Math 63CM Final Exam, Winter 2020

1. (i) Consider the system

$$x'(t) = -x(t) + x(t)y(t), \quad y'(t) = y(t).$$

Determine the stable manifold and the unstable manifolds corresponding to the equilibrium point $(0,0)$.

(ii) Consider the ODE

$$y'(t) = -G'(y(t)), \ y(0) = y_0,$$

with $G(y) = y^{2020} - y^2$, $y \in \mathbb{R}$. Characterize the possible ω-limit sets of the trajectory, and show that the ω-limit set of the trajectory depends on the starting point y_0.

2. (i) Assume that an $n \times n$ complex matrix $A \in \mathbb{C}^{n \times n}$ is similar to an upper triangular matrix B: there exists an invertible $n \times n$ matrix S so that $A = S^{-1}BS$. Show that the characteristic polynomials $\chi_A(\lambda)$ and $\chi_B(\lambda)$ of A and B are the same, and that the diagonal entries of B are the eigenvalues of A.

(ii) An $n \times n$ complex matrix $A \in \mathbb{C}^{n \times n}$ is dubious if the characteristic polynomial $\chi_A(\lambda)$ and its derivative $\chi'_A(\lambda) = \dfrac{d\chi_A(\lambda)}{d\lambda}$ have no common roots in $\mathbb{C}$. Show that dubious matrices are dense in the set of all $n \times n$ complex matrices.

3. Let the scalar-valued functions $r(t)$ and $v(t)$ be the solutions to

$$(1) \qquad\qquad r'(t) = v(t), \ v'(t) \ = -\frac{1}{r^2(t)},$$

with the initial condition $r(0) = r_0 > 0$ and $v(0) = v_0$.

(i) Show that if $v_0 < 0$, then there exists $T > 0$ such that $r(t) \to 0$ as $t \to T^-$.

(ii) Show that for each $r_0 > 0$ there exists $\bar{v}(r) > 0$ so that if $r(0) = r_0$ and $v(0) > \bar{v}(r)$ then we have $v(t) > v(0)/2$ for all $t \geq 0$. Hint: consider the first time t_0 such that $v(t_0) = v(0)/2$ and get a contradiction if $v(0)$ is sufficiently large. You may need to estimate $r(t)$ from below for all $0 \leq t \leq t_0$, and the integral

$$\int_0^t \frac{1}{r^2(s)} ds$$

from above.

4. Let $H(x,y)$ and $\Phi(x,y)$ be two smooth functions and consider a 2×2 system of ODE:

$$(2) \qquad \begin{aligned} \frac{dx(t)}{dt} &= \Phi(x(t), y(t)) \frac{\partial H(x(t), y(t))}{\partial y}, \\ \frac{dy(t)}{dt} &= -\Phi(x(t), y(t)) \frac{\partial H(x(t), y(t))}{\partial x}. \end{aligned}$$

(i) Show that for any $t \geq 0$ we have

$$H(x(t), y(t)) = H(x(0), y(0)),$$

so that the trajectory $(x(t), y(t))$ stays on a fixed level set of the function $H(x, y)$.

(ii) Consider the 2×2 system of ODE

$$x'(t) = ax(t) - bx(t)y(t), \quad y'(t) = -cy(t) + dx(t)y(t),$$

where a, b, c, $d \in \mathbb{R}$ are positive constants. Find the functions $H(x, y)$ and $\Phi(x, y)$ so that this system has the form (2).

(iii) Show that for any $x(0) > 0$ and $y(0) > 0$ there exists $R > 0$ so that for all $t \geq 0$ we have

$$x^2(t) + y^2(t) \leq R^2.$$

Hint: parts (i) and (ii) may be useful.

5. (i) Consider the ODE

$$(3) \qquad \frac{dz_n(t; x_0)}{dt} = \frac{2}{\pi} \arctan(n z_n(t; x_0)), \;\; z_n(0; x_0) = x_0,$$

with some $x_0 \in \mathbb{R}$. Show that the limit

$$Z(t; x_0) = \lim_{n \to +\infty} z_n(t; x_0)$$

exists for each $t \geq 0$. Find this limit and show that $Z(t; x_0)$ is not continuous at $x_0 = 0$ for each $t > 0$ fixed.

(ii*) [If you have time, extra credit.] Consider the ODE

$$(4) \qquad \frac{dx_n(t; x_0)}{dt} = -\frac{2}{\pi} \arctan(n x_n(t; x_0)), \;\; x(0; x_0) = x_0,$$

with some $x_0 \in \mathbb{R}$, and $n \geq 1$, for an unknown function $x_n(t; x_0)$. Show that the limit

$$X(t; x_0) = \lim_{n \to +\infty} x_n(t; x_0)$$

exists for each $t \geq 0$. Find this limit and show that $X(t; x_0)$ is a continuous function of x_0 for each $t \geq 0$ fixed.

Hint: the answer in both parts would not change if we would replace $(2/\pi) \arctan nx$ by any function $g(nx)$ such that $g(y)$ is odd and increasing, with $\lim_{y \to +\infty} g(y) = 1$. Think about what the graph of the function $g(nx)$ looks like for a large n.