Math 63CM Section 1 Handout

January 21, 2020

This handout is a collection of supplementary material to lecture; part of it is supposed to directly supplement lecture material, and the other part is supposed to introduce you to some more related mathematics. This handout is presented through exercises which range from easy to difficult. If you find any mistakes, let Kevin know!

1. Examples of the Inverse Function Theorem

Recall the statement of the inverse function theorem: suppose $f : U \to \mathbb{R}^n$ is continuously differentiable with $U \subseteq \mathbb{R}^n$ an open domain, and suppose for $x \in U$ we know $\det Df(x) \neq 0$. Then there exists a neighborhood $B_x \subseteq U$ of x and a neighborhood $Q_{f(x)} \subseteq f(U)$ such that $f : B_x \to Q_{f(x)}$ is invertible with its inverse continuously differentiable.

Besides the abstraction of the proof, we'll look into how to use the inverse function theorem to compute things.

Exercise 1.1. In the context of the above inverse function theorem, let $g : Q_{f(x)} \to B_x$ denote the continuously differentiable inverse to f. Show that for any $y \in Q_{f(x)}$, we have

$$[\mathrm{D}f](g(y)) \cdot [Dg](y) \ = \ \mathrm{Id}, \tag{1.1}$$

where Id is the identity matrix of size $n \times n$. Deduce that if Df is invertible on B_x, then

$$[Dg](y) \ = \ \{[Df](g(y))\}^{-1}. \tag{1.2}$$

So not only do we know that the inverse is continuously differentiable, we have a mechanism to compute what it is!

Exercise 1.2. Consider the function $f(x) = \sin(x)$ defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Observe that $g(x) = \sin^{-1}(x)$ defines an inverse to $f(x)$, so that $g(x) : (-1,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$. For any $x \in (-1,1)$, compute $g'(x)$; you may recognize it.

Do the same for $f(x) = \cos(x)$ and $f(x) = \tan(x)$; this also includes specifying domains for which the inverse function theorem applies!

Exercise 1.3. Consider the function $f(x) = \log x$ defined on $x \in (0, \infty)$. Find its derivative.

Exercise 1.4. Consider the function $f(x) = x^{-n}$ defined on $x \in (0, \infty)$. Find its derivative.

Exercise 1.5. We'll now look at a somewhat interesting case that shows you can't drop the continuously differentiable assumption for a weaker assumption, like just differentiable.

Consider the function $f(x) = x + 2x^2 \sin \frac{1}{x}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$. Show that this function is differentiable on $\mathbb{R}$, and find its derivative. Show its derivative at 0 is nonzero.

Suppose now that $f(x)$ is injective on $(0, \varepsilon)$ for any $\varepsilon > 0$. Deduce that it must be monotone, and thus its derivative cannot change sign on $(0, \varepsilon)$. Show that this is false, and thus $f(x)$ cannot be injective on $(0, \varepsilon)$ for any $\varepsilon > 0$. Thus, the inverse function theorem's claim cannot hold for f at the origin because of its differentiability but lack of continuously differentiable property.

2. Degree of a Continuous Function

We'll start with the winding number of a curve.

Definition 2.1. Given a parameterized curve $\gamma(t) : [0,1] \to \mathbb{R}^2$ that's closed, so $\gamma(0) = \gamma(1)$, and so that $\gamma(t) \neq 0$ for any $t \in [0,1]$, the *degree* or *winding number* of γ is defined to be

$$\deg(\gamma) \ = \ \frac{\theta(1) - \theta(0)}{2\pi}, \tag{2.1}$$

where $\theta(t) = \theta(\gamma(t))$, the angle from the origin of the point $\gamma(t) \in \mathbb{R}^2$.

The winding number of a curve is supposed to capture the total number of times that a curve "wraps around the origin".

This is also the degree of the function $\gamma(t) : [0,1] \to \mathbb{R}^2$, which can be realized as a function $\gamma(t) : S^1 \to \mathbb{R}^2$ where S^1 is the circle because of the closed condition. We'll detour and introduce the degree of a function.

Definition 2.2. Suppose $K \subseteq \mathbb{R}^n$ is a compact subset, and let $f : K \to \mathbb{R}^n$ denote a continuously differentiable function. Suppose $p \in f(K)$ is a point such that for any $x \in K$ such that $f(x) = p$, we have $\det Df(x) \neq 0$, and suppose the set of $x \in K$ so that $f(x) = p$ is finite. Then the degree of f at p is

$$\deg_f(p) = \sum_{x \in K : f(x) = p} \operatorname{sgn}(\det Df(x)). \tag{2.2}$$

The point is that if K is connected, it should not depend on the point p but rather just the function f.

Why is this a useful definition, and why is it even the same? Let's think about it.

Exercise 2.3. Consider the function $f(r,\theta) : (r\cos\theta, r\sin\theta) \mapsto (r\cos 2\theta, r\sin 2\theta)$ as a function from the annulus U to itself, where U is the annulus from HW2.

Show that for every $p \in U$, there exist exactly two points $x \in U$ such that $f(x) = u$. Moreover, show that this map satisfies $Df(x) = 2$ for all $x \in U$. In particular, deduce that $\deg(f) = 2$.

On the other hand, consider the closed curve defined by $\gamma(t) = f(r, 2\pi t)$, for $t \in [0,1]$. Show that $\deg(\gamma) = 2$ for any $r \in (\frac{1}{2}, 1)$.

Here's an instance for why it might be useful to consider the second definition of degree; it generalizes to higher dimensions. Also, it provides some means to prove the following statement that we'll have to accept for now.

Theorem 2.4. *Suppose f, g are continuously differentiable functions on $K \subseteq \mathbb{R}^n$, and define $h_t = tf + (1-t)g$ for $t \in [0,1]$ as a family of functions. Suppose for each $t \in [0,1]$ that p satisfies the conditions in Definition 2.2. Then $\deg h_t$ is constant in $t \in [0,1]$.*

This isn't so easy given what I've told you, but we'll see how to apply it.

Exercise 2.5. Consider $K = \overline{B(0,1)}$ the closed ball of radius 1 in $\mathbb{R}^n$, and consider any continuously differentiable function $f : K \to K$. Define the family of functions

$$h_t(x) = \frac{x - tf(x)}{\sup_{y \in K} |y - tf(y)|} \tag{2.3}$$

Observe that for $t = 0$, this function is the identity map $i : K \to K$. Show that $\deg(i) = 1$, and in particular $\deg_i(0) = 1$. Assuming that 0 satisfies the conditions of Definition 2.2, show that $\deg_f(0) = 1$, so in particular there exists $x \in K$ such that $h_t(x) = 0$. Deduce that f has a fixed point.