

Math 63CM Section 3 Handout

January 24, 2020

This handout is a collection of supplementary material to lecture; part of it is supposed to directly supplement lecture material, and the other part is supposed to introduce you to some more related mathematics. This handout is presented through exercises which range from easy to difficult. If you find any mistakes, let Kevin know!

1. GRONWALL INEQUALITY

The principle set up for Gronwall is the following integral inequality: suppose a *positive* continuous function $x(t)$ satisfies, for some positive constants $a, C > 0$:

$$x(t) \leq a + C \int_0^t x(s) ds \quad (1.1)$$

for all $t \in [0, T]$ for some $T > 0$; the endpoints are not very important here.

Exercise 1.1. Check that $x(t) = ae^{Ct}$ satisfies the inequality. Check that it is an inequality

Thus, we expect that $x(t)$ is smaller than some exponential in time, or maybe even less than ae^{Ct} . This is the point of the following exercise, which provides *Gronwall's inequality*.

Exercise 1.2. Define $F(t) = \int_0^t x(s) ds$. Check that

$$F'(t) \leq a + CF(t). \quad (1.2)$$

Now define $G(t) = e^{-Ct}F(t)$. Check that

$$G'(t) \leq ae^{-Ct}. \quad (1.3)$$

Deduce $G(t) \leq \frac{a}{C} - \frac{a}{C}e^{-Ct}$, and thus $F(t) \leq \frac{a}{C}e^{Ct} - \frac{a}{C}$. From this, prove that $x(t) \leq ae^{Ct}$.

There's a second way to obtain the Gronwall inequality by an iterative procedure. Both methods have their ups and downs.

Exercise 1.3. From the principle inequality, deduce

$$x(t) \leq a + aCt + C^2 \int_0^t \int_0^s x(r) dr ds. \quad (1.4)$$

Inductively prove that

$$x(t) \leq a \sum_{\ell=0}^k \frac{C^\ell t^\ell}{\ell!} + C^{k+1} \int_0^t \int_0^{s_1} \dots \int_0^{s_k} x(r) dr ds_k \dots ds_1. \quad (1.5)$$

Deduce that $x(t) \leq a \sum_{\ell=0}^{\infty} \frac{C^\ell t^\ell}{\ell!} = ae^{Ct}$.

There's a generalization of Gronwall to inequalities of the form

$$x(t) \leq a(t) + \int_0^t b(s)x(s) ds, \quad (1.6)$$

where $a(t)$ and $b(t)$ are continuous, non-negative functions.

Exercise 1.4. Use either of the two methods above to show that

$$x(t) \leq a(t) + \int_0^t a(s)b(s)e^{\int_0^s b(r) dr} ds. \quad (1.7)$$

2. COMPARISON PRINCIPLE

This part will be reminiscent of Problem 5 on HW3. Let's start with a family of functions $F_\varepsilon(x)$ indexed by $\varepsilon \rightarrow 0$, and we'll consider the sequence of ODEs and their solutions

$$x'_\varepsilon(t) = F_\varepsilon(x_\varepsilon(t)), \quad x_\varepsilon(0) = x(0). \quad (2.1)$$

We'll make two assumptions:

- The F_ε -functions are uniformly Lipschitz, i.e. they are globally Lipschitz, and the Lipschitz constants/norms are independent of $\varepsilon \rightarrow 0$.
- As $\varepsilon \rightarrow 0$, the functions $F_\varepsilon \rightarrow F_\infty$ for some limiting function F_∞ uniformly on compact sets in t .

Exercise 2.1. Convince yourself that F_∞ is globally Lipschitz.

The initial conditions to the ODEs are the same for all ε , and indeed we've already seen continuity of solutions to an ODE with respect to the initial condition. We'll now see continuity of solutions to the ODE with respect to the functions F_ε , namely we'll see the solutions for F_ε converges to the solution for F_∞ .

Exercise 2.2. From the global existence and uniqueness theory, show that for any $T > 0$, uniformly in $\varepsilon \rightarrow 0$, we have $|x_\varepsilon(t)| \leq C_T$ for all $|t| \leq T$, where C_T depends only on T (and the initial condition).

Hint: consider the integral equation

$$x_\varepsilon(t) - x(0) = \int_0^t F_\varepsilon(x_\varepsilon(s)) \, ds = \int_0^t F_\varepsilon(x_\varepsilon(s)) - F_\varepsilon(x(0)) \, ds + F_\varepsilon(x(0)) \cdot t, \quad (2.2)$$

and apply the Lipschitz property for F_ε , a bound for the remaining term on the RHS in terms of T and $x(0)$, and the Gronwall inequality.

Exercise 2.3. To show that $x_\varepsilon(t) \rightarrow x_\infty(t)$ where $x_\infty(t)$ solves the ODE for F_ε replaced by F_∞ uniformly on compact sets in t , it suffices to show that for any K , $x_\varepsilon(t) \rightarrow x_\infty(t)$ converges uniformly on $[-K, K]$; convince yourself of this.

Show the convergence on $[-K, K]$.

Hint: consider the integral equation for the difference $x_\varepsilon(t) - x_\infty(t)$:

$$x_\varepsilon(t) - x_\infty(t) = \int_0^t F_\varepsilon(x_\varepsilon(s)) - F_\infty(x_\infty(s)) \, ds = \int_0^t F_\varepsilon(x_\varepsilon(s)) - F_\infty(x_\varepsilon(s)) \, ds + \int_0^t F_\infty(x_\varepsilon(s)) - F_\infty(x_\infty(s)) \, ds. \quad (2.3)$$

For the first quantity, from Exercise 2.2 deduce that this first quantity converges to 0 as $\varepsilon \rightarrow 0$ uniformly over $t \in [-K, K]$. For the second quantity, use the Lipschitz property and the Gronwall inequality.

We'll now actually see an example of the comparison principle (see HW3, Problem 5).

Consider the equation $x'(t) = e^{x(t)^2}$ with $x(0) = 0$. We can't explicitly solve for the solution to this ODE easily (or even at all, maybe), but we will show that it has finite time-horizons.

Exercise 2.4. Show that $1 + x^2 \leq e^{x^2}$ for all $x \in \mathbb{R}$. Show that the solution to $y'(t) = 1 + y(t)^2$ with $y(0) = 0$ cannot be extended to $t \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. Deduce that $z'(t) = e^{z(t)^2}$ blows up as $t \rightarrow \frac{\pi}{2}$ when equipped with initial condition $z(0) \geq 0$.