \in \mathbb{C}^n$

$$i\frac{dv}{dt} = Av + D(t)v, \;\; v(0) = z \in \mathbb{C}^n,$$

where $D(t)$ is a diagonal matrix with the entries $D(t) = \text{diag}(|v_1(t)|^2, \ldots, |v_n(t)|^2)$. Show that $\|v(t)\| = \|z\|$ for all $t \geq 0$.