

Math 63CM: Practice problems for midterm 2

1. (i) Let A be an $n \times n$ complex matrix. Prove that every eigenvalue λ of A satisfies $|\lambda| \leq \|A\|_{op}$.
- (ii) Let all eigenvalues of a matrix A satisfy $|\lambda| \leq 2$. Show that for any initial condition x_0 there exists $C_0 > 0$ so that the solution to $x'(t) = Ax(t)$ satisfies $\|x(t)\| \leq C \cosh(2t)$ for all $t \in \mathbb{R}$.
2. Let A be an $n \times n$ complex matrix and $q(\lambda)$ be a polynomial with complex coefficients. Prove that μ is an eigenvalue of $q(A)$ if and only if $\mu = q(\sigma)$ where σ is an eigenvalue of A . Hint: First consider the case where A is an upper triangular matrix.
3. (i) Suppose A is a 2×2 complex-valued matrix such that all its eigenvalues have non-positive real parts, that is, every eigenvalue λ satisfies $\operatorname{Re}(\lambda) \leq 0$. Consider the ODE $x'(t) = Ax$ in \mathbb{C}^2 . Prove that there exists a constant $C > 0$ such that for all initial values $x(0) = x_0$ with $\|x_0\| < 1$, one has $\|x(t)\| \leq Ct$ for all $t \geq 1$.
- (ii) Give an example of a matrix A as in part (i) and an initial condition x_0 so that there exists a sequence $t_k \rightarrow +\infty$ such that $\|x(t_k)\|/t_k \geq 1$ for all $k \geq 1$.
4. (i) Assume all eigenvalues λ of an $n \times n$ complex matrix A have strictly negative real part. Let $x(t)$ be the solution to $x'(t) = Ax$, $x(0) = y$. Show that

$$A^{-1}y = \int_0^{\infty} x(t)dt.$$

- (ii) In the above setting, assume that all eigenvalues λ of A satisfy $\operatorname{Re}\lambda < -1$. Show that there exists $C > 0$ that does not depend on y so that

$$\|A^{-1}y - \int_T^{\infty} x(t)dt\| \leq C\|y\|e^{-T}.$$

5. (i) Let $f(t)$ be a periodic function with period 1: $f(t+1) = f(t)$. Assume all eigenvalues λ of an $n \times n$ complex matrix A have strictly negative real part. Show that there exists a periodic solution to $x'(t) = Ax(t) + f(t)$.
- (ii) Let $x(t)$ be the solution to $x'(t) = Ax + f(t)$, $x(0) = y$, with a given $y \in \mathbb{C}^n$. Show that there exist $C > 0$ that may depend on y , and $\alpha > 0$ that does not depend on y , so that

$$\|x(t+1) - x(t)\| \leq Ce^{-\alpha t}, \text{ for all } t > 0.$$

6. Consider the ODE $x'(t) = x(t)(1 - x(t))$, $x(0) = x_0$. Show that $x = 1$ is an asymptotically stable equilibrium, and $x = 0$ is an unstable equilibrium: given any $\varepsilon > 0$ there exists $T > 0$ that depends on x_0 , so that $x(t)$ is not in $(-\varepsilon, \varepsilon)$ for all $t \geq T$.