Homework # 1.

Due in class on Friday, January 10.

Recall that the left limit of a function f at a point c is defined as follows:

$$\lim_{x \to c^-} f(x) = \ell$$

if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any $x \in (c - \delta, c)$ we have $|f(x) - \ell| < \varepsilon$. The right limit

$$\lim_{x \to c^+} f(x)$$

is defined very similarly.

1. Let $f(x)$ be a monotonic function on an interval $[a, b]$. (i) Let $c \in (a, b)$. Show that the left and right limits

$$l(c) = \lim_{x \to c^-} f(x), \ r(c) = \lim_{x \to c^+} f(x)$$

exist.
(ii) Show that for any $n \in \mathbb{N}$ there exist finitely many points c in $[a, b]$ such that $|l(c) - r(c)| > 1/n$.
(iii) Show that f is continuous on $[a, b]$ except possibly at a finite or countable set of points. Recall that an infinite set S is countable if there is one-to-one map (that is, a bijection) between S and $\mathbb{N}$.

2. Let a_n and b_n be two Cauchy sequences in a metric space (X, d). Is it true that the sequence of real numbers $\alpha_n = d(a_n, b_n)$ must have a limit? Does (X, d) need to be a complete metric space for this to be true?

3. Let a function f be continuously differentiable on an interval $[a, b]$. Show that f can be represented as a difference of two non-decreasing functions on $[a, b]$.

4. (i) Show, without using the contraction mapping principle, that if all eigenvalues λ_k, $k = 1, \ldots n$, of a symmetric $n \times n$ matrix A satisfy $|\lambda_k| < 1$, then the map $F(x) = y + Ax : \mathbb{R}^n \to \mathbb{R}^n$ with some $y \in \mathbb{R}^n$ fixed, has a unique fixed point.
(ii) Give an example of a mapping A that maps a complete metric space X to itself, so that $d(A(x_1), A(x_2)) < d(x_1, x_2)$ for all x_1, x_2 but A does not have a fixed point. Hint: let ℓ_1 be the space of all sequences $\{x_n\}$ with $x_n \in \mathbb{R}$ such that $\sum_{k=1}^{\infty} |x_k| < \infty$, with the norm

$$\|x\| = \sum_{k=1}^{\infty} |x_k|.$$

For a fixed $y \in \ell_1$ consider a map $A : \ell_1 \to \ell_1$, with $[A(x)]_k = y_k + \lambda_k x_k$. Find $y \in \ell_1$ and numbers λ_k so that $\|A(x) - A(u)\|_{\ell_1} < \|x - u\|_{\ell_1}$ for all $x, u \in \ell_1$ but there is no $x \in \ell_1$ so that $A(x) = x$.