NOTES ON SIMPLIFICATION OF MATRICES

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These notes discuss how to "simplify" an $(n \times n)$ matrix. In particular, we expand on some of the material from the textbook (with some repetition). Part of the exposition is inspired by some notes of Eliashberg, which you may easily find on the internet. From the point of view of the larger context of the course, the goal is to have a way to compute e^A for a given matrix A, which is useful in solving linear ODEs. **I will likely be modifying these notes as I lecture. Meanwhile, if you have any comments or corrections, even very minor ones, please let me know**.

Before we proceed, let's start with a few observations:

(1) To compute e^A, it's useful to find some B which is *similar* to A, i.e., $B = S^{-1}AS$ for some S, such that e^B is easier to compute. This is because (**Exercise**)
$$e^A = Se^B S^{-1}.$$
(2) Now, in terms of computing exponentials, the easiest matrices are diagonal ones. If $\Lambda = diag(\lambda_1, \dots \lambda_n)$, then (**Exercise**)
$$e^\Lambda = diag(e^{\lambda_1}, \dots, e^{\lambda_n}).$$
Combining this with the first observation, we have an easy way to compute the exponentials of diagonalizable matrices (cf. Definition 1.4).
(3) Another class of matrices whose exponentials are easy to compute are *nilpotent* matrices. Here, we say that N is *nilpotent* if $N^k = 0$ for some $k \in \mathbb{N}$. For a nilpotent matrix, the power series that defines e^N becomes a finite sum and is therefore easier to compute.

Not all matrices are diagonalizable (Why? **Exercise**). The main result that we will get to is that all matrices can be decomposed as the sum of a diagonalizable matrix B and a nilpotent matrix N such that B and N commute. This gives a way of computing the exponential of any matrix.

Even if one is only interested in real matrices, as we will see, complex numbers enter naturally (cf. Remark 1.2). **From now on, we work with complex matrices and vectors.**

1. Some results on diagonalizability

We begin by recalling the following definitions from 61CM:

Definition 1.1 (Eigenvalues and eigenvectors)**.** Let A be an $(n \times n)$ matrix. We say that λ is an *eigenvalue* of A if there exists $v \neq 0$ such that
$$Av = \lambda v.$$
If λ is an eigenvalue, any v satisfying $Av = \lambda v$ (including 0) is called an *eigenvector*.

Remark 1.2. Some remarks are in order:

(1) λ is an eigenvalue iff $ker(\lambda I - A) \neq \{0\}$ iff $\det(\lambda I - A) = 0$.
(2) (Complex) eigenvalues always exist. This is because $\det(\lambda I - A) = 0$ is a polynomial in λ, which always has a root in $\mathbb{C}$ by the fundamental theorem of algebra. In fact, for an $(n \times n)$ matrix A, we always have n roots if we count multiplicity. This is the main reason we work over $\mathbb{C}$ instead of $\mathbb{R}$.

Definition 1.3 (Similarity)**.** Two $(n \times n)$ matrices A and B are similar if there exists an invertible $(n \times n)$ matrix S such that $S^{-1}AS = B$.

Definition 1.4 (Diagonalizability)**.** A matrix is *diagonalizable* if it is similar to a diagonal matrix.

One important (sufficient but not necessary) criterion for diagonalizability is given by the spectral theorem from 61CM:

Theorem 1.5 (Spectral theorem)**.** *A real symmetric matrix is diagonalizable. Moreover, given a real symmetric matrix A, one can find a real $(n \times n)$ matrix Q such that $Q^{-1}AQ$ is diagonal and $Q^TQ = I$.*

We will not repeat the proof here.

In general, we have the following characterization:

Lemma 1.6. *An $(n \times n)$ matrix A is diagonalizable if and only if $\mathbb{C}^n$ admits a basis which consists of eigenvectors of A.*

Proof. **"Only if"** Suppose A is diagonalizable. Then $S^{-1}AS = D$ for some diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and some invertible matrix S. Let $v_i \in \mathbb{C}^n, i = 1, \dots, n$, be the columns of S, i.e.,

$$S = \left[\begin{array}{c|c|c|c|c} v_1 & v_2 & \dots & \dots & v_n \end{array}\right].$$

Since S is invertible, $\{v_i\}_{i=1}^n$ forms a basis. Moreover,

$$\left[\begin{array}{c|c|c|c|c} Av_1 & Av_2 & \dots & \dots & Av_n \end{array}\right] = AS = SD = \left[\begin{array}{c|c|c|c|c} \lambda_1 v_1 & \lambda_2 v_2 & \dots & \dots & \lambda_n v_n \end{array}\right].$$

Hence, comparing each column, we have $Av_i = \lambda_i v_i$. In other words, the basis $v_1, \dots, v_n$ consists of eigenvectors.

"If" This is similar. Given a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A, define

$$S = \left[\begin{array}{c|c|c|c|c} v_1 & v_2 & \dots & \dots & v_n \end{array}\right].$$

Then a similar computation as the "only if" part shows that $S^{-1}AS$ is diagonal. $\square$

A corollary is the following (sufficient but not necessary) criterion for diagonalizability:

Corollary 1.7. *Suppose all the eigenvalues of an $(n \times n)$ matrix A are distinct, then A is diagonalizable.*

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the distinct eigenvalues and let $\{v_i\}_{i=1}^n \subset \mathbb{C}^n \setminus \{0\}$ be corresponding eigenvectors. We prove that $\{v_i\}_{i=1}^n$ is linearly independent by induction.

Base case Since $v_1 \neq 0$, $\{v_1\}$ is linearly independent.

Induction step Suppose $\{v_1, \dots v_{k-1}\}$ are linearly independent. Let

$$\sum_{i=1}^k c_i v_i = 0.$$

Applying A, we get

$$\sum_{i=1}^k c_i \lambda_i v_i = 0.$$

Multiply the first equation by λ_k and subtract the second equation from it. We obtain

$$\sum_{i=1}^{k-1} c_i(\lambda_k - \lambda_i)v_i = 0.$$

By linear independence of $\{v_1, \dots, v_{k-1}\}$, $c_i(\lambda_k - \lambda_i) = 0$ for all $i \in \{1, \dots, k-1\}$. Since $\lambda_k \neq \lambda_i$, we have $c_i = 0$ for all $i \in \{1, \dots, k-1\}$. This then implies also that $c_k = 0$. Hence, $\{v_1, \dots v_k\}$ are linearly independent.

By induction, $\{v_1, \dots v_n\}$ is linearly independent. It therefore forms a basis of $\mathbb{C}^n$. By Lemma 1.6, A is diagonalizable. $\square$

In general, not all matrices are diagonalizable (see homework 3 for an example). Nevertheless, we have Propositions 1.8 and Proposition 1.9 below.

Proposition 1.8. *Every matrix is similar to an upper triangular matrix.*

Proof. Let us prove this by induction on the size of the matrix. Clearly all 1×1 matrices are upper triangular.

Suppose that for some $k \geq 2$, every $(k-1) \times (k-1)$ matrix is similar to an upper triangular matrix.

Let A be an $k \times k$ matrix. By Remark 1.2, A has an eigenvector, say, x with eigenvalue λ. Complete x to a basis $\{x, y_2, \dots, y_k\}$. Consider now the matrix

$$S_1 = \left[\begin{array}{c|c|c|c|c} \\ x & y_2 & \dots & \dots & y_k \\ \\ \end{array} \right].$$

Then

$$S_1^{-1}AS_1 = \left[\begin{array}{c|ccc} \lambda & * & \dots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array} \right],$$

where B is a $(k-1) \times (k-1)$ matrix (which we do not know much about) and $*$ are entries that we have no information about. By the induction hypothesis, there exists a $(k-1) \times (k-1)$ matrix $\bar{S}_2$ such that $\bar{S}_2^{-1} B \bar{S}_2$ is upper triangular. Now let S_2 be a $k \times k$ matrix defined by

$$S_2 = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \bar{S}_2 & \\ 0 & & & \end{array} \right].$$

Let $S = S_1 S_2$. We compute

$$\begin{aligned} S^{-1}AS =& S_2^{-1}S_1^{-1}AS_1S_2 = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \bar{S}_2^{-1} & \\ 0 & & & \end{array} \right] \left[\begin{array}{c|ccc} \lambda & * & \dots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array} \right] \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \bar{S}_2 & \\ 0 & & & \end{array} \right] \\ =& \left[\begin{array}{c|ccc} \lambda & * & \dots & * \\ \hline 0 & & & \\ \vdots & & \bar{S}_2^{-1}B\bar{S}_2 & \\ 0 & & & \end{array} \right], \end{aligned}$$

which is an upper triangular matrix. $\square$

Proposition 1.9. *The set of diagonalizable matrices is dense in the set of all matrices.*

Proof. Given an $(n \times n)$ matrix A and an $\epsilon > 0$, we want to find a matrix B with $\|A - B\|_{op} < \epsilon$ such that B is diagonalizable.

To do this, first, we find S such that $S^{-1}AS$ is upper triangular. Then the eigenvalues of A are the diagonal entries of $S^{-1}AS$ (Why? **Exercise**). Now one can find an upper triangular matrix $\bar{B}$ by perturbing the diagonal entries of $S^{-1}AS$ such that $\|\bar{B} - S^{-1}AS\|_{op} < \frac{\epsilon}{\|S\|_{op}\|S^{-1}\|_{op}}$ and all the eigenvalues of $\bar{B}$ are distinct. Define $B = S\bar{B}S^{-1}$. We check that

- $\|A - B\|_{op} = \|SS^{-1}(A - B)SS^{-1}\|_{op} \leq \|S\|_{op}\|S^{-1}AS - \bar{B}\|_{op}\|S^{-1}\|_{op} < \epsilon$.
- B is diagonalizable: $\bar{B}$ is diagonalizable by Corollary 1.7. Since B is similar to $\bar{B}$, B is diagonalizable.

$\square$

2. Cayley–Hamilton theorem

Definition 2.1. Let $p(\lambda) = a_n\lambda^n + \cdots + a_0$ with $a_i \in \mathbb{C}$ for $i = 1, \ldots, n$ be a polynomial. Then for an $(n \times n)$ matrix A, $p(A)$ is the $(n \times n)$ matrix

$$p(A) = a_nA^n + \cdots + a_0I.$$

The following is a fundamental result:

Theorem 2.2 (Cayley–Hamilton)**.** *Let $\chi_A(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of A. Then $\chi_A(A) = 0$.*

Proof. **Step 1.** First, this is true for diagonal matrices. To see that, let A be a diagonal matrix,

$$A = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{bmatrix}.$$

It is easy to compute that $\chi_A(\lambda) = \Pi_{i=1}^n(\lambda - \lambda_i)$. We now check

$$\begin{aligned} &\chi_A(A) = \Pi_{i=1}^n(A - \lambda_iI) \\ &= \begin{bmatrix} 0 & & & \\ & \lambda_2 - \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_2 \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 - \lambda_n & & & \\ & \lambda_2 - \lambda_n & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot (\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) & & & \\ & (\lambda_2 - \lambda_1) \cdot 0 \cdots (\lambda_2 - \lambda_n) & & \\ & & \ddots & \\ & & & (\lambda_n - \lambda_1) \cdot (\lambda_n - \lambda_2) \cdots 0 \end{bmatrix} = 0. \end{aligned}$$

Step 2. We claim that a consequence of Step 1 is that $\chi_A(A) = 0$ for all diagonalizable A. To see this, suppose $S^{-1}AS = D$ for some diagonal matrix D. Then

$$\det(\lambda I - A) = \det(S^{-1})\det(\lambda I - A)\det(S) = \det(S^{-1}(\lambda I - A)S) = \det(\lambda I - D).$$

Hence $\chi_A(\lambda) = \chi_D(\lambda)$. Now suppose

$$\chi_A(\lambda) = \chi_D(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Then

$$\begin{aligned} \chi_A(A) =& A^n + a_{n-1}A^{n-1} + \cdots + a_0I = SS^{-1}(A^n + a_{n-1}A^{n-1} + \cdots + a_0I)SS^{-1} \\ =& S\left((S^{-1}AS)^n + a_{n-1}(S^{-1}AS)^{n-1} + \cdots + a_0I\right)S^{-1} \\ =& S\chi_D(D)S^{-1}. \end{aligned}$$

By Step 1, $\chi_D(D) = 0$. Hence, $\chi_A(A) = 0$.

Step 3. $\chi_A(A)$ depends on A continuously. Therefore, the conclusion of the theorem follows from the density of diagonalizable matrices (Proposition 1.9). $\square$

3. Generalized eigenspaces

The Cayley–Hamilton theorem gives us a way to decompose $\mathbb{C}^n$ into generalized eigenspaces.

Definition 3.1. Let A be an $(n \times n)$ matrix. We say that v is a generalized eigenvector if there exists an eigenvalue λ and a $k \in \mathbb{N}$ such that $v \in ker(\lambda I - A)^k$.

Definition 3.2. Let A be an $(n \times n)$ matrix and λ be an eigenvalue. The generalized eigenspace associate to λ is given by

$$\{v \in \mathbb{C}^n : (\lambda I - A)^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

Definition 3.3. Given two polynomials f and g, we say that h is a greatest common divisor of f and g if h is a non-zero polynomial of maximal degree which divides both f and g.

Proposition 3.4 (Bézout's identity)**.** *Let f, g be polynomials. Then there exists a greatest common divisor which is unique up to multiplication by constants. Moreover, for any greatest common divisor h, there exists polynomials p and q such that*

$$pf + qg = h.$$

Proof. Consider the set $S = \{pf + qg : p, q \text{ polynomials}\}$. Let $h = p_h f + q_h g$ be a non-zero polynomial in S with minimal degree. We claim that h divides any polynomial in S.

Suppose not. Then there are some p_d and q_d such that

$$p_d f + q_d g = sh + r$$

for some polynomials s and r, with $r \neq 0$ and $deg(r) < deg(h)$. But then

$$r = (p_d - p_h s)f + (q_d - q_h s)g$$

is a non-zero element in S with a smaller degree than h, which is a contradiction.

We now claim that h is a greatest common divisor of f and g. Suppose c is a common divisor of f and g, say $f = cp_c$ and $g = cq_c$ for some polynomials p_c and q_c. Then

$$h = p_h f + q_h g = p_h p_c c + q_h q_c c = (p_h p_c + q_h q_c)c.$$

Hence, c divides h. In other words, any other common divisor of f and g must divide h. This shows that h is a greatest common divisor. The same argument also shows the uniqueness of the greatest common divisor up to multiplicative constant.

Finally, Bézout's identity follows from taking $p = p_h$ and $q = q_h$. $\square$

Definition 3.5. We denote by (f, g) the unique monic (i.e., with leading coefficient 1) polynomial which is a greatest common divisor of f and g.

Theorem 3.6. *Let V be a finite dimensional vector space, $A : V \to V$ be a linear map and f, g be polynomials such that $(f, g) = 1$ and $f(A)g(A) = 0$. Then*

$$V = ker(f(A)) \oplus ker(g(A)),$$

i.e., for every $v \in V$, there exist unique $x \in ker(f(A))$ and $y \in ker(g(A))$ such that $v = x + y$.

Proof. By Proposition 3.4, there exist polynomials p and q such that $pf + qg = 1$. Now given $v \in V$, define

$$x = q(A)g(A)v, \quad y = p(A)f(A)v.$$

By definition,

$$v = Iv = (p(A)f(A) + q(A)g(A))v = y + x$$

and

$$f(A)x = q(A)f(A)g(A)v = 0, \quad g(A)y = p(A)f(A)g(A)v = 0.$$

Notice that we have used that any two polynomials of A commute. We have thus shown that for every $v \in V$, there exist $x \in ker(f(A))$ and $y \in ker(g(A))$ such that $v = x + y$.

Finally, suppose $v = x + y = x' + y'$, where x, $x' \in ker(f(A))$ and y, $y' \in ker(g(A))$ Then $x - x' = y - y' \in ker(f(A)) \cap ker(g(A))$. Then

$$x - x' = (f(A)p(A) + g(A)q(A))(x - x') = 0.$$

Hence, $x = x'$ and consequently $y = y'$. $\square$

Iterating this, we have

Corollary 3.7. *Let V be a finite dimensional vector space, $A : V \to V$ be a linear map and $f = f_1 \times \cdots \times f_k$, where*

- *f_i are polynomials with $(f_i, f_j) = 1$ whenever $i \neq j$,*
- *$f(A) = 0$.*

Then

$$V = ker(f_1(A)) \oplus \cdots \oplus ker(f_k(A)).$$

In particular, using Cayley–Hamilton theorem, we can decompose $\mathbb{C}^n$ into generalized eigenspaces:

Corollary 3.8. *Let A be an $(n \times n)$ matrix and suppose the characteristic polynomial can be written as*

$$det(\lambda I - A) = \Pi_{i=1}^k (\lambda - \lambda_i)^{\nu_i},$$

where the λ_i's are distinct and $\sum_{i=1}^k \nu_i = n$. Then

$$\mathbb{C}^n = ker(A - \lambda_1 I)^{\nu_1} \oplus ker(A - \lambda_2 I)^{\nu_2} \oplus \cdots \oplus ker(A - \lambda_k I)^{\nu_k}. \tag{3.1}$$

This also implies the following:

Corollary 3.9. *$\mathbb{C}^n$ admits a basis of generalized eigenvectors.*

4. Decomposition into diagonalizable and nilpotent matrices

Theorem 4.1. *Let A be an $(n \times n)$ matrix. Then there exist $(n \times n)$ matrices L and N such that*

(1) $A = L + N$,
(2) L is diagonalizable,
(3) N is nilpotent, in fact, $N^n = 0$,
(4) $LN = NL$.

Proof. Recall that $\mathbb{C}^n$ admits a decomposition (3.1). Define L so that for any $v \in ker(A - \lambda_i I)^{\nu_i}$, $Lv = \lambda_i v$. It is easy to check that L is linear and therefore is given by a matrix.

We now check the following claims:

Claim 1: L is diagonalizable. By definition $\mathbb{C}^n$ admits a basis of eigenvectors of L. Hence, L is diagonalizable by Lemma 1.6.

Claim 2: $A - L$ is nilpotent. We will show that for $\nu = \max\{\nu_1, \ldots, \nu_k\}$, $(A - L)^\nu v = 0$ for all $v \in \mathbb{C}^n$. This then implies $(A - L)^\nu = 0$. Since $\nu \leq n$, we have $N^n = 0$.

By (3.1), it suffices to consider $v \in ker(A - \lambda_i I)^{\nu_i}$ for some i. By definition of L, $(A - L)v = (A - \lambda_i I)v$. Now clearly $(A - \lambda_i I)v \in ker(A - \lambda_i I)^{\nu_i}$, so we have $(A - L)^2 v = (A - \lambda_i I)^2 v$. Iterating this, we have $(A - L)^\nu = (A - \lambda_i I)^\nu v = 0$ since $\nu \geq \nu_i$ and $v \in ker(A - \lambda_i I)^{\nu_i}$.

Claim 3: $N = A - L$ and L commute. As before, by (3.1), it suffices to check that $NLv = LNv$ for $v \in ker(A - \lambda_i I)^{\nu_i}$ for some i. Now

$$NLv = (A - \lambda_i I)\lambda_i v = \lambda_i (A - \lambda_i I)v = LNv.$$

□

Theorem 4.2. *The decomposition in Theorem 4.1 is unique.*

Proof. Recall that $\mathbb{C}^n$ admits a decomposition (3.1). We want to prove that if $A = L + N$, where L is diagonalizable, $N^k = 0$ for some $k \in \mathbb{N}$ and $NL = LN$, then $Lv = \lambda_i v$ for any $v \in ker(A - \lambda_i I)^{\nu_i}$.

We compute, for $v \in ker(A - \lambda_i I)^{\nu_i}$, that

$$\begin{aligned}(L - \lambda_i I)^{n+k} v =& (A - \lambda_i I - N)^{2n} v \\ =& \sum_{j=0}^{2n} \frac{(n+k)!}{j!(n+k-j)!} (-N)^j (A - \lambda_i I)^{n+k-j} v \\ =& \sum_{j=0}^{n-1} \frac{(n+k)!}{j!(n+k-j)!} (-N)^j (A - \lambda_i I)^{n+k-j} v \\ =& 0.\end{aligned}$$

Here, we have used

(1) N and A commute since $NA = N(L+N) = NL + N^2 = LN + N^2 = (L+N)N = AN$,
(2) $N^k = 0$,
(3) Since $(A - \lambda_i I)^{\nu_i} v = 0$ and $\nu_i \leq n$, we have $(A - \lambda_i I)^{\ell} v = 0$ for all $\ell \geq n$.

Now, $(L - \lambda_i I)^{n+k} v = 0$. Since L is diagonalizable, $(L - \lambda_i I)v = 0$. This is what we want to show. □

5. Example

Consider

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Eigenvalues We first find its eigenvalues.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & -1 & 0 \\ 1 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 1 & \lambda \end{bmatrix} = \lambda\lambda(\lambda^2+1) - 1(-1)(\lambda^2+1) = (\lambda^2+1)^2.$$

Therefore, the eigenvalues are $\pm i$, each with multiplicity 2.

Generalized eigenvectors We now find the generalized eigenvectors.

$$iI - A = \begin{bmatrix} i & -1 & -1 & 0 \\ 1 & i & 0 & -1 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{bmatrix}$$

$$(iI - A)^2 = \begin{bmatrix} i & -1 & -1 & 0 \\ 1 & i & 0 & -1 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{bmatrix} \begin{bmatrix} i & -1 & -1 & 0 \\ 1 & i & 0 & -1 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{bmatrix} = \begin{bmatrix} -2 & -2i & -2i & 2 \\ 2i & -2 & -2 & -2i \\ 0 & 0 & -2 & -2i \\ 0 & 0 & 2i & -2 \end{bmatrix}$$

By inspection,

$$ker((iI - A)^2) = span \left\{ \begin{bmatrix} 0 \\ 0 \\ i \\ -1 \end{bmatrix}, \begin{bmatrix} i \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{5.1}$$

Similarly,

$$-iI - A = \begin{bmatrix} -i & -1 & -1 & 0 \\ 1 & -i & 0 & -1 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix}.$$

We compute

$$(-iI - A)^2 = \begin{bmatrix} -i & -1 & -1 & 0 \\ 1 & -i & 0 & -1 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} \begin{bmatrix} -i & -1 & -1 & 0 \\ 1 & -i & 0 & -1 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} = \begin{bmatrix} -2 & -2i & 2i & 2 \\ -2i & -2 & -2 & 2i \\ 0 & 0 & -2 & 2i \\ 0 & 0 & -2i & -2 \end{bmatrix}.$$

By inspection,

$$ker((-iI - A)^2) = span \left\{ \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{5.2}$$

It is easy to check that indeed

$$\mathbb{C}^n = ker((iI - A)^2) \oplus ker((-iI - A)^2).$$

We now decompose A into its diagonalizable and nilpotent parts. Define L according to the proof of Theorem 4.1. We check that

$$L\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=(-\frac{i}{2})L\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}-\frac{i}{2}L\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=(-\frac{i}{2})i\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}-(\frac{i}{2})(-i)\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\\0\end{bmatrix},$$

$$L\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=(-\frac{1}{2})L\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}+\frac{1}{2}L\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=-(\frac{1}{2})i\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}+(\frac{1}{2})(-i)\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},$$

$$L\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=(-\frac{i}{2})L\begin{bmatrix}0\\0\\i\\-1\end{bmatrix}(-\frac{i}{2})L\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=-(\frac{i}{2})i\begin{bmatrix}0\\0\\i\\-1\end{bmatrix}-(\frac{i}{2})(-i)\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-1\end{bmatrix},$$

$$L\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=-\frac{1}{2}L\begin{bmatrix}0\\0\\i\\-1\end{bmatrix}+\frac{1}{2}L\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=-(\frac{1}{2})i\begin{bmatrix}0\\0\\i\\-1\end{bmatrix}+(\frac{1}{2})(-i)\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}.$$

Therefore,

$$L=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}.$$

We check that

$$N=A-L=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

which is obviously nilpotent.

Taking exponential using the previous decomposition

L is diagonalizable. More precisely, given the above computations, we know that for

$$S=\begin{bmatrix}0&i&0&i\\0&-1&0&1\\i&0&i&0\\-1&0&1&0\end{bmatrix},$$

we have

$$S^{-1}LS=\begin{bmatrix}i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-i\end{bmatrix}.$$

We check that

$$S^{-1}=\begin{bmatrix}0&0&-\frac{i}{2}&-\frac{1}{2}\\-\frac{i}{2}&-\frac{1}{2}&0&0\\0&0&-\frac{i}{2}&\frac{1}{2}\\-\frac{i}{2}&\frac{1}{2}&0&0\end{bmatrix}.$$

We check that

$$
\begin{aligned}
S^{-1}LS &= \begin{bmatrix} 0 & 0 & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \\ i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & -i & 0 & -i \\ -1 & 0 & 1 & 0 \\ -i & 0 & -i & 0 \end{bmatrix}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
e^{Lt} =& S \begin{bmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{-it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{bmatrix} S^{-1} \\
=& \begin{bmatrix} 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \\ i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{-it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \\
=& \begin{bmatrix} 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \\ i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{i}{2}e^{it} & -\frac{1}{2}e^{it} \\ -\frac{i}{2}e^{it} & -\frac{1}{2}e^{it} & 0 & 0 \\ 0 & 0 & -\frac{i}{2}e^{-it} & \frac{1}{2}e^{-it} \\ -\frac{i}{2}e^{-it} & \frac{1}{2}e^{-it} & 0 & 0 \end{bmatrix} \\
=& \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix}.
\end{aligned}
$$

Also, since

$$
N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0,
$$

we have,

$$
e^{tN} = I + tN = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Putting all these together

$$
e^{tA} = e^{tL}e^{tN} = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix}.
$$

6. Jordan canonical form

It is useful to further analyse nilpotent matrices. This will also lead us to the notion of Jordan canonical forms, which gives us another way to compute exponentials of matrices.

We begin with the analysis of nilpotent matrices. One example of nilpotent matrices is the following $m \times m$ matrix:

$$J_m := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}.$$

An easy computation shows that

Lemma 6.1.

$$J_m^m = 0.$$

Proposition 6.2. *Let N be an $(n \times n)$ nilpotent matrix. Then N is similar to a matrix J, which is a block diagonal matrix with blocks $J_{n_1+1}, \dots, J_{n_k+1}$, where $(n_1+1) + \dots + (n_k+1) = n$.*

Proof. It suffices to find a basis $v_1, Nv_1, \dots, N^{n_1}v_1, v_2, Nv_2, \dots, N^{n_2}v_2, \dots, v_k, \dots, N^{n_k}v_k$, where

$$N^{n_1+1}v_1 = N^{n_2+1}v_2 = \dots = N^{n_k+1}v_k = 0. \tag{6.1}$$

Then for

$$S = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} N^{n_1}v_1 & \dots & Nv_1 & v_1 & N^{n_2}v_2 & \dots & v_2 & \dots & N^{n_k}v_k & \dots & v_k \end{array} \right],$$

$S^{-1}AS$ is in the desired form.

Step 1: Choosing v_1. Let v_1 be a non-zero vector. Since N is nilpotent, there exists $n_1 \in \mathbb{N} \cup \{0\}$ such that $N^{n_1}v_1 \neq 0$ and $N^{n_1+1}v_1 = 0$. We claim that $v_1, Nv_1, \dots, N^{n_1}v_1$ are linearly independent. We prove this by induction. First $N^{n_1}v_1$ is linearly independent since it is non-zero. Now, suppose $N^{n_1}v_1, N^{n_1-1}v_1, \dots, N^{n_1-j}v_1$ are linearly independent. Suppose

$$c_{n_1-j-1}N^{n_1-j-1}v_1 + c_{n_1-j}N^{n_1-j}v_1 + \dots + c_{n_1}N^{n_1}v_1 = 0. \tag{6.2}$$

Multiplying by N on the left, we have

$$c_{n_1-j-1}N^{n_1-j}v_1 + c_{n_1-j}N^{n_1-j+1}v_1 + \dots + c_{n_1-1}N^{n_1}v_1 = 0,$$

since $N^{n_1+1}v_1 = 0$. By linear independence of $N^{n_1}v_1, N^{n_1-1}v_1, \dots, N^{n_1-j}v_1$, we have

$$c_{n_1-j-1} = c_{n_1-j}N^{n_1-j+1} = \dots = c_{n_1-1} = 0.$$

Using (6.2), we also have $c_{n_1} = 0$. Hence, $N^{n_1}v_1, N^{n_1-1}v_1, \dots, N^{n_1-j}v_1, N^{n_1-j-1}v_1$ are linearly independent. The claim follows from induction.

If $v_1, Nv_1, \dots, N^{n_1}v_1$ form a basis, we are done. Otherwise, proceed to Step 2.

Step 2. Choosing v_2. Take v_2 so that $v_1, Nv_1, \dots, N^{n_1}v_1, v_2$ are linearly independent. Take n_2 such that $N^{n_2}v_2 \neq 0$ but $N^{n_2+1}v_2 = 0$. Without loss of generality, assume $n_2 \leq n_1$ (otherwise, relabel).

There are now two cases: either

(1) $N^{n_1}v_1$ and $N^{n_2}v_2$ are linearly independent; or
(2) $N^{n_1}v_1 = aN^{n_2}v_2$ for some $a \in \mathbb{C} \setminus \{0\}$.

In case (1), we claim that $v_1, \dots, N^{n_1}v_1, v_2, \dots, N^{n_2}v_2$ are linearly independent. We will prove inductively in j that $v_1, \dots, N^{n_1}v_1, N^{n_2-j}v_2, \dots, N^{n_2-j}v_2, \dots, N^{n_2}v_2$ are linearly independent. We already know from above that $v_1, \dots, N^{n_1}v_1$ are linearly independent. Suppose that $v_1, \dots, N^{n_1}v_1, N^{n_2-j}v_2, \dots, N^{n_2-j}v_2, \dots, N^{n_2}v_2$ are linearly independent for some $j \geq -1$ (with $j = -1$ meaning that none of the v_2 terms are present). Let

$$c_1v_1 + \dots + c_{n_1}N^{n_1}v_1 + d_{n_2-j-1}N^{n_2-j-1} + d_{n_2-j}N^{n_2-j} + \dots + d_{n_2}N^{n_2}v_2 = 0,$$

Applying N and using the induction hypothesis, we have $c_1 = \dots = c_{n_1-1} = 0$, $d_{n_2-j-1} = \dots = d_{n_2-1} = 0$. This implies

$$c_{n_1}N^{n_1}v_1 + d_{n_2}N^{n_2}v_2 = 0.$$

Since $N^{n_1}v_1$ and $N^{n_2}v_2$ are linearly independent, $c_{n_1} = d_{n_2} = 0$. Hence,

$$v_1, \dots, N^{n_1}v_1, N^{n_2-j-1}v_2, \dots, N^{n_2-j}v_2, \dots, N^{n_2}v_2$$

are linearly independent. This completes the induction.

In case (2), we let $v_2' = v_2 - a^{-1}N^{n_1-n_2}v_1$. Notice that this is still linearly independent with $v_1, \dots v_{n_1}$ and there exists $n_2' < n_2$ such that $N^{n_2}v_2' \neq 0$ and $N^{n_2'}v_2' = 0$. Now we consider the following cases, either either

(1') $N^{n_1}v_1$ and $N^{n_2'}v_2'$ are linearly independent; or
(2') $N^{n_1}v_1 = a'N^{n_2'}v_2'$ for some $a \in \mathbb{C} \setminus \{0\}$.

In case (1'), we then argue as above to show that $v_1, \dots, N^{n_1}v_1, v_2', \dots, N^{n_2'}v_2'$ are linearly independent. In case (2'), we then define $v_2'' = v_2' - a'^{-1}N^{n_1-n_2'}v_2'$ and then repeat this process. Since $n_2 > n_2' > \dots$, this process must come to an end. At the end, we will obtain ${v_2}'^{\cdots\prime}$ such that $v_1, \dots, N^{n_1}v_1, {v_2}'^{\cdots\prime}, \dots, N^{{n_2}'^{\cdots\prime}}{v_2}'^{\cdots\prime}$ are linearly independent. We now abuse notation to rename ${v_2}'^{\cdots\prime}$ and ${n_2}'^{\cdots\prime}$ as v_2 and n_2.

Now if $v_1, \dots, N^{n_1}v_1, v_2, \dots, N^{n_2}v_2$ form a basis, we are done. Otherwise, we continue this process by finding v_3, v_4, etc., and obtain a basis. □

Remark 6.3. Let us note that the proof above also suggests an *algorithm* for finding a basis

$$v_1, Nv_1, \dots, N^{n_1}v_1, v_2, Nv_2, \dots, N^{n_2}v_2, \dots, v_k, \dots, N^{n_k}v_k$$

satisfying (6.1).

Definition 6.4. The following $(m \times m)$ matrix is known as a Jordan block

$$\lambda I + J_m := \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}.$$

Definition 6.5. A matrix is said to be in *Jordan canonical form* if it is a block diagonal matrix with Jordan blocks.

Corollary 6.6. *Any $(n \times n)$ matrix A is similar to a matrix in Jordan canonical form.*

Proof. We pick a basis of generalized eigenvectors as follows. First recall (3.1). Since $A - \lambda_i I$ is nilpotent for any $\ker((A - \lambda_i I)^{\nu_i})$, by Proposition 6.2, we can find a basis of $\ker((A - \lambda_i I)^{\nu_i})$ which takes the form

$$v_{i,1}, (A - \lambda_i I)v_{i,1}, \dots, (A - \lambda_i I)^{n_{i,1}}v_{i,1}, \dots, v_{i,k_i}, (A - \lambda_i I)v_{i,k_i}, \dots, (A - \lambda_i I)^{n_{i,k_i}}v_{i,k_i}.$$

By (3.1), putting all these bases together for all generalized eigenspaces $\ker((A - \lambda_i I)^{\nu_i})$, we obtain a basis of $\mathbb{C}^n$. Finally, defining S by

$$S = \left[\begin{array}{c|c|c|c|c|c|c} (A - \lambda_1 I)^{n_{1,1}}v_{1,1} & \dots & v_{1,1} & \dots & (A - \lambda_m I)^{n_{m,k_m}}v_{m,k_m} & \dots & v_{m,k_m} \end{array} \right],$$

one checks that $S^{-1}AS$ is in the Jordan canonical form.

□

6.1. **Exponentials of matrices in Jordan canonical form.** First, we note that

$$e^{J_m t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!} \\ \vdots & \ddots & \ddots & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

Take a Jordan block $\lambda I + J_m$. Notice that $\lambda I J_m = J_m \lambda I$. Hence,

$$e^{(\lambda I + J_m)t} := \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \dots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda t} \\ \vdots & \ddots & \ddots & e^{\lambda t} & te^{\lambda t} \\ 0 & \dots & \dots & 0 & e^{\lambda t} \end{bmatrix}.$$

6.2. **Example with Jordan canonical form.** Let us go back to the example in Section 5 and find the Jordan canonical form of A. Let us first find a basis of $\mathbb{C}^n$ following the algorithm suggested in the proof of Proposition 6.2. Take $\begin{bmatrix} 0 \\ 0 \\ i \\ -1 \end{bmatrix}$, an element in the generalized eigenspace in (5.1). We compute

$$(A - iI)\begin{bmatrix} 0 \\ 0 \\ i \\ -1 \end{bmatrix} = \begin{bmatrix} -i & 1 & 1 & 0 \\ -1 & -i & 0 & 1 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ i \\ -1 \end{bmatrix} = \begin{bmatrix} i \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Now consider the vector $\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$ in the generalized eigenspace in (5.2). We compute

$$(A + iI)\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 1 & 1 & 0 \\ -1 & i & 0 & 1 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, for S defined as

$$S = \begin{bmatrix} i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \\ 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

we have

$$S^{-1}AS = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Let us note that this gives another way to compute e^{At}. First, we compute S^{-1}:

$$S^{-1} = \begin{bmatrix} -\frac{i}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \end{bmatrix}.$$

Then, we have

$$
\begin{aligned}
e^{At} =& S \begin{bmatrix} e^{it} & t & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{-it} & t \\ 0 & 0 & 0 & e^{-it} \end{bmatrix} S^{-1} \\
=& \begin{bmatrix} i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \\ 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{it} & te^{it} & 0 & 0 \\ 0 & e^{it} & 0 & 0 \\ 0 & 0 & e^{-it} & te^{-it} \\ 0 & 0 & 0 & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \end{bmatrix} \\
=& \begin{bmatrix} i & 0 & i & 0 \\ -1 & 0 & 1 & 0 \\ 0 & i & 0 & i \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{i}{2}e^{it} & -\frac{1}{2}e^{it} & -\frac{it}{2}e^{it} & -\frac{t}{2}e^{it} \\ 0 & 0 & -\frac{i}{2}e^{it} & -\frac{1}{2}e^{it} \\ -\frac{i}{2}e^{-it} & \frac{1}{2}e^{-it} & -\frac{it}{2}e^{-it} & \frac{t}{2}e^{-it} \\ 0 & 0 & -\frac{i}{2}e^{-it} & \frac{1}{2}e^{-it} \end{bmatrix} \\
=& \begin{bmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix}.
\end{aligned}
$$