

Math 63CM Section 5 Handout

February 7, 2020

1. DIAGONALIZING MATRICES

Exercise 1.1. Consider the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \quad (1.1)$$

is it diagonalizable? Why? Diagonalize it! More precisely, find an invertible matrix Q and a diagonal matrix D such that

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = QDQ^{-1}.$$

Exercise 1.2. Consider the matrix

$$\begin{pmatrix} 1 & 9 \\ 1 & 1 \end{pmatrix}; \quad (1.2)$$

diagonalize this matrix. Do the same for the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 7 \\ 0 & 6 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \quad (1.3)$$

Exercise 1.3. Something a bit more "abstract", but not quite, because you'll need to do similar calculations!

Consider the matrix

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}; \quad (1.4)$$

is this diagonalizable? Why or why not? What about the matrix $a\text{Id} + T$ with T a nonzero strictly upper triangular matrix (nonzero upper triangular matrix with zero diagonal entries)?

2. FINDING THE UPPER TRIANGULAR MATRIX

There are two meanings to finding an "upper triangular form" of a matrix. The first is the one we've been talking about in this class so far.

Exercise 2.1. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix}. \quad (2.1)$$

Find an invertible matrix Q and an upper triangular matrix B such that $A = QBQ^{-1}$. We'll show you one algorithm given by the following steps.

- Show the characteristic polynomial of A is given by $p_A(\lambda) = (\lambda - 2)^3$; deduce that 2 is the only eigenvalue of A .
- Show that any eigenvector of A with eigenvalue of 2 is given by a scalar multiple of $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. For the following step, it will only suffice to know that this vector is an eigenvector, not that it's unique up to scalar multiplication, but this is a good exercise anyway.

- Find any two vectors in \mathbb{R}^3 such that these two vectors combined with $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ form a basis of \mathbb{R}^3 . Call these vectors v_2, v_3 , and define the matrix $Q = (v_1 \ v_2 \ v_3)$. Why is Q invertible? Show $Q^{-1}AQ$ has its first column a scalar multiple of the first basis vector.

To be concrete, one can choose $v_2 = (1 \ -1 \ 0)$ and $v_3 = (0 \ 0 \ 1)$, and then we have

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.2)$$

Find Q^{-1} ; it should be given by

$$Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.3)$$

How did I find this?

Next, we should have

$$Q^{-1}AQ = \begin{pmatrix} 2 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & 3 \end{pmatrix}. \quad (2.4)$$

- This is not quite what we want, because this last matrix is not upper triangular. However, it is promising because we see the 2 in the first entry. Call this matrix B_1 , and define

$$A_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ -2 & 3 \end{pmatrix}. \quad (2.5)$$

We'll want to do the same with A_2 . In particular, if we can find Q_2 such that $Q_2^{-1}A_2Q_2$ is upper triangular, then upon defining

$$Q'_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q_2(1,1) & Q_2(1,2) \\ 0 & Q_2(2,1) & Q_2(2,2) \end{pmatrix}, \quad (2.6)$$

convince yourself that $[Q'_2]^{-1}B_1Q'_2$ is upper triangular. This is what we want!

- So let's put A_2 in upper triangular form. Unsurprisingly, its characteristic polynomial is $(\lambda-2)^2$, and an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. So let $Q_2 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, and show that $Q_2^{-1}A_2Q_2$ is upper triangular.

Exercise 2.2. The other upper triangular matrix one can get is through row reduction, or Gaussian elimination. For example, consider the matrix $\begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Show it's row reduced form is $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Compare the determinants of the two matrices.

In general, row reduction does not preserve general quantities associated to matrices (compare the characteristic polynomials of the previous two matrices, and even the traces). But it's good for determinants! You just have to keep track of how often you swap rows/columns (even or odd number of times?) and avoid replacing any row/column by its scalar multiple! Or if you do this replacement, keep track of which scalars you're multiplying rows/columns by.