Homework # 1.

Math 63CM Homework # 2

Due in class on Friday, January 17.

1. Suppose A, B are similar $n \times n$ matrices, that is, there is an $n \times n$ invertible matrix C with $B = C^{-1}AC$. Prove that A, B have the same eigenvalues. Hint: Show that for all $\lambda \in \mathbb{R}$ we have $\det(B - \lambda I) \equiv \det(A - \lambda I)$.

2. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a $C^1(U)$ map. Show that if $\det Df(x) \neq 0$ for each $x \in U$, then $f(U)$ is open.

3. Let U be the annular region $\{x \in \mathbb{R}^2 : 1/2 < \|x\| < 1\}$. Prove that U is open and give an example of a C^1 function $f : U \to U$ such that $\det Df(x) > 0$ for every $x \in U$ and such that f is not one-to-one on U.

4. Ellipsoids: Let A be an $n \times n$ invertible matrix (not necessarily symmetric) and let $E = \{x \in \mathbb{R}^n : \|Ax\| \leq 1\}$. Prove that E is an *ellipsoid*: there exists an orthogonal matrix Q and positive real numbers $\lambda_1, \ldots, \lambda_n$ so that E is the set of all $z \in \mathbb{R}^n$ that can be written as $z = Qy$, with $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} \lambda_j y_j^2 \leq 1.$$

Hint: Start by checking that $\|Ax\| \leq 1$ if and only if $(x \cdot (A^T Ax)) \leq 1$, and think about applying the Spectral Theorem to $A^T A$.

5. Let $x(t)$ be the solution to the initial value problem $x' = x$, $x(0) = 1$.
You can easily check that its solution is $x(t) = \exp t$ but please do not use this fact answering the questions below.
(i) Show that the function $x(t)$ is actually infinitely differentiable and its Taylor series around any point converges to it.
(ii) Show that the solution satisfies $x(t + s) = x(t)x(s)$, $t, s \in \mathbb{R}$. Hint: For a fixed $s \in \mathbb{R}$, show that both sides solve the same ODE with the same initial condition at $t = 0$.
(iii) Calculate explicitly the iteration sequence given by the mapping

$$Tx(t) = 1 + \int_0^t x(\tau)\, d\tau,$$

starting with $x_0(t) = 1$. (This is the iteration if you solve $x' = x$ with $x(0) = 1$ and apply the proof of the contraction mapping theorem with $x_0(t) = 1$ – a reasonable first guess as it satisfies the initial condition.)

6. (i) Show that the function $g(x) = e^{-\pi x^2}$ satisfies an ODE

$$g'(x) + 2\pi x g(x) = 0,\ \ g(0) = 1.$$

(ii) Given a real number $k \in \mathbb{R}$ define

$$u(k) = \int_{-\infty}^{\infty} \cos(2\pi kx) e^{-\pi x^2} dx.$$

Show that $u(k)$ satisfies

$$u'(k) + 2\pi k u(k) = 0,\ \ u(0) = 1.$$

Be careful with how you justify differentiation of the integral.
(iii) Show that $u(k) = e^{-\pi k^2}$.