Math 63CM Section 4 Handout

January 30, 2020

This handout is a collection of supplementary material to lecture; part of it is supposed to directly supplement lecture material, and the other part is supposed to introduce you to some more related mathematics. This handout is presented through exercises which range from easy to difficult. If you find any mistakes, let Kevin know!

1. Linear Algebra Review

Definition 1.1. Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, the kernel and image of T are defined by

$$\ker(T) = \{v \in \mathbb{R}^n : T(v) = 0\}, \quad \mathrm{Im}(T) = \{w \in \mathbb{R}^m : \exists v \in \mathbb{R}^n : T(v) = w\} = T(\mathbb{R}^n). \tag{1.1}$$

Exercise 1.2. Show that $\ker(T) \subseteq \mathbb{R}^n$ and $\mathrm{Im}(T) \subseteq \mathbb{R}^m$ are subspaces.

Exercise 1.3. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$. Show T is invertible if and only if $\ker(T) = 0$ if and only if $\mathrm{Im}(T) = \mathbb{R}^n$, and that $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is linear.

Definition 1.4. Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to a matrix $A = (a_{ij})_{i,j=1}^n$, then the trace and determinant are defined to be

$$\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii}, \quad \det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}. \tag{1.2}$$

Exercise 1.5. Show that for any two $n \times n$ matrices, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Deduce that no pair of matrices can satisfy $AB - BA = \mathrm{Id}$.

Exercise 1.6. Suppose A has a column (or row) consisting of only 0s. Show that $\det(A) = 0$.

Exercise 1.7. It's a fact that the determinant of A changes sign whenever two rows or two columns are swapped. It's also a fact that $\det(AB) = \det(A)\det(B)$.

Moreover, if we multiply any individual row or column by a constant $c \in \mathbb{R}$, then the entire determinant is scaled by c. Also, if we replace any column by adding any linear combination of the other columns, then the determinant does not change.

Lastly, it's a fact that $\det(A) = 0$ if and only if A is not invertible (use the previous pair of facts and the previous exercise to show this).

Exercise 1.8. Show that if A is a diagonal matrix, that $\det(A)$ is the product of its diagonal entries.

Definition 1.9. An eigenvalue of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is any $\lambda \in \mathbb{R}$ for which $T(v) = \lambda v$ for some nonzero vector $v \in \mathbb{R}^n$. The vector v is a λ-eigenvector if λ is an eigenvalue of T and $T(v) = \lambda v$, and the set of all λ-eigenvectors is the λ-eigenspace of T. The set of all eigenvalues of T is its spectrum.

Exercise 1.10. Convince yourself that $\ker(T) = 0$ if and only if λ is not an eigenvalue of T.

Show that the spectrum of T consists of exactly those λ such that $\ker(\lambda \mathrm{Id} - T)$ is a nonzero subspace.

Exercise 1.11. Show that the λ-eigenspace of a linear map T is a subspace of $\mathbb{R}^n$ for every eigenvalue of T.

Definition 1.12. The characteristic polynomial of T is the function $p(\lambda) = \det(\lambda \mathrm{Id} - A)$.

Exercise 1.13. Show that λ_0 is an eigenvalue of A if and only if $p(\lambda_0) = 0$. In particular, the roots of $p(\lambda)$ form exactly the spectrum of A.

Theorem 1.14. *If A is real symmetric, then A is diagonalizable, i.e. for an orthogonal matrix O and a diagonal matrix D, we have $A = O^T DO$. The eigenvalues of A are the diagonal entries of D, and the i-th column of O is an eigenvector of A with eigenvalue λ_i, the i-th diagonal entry of D. Moreover, the eigenvalues λ_i are real, and the eigenvectors have real entries.*

Exercise 1.15. Show that $A = Q^{-1}DQ$ for some matrix Q is equivalent to $p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$ where $\lambda_1, \ldots, \lambda_n$ are the diagonal entries of D.

Exercise 1.16. Find a counterexample to the spectral theorem above if A is not assumed to be symmetric.

Exercise 1.17. Suppose that $A = Q^{-1}DQ$ for some matrix Q and diagonal matrix D. Show that $\mathrm{Tr}(A) = \sum_{i=1}^n \lambda_i$ and $\det(A) = \prod_{i=1}^n \lambda_i$.

Exercise 1.18. Suppose A is real symmetric. Show that any pair of eigenvectors with distinct eigenvalues are orthogonal.

Exercise 1.19. Recall the exponential of a matrix A is defined as $\mathrm{e}^A = \sum_{k=0}^\infty \frac{1}{k!} A^k$, where $A^0 = \mathrm{Id}$.

Suppose $A = Q^{-1}DQ$ for some diagonal matrix D and invertible matrix Q. Show that $\mathrm{e}^A = Q^{-1}\mathrm{e}^D Q$, and thus find the eigenvalues of e^A.

Show that $\det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{Tr}(A)}$.

2. Volume Preserving Maps

Definition 2.1. Define the two-dimensional baker's map by

$$F : [0,1]^2 \to [0,1]^2 : \quad F(x,y) = \begin{cases} (2x, \frac{1}{2}y) & x \in [0, \frac{1}{2}] \\ (2-2x, 1-\frac{1}{2}y) & x \in [\frac{1}{2}, 1] \end{cases} \tag{2.1}$$

Exercise 2.2. Show that F is volume-preserving. More precisely, show that if $B = [a,b] \times [c,d] \subseteq [0,1]^2$ is any box in the unit square, then $F(B) = \{(x,y) \in [0,1]^2 : \exists (w,z) \in B : F(w,z) = (x,y)\}$ has the same volume as B.