Math 63CM Homework # 3

Due in class on Friday, January 24.

1. (i) Find the unique solution to the ODE $y'(t) = ay(t)$, with the initial condition $y(0) = y_0$, where $a \in \mathbb{R}$ and $y_0 \in \mathbb{R}$ are given numbers.
(ii) Let $a(t)$ be a given continuous function. Explain why the initial value problem $y'(t) = a(t)y(t)$, with the initial condition $y(0) = y_0$, has a unique solution for all $t \in \mathbb{R}$. Show that if $y_0 > 0$ then $y(t) > 0$ for all $t \in \mathbb{R}$ and if $y_0 < 0$ then $y(t) < 0$ for all $t \in \mathbb{R}$.
(iii) Find this solution. Hint: it may help to look at $z(t) = \log y(t)$ if $y_0 > 0$ and at $z(t) = -\log y(t)$ if $y_0 < 0$.

2. Given an ODE

$$u'(t) = F(u(t)), \quad u(0) = u_0,$$

define

$$T_- := \inf\{T : \text{a solution } u(t) \text{ exists for } t \in (T, 0]\}$$

and

$$T_+ := \sup\{T : \text{a solution } u(t) \text{ exists for } t \in [0, T)\}.$$

(i) Consider the ODE for $u : I \subset \mathbb{R} \to \mathbb{R}$.

$$u'(t) = (u(t))^2, \quad u(0) = u_0.$$

For each u_0, find T_- and T_+. Conclude that the ODE has a global solution (i.e., a solution $u : \mathbb{R} \to \mathbb{R}$) if and only if $u_0 = 0$.
(ii) Find a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that for some $u_0 \in \mathbb{R}$, the solution to

$$u'(t) = F(u(t)), \quad u(0) = u_0$$

has T_-, T_+ both finite.

3. Consider the initial value problem

$$u'(t) = v(t), \quad v'(t) = -4u(t), \quad u(0) = 0, \quad v(0) = 2.$$

(i) Show, by explicit computation, that $u(t) = \sin 2t$, $v(t) = 2\cos 2t$ is a solution.
(ii) Consider the Picard's iteration: $u_0(t) = 0$, $v_0(t) = 2$ and

$$u_k(t) = \int_0^t v_{k-1}(s)\,ds, \;\; v_k(t) = 2 - 4\int_0^t u_{k-1}(s)\,ds,$$

for $k \geq 1$. Show explicitly that there exists $\varepsilon > 0$ such that for $|t| < \varepsilon$, $u_k(t) \to \sin 2t$, $v_k(t) \to 2\cos 2t$ as $k \to \infty$. Hint: Compare $u_k(t)$ with the Taylor's series of $\sin 2t$ around $t = 0$. You may use any results from 61CM provided they are clearly stated.

5. (i) Let $F_1(x)$ and $F_2(x)$ be two continuously differentiable and uniformly Lipschitz functions on $\mathbb{R}$ such that $F_1(x) < F_2(x)$ for all $x \in \mathbb{R}$. Consider the unique solutions to the ODEs

$$x_1'(t) = F_1(x_1(t)), \; x_2'(t) = F_2(x_2(t)),$$

with the initial conditions $x_1(0) = a$, $x_2(0) = b$ and $a < b$. Show that $x_1(t) \leq x_2(t)$ for all $t > 0$.
(ii) Consider the same question as in part (i) but with the weaker assumptions that $F_1(x) \leq F_2(x)$

for all $x \in \mathbb{R}$ and that $a \leq b$. Show that we still have $x_1(t) \leq x_2(t)$ for all $t > 0$. Hint: consider solutions to

$$y_n'(t) = F_2(y_n(t)) + \frac{1}{n},\ y_n(0) = b + \frac{1}{n},$$

show that $y_n(t) > x_1(t)$ and pass to the limit $n \to +\infty$. Explain carefully why $y_n(t)$ converges to $x_2(t)$ as $n \to +\infty$, and in which sense this convergence holds.

6. In the statement of the Arzela–Ascoli theorem proven in class, we considered a sequence of functions f_n with $f_n : J \to \mathbb{R}^n$ for some compact interval $J \subset \mathbb{R}$ and assumed that f_n satisfy

$$\begin{gathered}\forall \epsilon > 0,\ \exists \delta > 0,\ \forall n \in \mathbb{N}, \\ |t-s| < \delta,\ t, s \in J \implies |f_n(t) - f_n(s)| < \epsilon.\end{gathered} \tag{1}$$

Prove that the following (weaker-looking) condition

$$\begin{gathered}\forall \epsilon > 0, \forall t \in J,\ \exists \delta > 0,\ \forall n \in \mathbb{N}, \\ |t-s| < \delta,\ s \in J \implies |f_n(t) - f_n(s)| < \epsilon\end{gathered} \tag{2}$$

in fact implies (1).

7. Construct a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that f_n is uniformly bounded but there is <u>no</u> subsequence of f_n which converges uniformly to a continuous limit. Show explicitly that the sequence f_n you constructed is not equicontinuous.

8. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Suppose there is a $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$\|F(x)\| \leq C\|x\|. \tag{3}$$

Show that any solution to $x'(t) = F(x(t))$, $x(0) = x_0$ can be extended for all time, i.e. $J^\star_{x_0} = \mathbb{R}$.

9. Consider Newton's equation $y''(t) = -(\nabla V)(y(t))$ for some smooth $V : \mathbb{R}^n \to \mathbb{R}$. Suppose $y(t)$ is a solution with the maximal time interval of existence $(-\infty, T)$, where $T < \infty$. Show that there exists a sequence $t_n \to T$ such that $V(y(t_n)) \to -\infty$.