Math 63CM Homework # 4

Due in section on Friday, February 6.

1. Recall that the operator norm of an $n \times n$ matrix A is defined as

$$\|A\|_{\text{op}} := \sup_{\|x\|=1} \|Ax\|.$$

(i) Show that for any $x \in \mathbb{R}^n$ we have $\|Ax\| \leq \|A\|_{op}\|x\|$.
(ii) Show that if A and B are two $n \times n$ matrices, then $\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}$.
(iii) Let a_{ij}, $1 \leq i, j \leq n$ be the entries of A. Show that

$$\|A\|_{\text{op}} \leq \Big(\sum_{i,j=1}^{n} a_{ij}^2 \Big)^{\frac{1}{2}}.$$

Does equality hold in general? Justify your answer.
(iv) Show that

$$\max_j \Big(\sum_{i=1}^{n} a_{ij}^2 \Big)^{\frac{1}{2}} \leq \|A\|_{\text{op}}.$$

2. Recall that $\|\cdot\|$ is a norm on a vector space X (defined over $\mathbb{R}$) if
(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
(2) $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{R}$ and $x \in X$,
(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
Let X be a finite-dimensional vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Show that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \text{ for all } x \in X.$$

3. (i) Give an example of two (2×2) real matrices A and B such that $e^{A+B} \neq e^A e^B$. [Hint: You may find it easier to look for A and B with $A^2 = B^2 = 0$.]
(ii) Prove that the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable.
4. Let A be an $(n \times n)$ complex matrix. Define $\text{tr}A = \sum_{i=1}^{n} a_{ii}$.
(i) Let C and D be two $n \times n$ complex matrices. Show that $\text{Tr}(CD) = \text{Tr}(DC)$.
(ii) Suppose $B = S^{-1}AS$ for some S. Show that $\text{tr}A = \text{tr}B$.
(iii) Prove that $\text{tr}A$ equals to the sum of the eigenvalues of A (where the eigenvalues are counted with multiplicities).
(iv) Show that $\det e^A = e^{tr\,A}$. [Hint: explain why it suffices to consider the case where A is an upper triangular matrix and use this fact.]
4. Problem 2.3 in Brendle. Fully simplify your answer — in particular, evaluate any matrix exponentials.
5. Problem 2.4 in Brendle.
6. Problem 2.7 in Brendle.

7. (i) Let $\psi(x,y)$ be a smooth globally Lipschitz function defined on $\mathbb{R}^2$ and define the flow $F(x,y)$ with the components

$$F(x,y) = (F_1(x,y), F_2(x,y)) = \Big(\frac{\partial\psi(x,y)}{\partial y}, -\frac{\partial\psi(x,y)}{\partial x}\Big).$$

Show that $\nabla \cdot F(x,y) = 0$.
(ii) Let $X(t)$ and $Y(t)$ solve the system of ODEs

$$\frac{dX(t)}{dt} = F_1(X(t), Y(t)), \quad \frac{dY(t)}{dt} = F_2(X(t), Y(t)),$$

with the initial condition $X(0) = x$, $Y(0) = y$. Show that $\psi(X(t), Y(t)) = \psi(x,y)$ for all $t \geq 0$.
(iii) Let $H(x,y)$ be a function such that

$$\frac{\partial H(x,y)}{\partial x}\frac{\partial\psi(x,y)}{\partial y} - \frac{\partial H(x,y)}{\partial y}\frac{\partial\psi(x,y)}{\partial x} = 0 \ \text{ for all } (x,y) \in \mathbb{R}^2.$$

Show that then $H(X(t), Y(t)) = H(X(0), Y(0))$ for all $t \geq 0$.