# Spring 2018 Math 63CM: Midterm 1 edited for 2020 material

Note: this was a one hour exam, the exam on Tuesday will be 1.5 hours.

1. Find the solution to
$$\begin{cases} x'(t) = 1 + (x(t))^2 \\ x(0) = 0 \end{cases},$$
where $x$ is a real-valued function. In addition, determine its maximal interval of existence.

2. Let $f_n\colon [0,1] \to \mathbb{R}$ be an equicontinuous sequence of functions, converging pointwise to a function $f$. Prove that the convergence is in fact uniform.

3. Consider the ODE
$$\begin{cases} x'(t) = F(x(t)) \\ x(0) = x_0 \end{cases},$$
where $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and $x_0 \in \mathbb{R}^n$.

   Let $H : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $(\nabla H(x) \cdot F(x)) = 0$ for all $x \in \mathbb{R}^n$. Here, $(v \cdot w) = v_1 w_1 + \cdots + v_n w_n$ is the standard inner product in $\mathbb{R}^n$:.

   (a) Let $x(t)$ be the unique solution to the initial value problem above on a time interval $[-s, s]$, with some $s > 0$. Prove that $H(x(t)) = H(x_0)$ for all $-s \leq t \leq s$.

   (b) Suppose that for every $E \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : H(x) = E\} \subset \mathbb{R}^n$ is compact. Prove that the solution exists for all $t \in \mathbb{R}$.

4. Let $F_k\colon \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of smooth functions such that
$$\sup_{k \in \mathbb{N}} \sup_{z \in \mathbb{R}^n} \|F_k(z)\| \leq 1.$$
Suppose $F_k \to F$ uniformly. Let $y_0 \in \mathbb{R}^n$ and $x_k\colon [0,1] \to \mathbb{R}^n$ be a sequence of continuous functions such that

   - $x_k'(t) = F_k(x_k(t))$ for $t \in (0,1)$,
   - $x_k(0) = y_0$.

   Prove that there is a subsequence $\{x_{k_\ell}(t)\}$ and a function $x\colon [0,1] \to \mathbb{R}^n$ such that $x_{k_\ell} \to x$ uniformly on $[0,1]$ as $\ell \to \infty$ and $x(t)$ satisfies

   - $x'(t) = F(x(t))$ for $t \in (0,1)$,
   - $x(0) = y_0$.