Math 63CM Section 1 Handout

January 10, 2020

This handout is a collection of supplementary material to lecture; part of it is supposed to directly supplement lecture material, and the other part is supposed to introduce you to some more related mathematics. This handout is presented through exercises which range from easy to difficult. If you find any mistakes, let Kevin know!

1. Examples of ODEs

Exercise 1.1. We'll look into some more the general theory of existence for ODEs with concrete examples.

(1) Convince yourself (or prove!) that the function $f(x) = x^{\frac{1}{2}}$ is not globally Lipschitz on $[0, \infty)$. Prove that this function *is* globally Lipschitz on the interval $[\varepsilon, \infty)$ for any fixed $\varepsilon > 0$.

(2) Consider the ODE

$$\begin{cases} \dot{y}(t) = y(t)^{\frac{1}{2}} + \alpha \\ y(0) = \beta \end{cases} \tag{1.1}$$

Recall that for $\alpha = \beta = 0$ that there do not exist *unique* solutions. For which pairs α, β do there exist unique solutions? For which pairs α, β do there not exist solutions?

Exercise 1.2. This exercise is slightly different, in that we do not only want to show existence or non-existence of the solution to an ODE but we want to know something about the solution.

(1) Convince yourself that the function $f(x) = x(1-x)(1+x)$ is *not* globally Lipschitz on $\mathbb{R}$; prove that it is locally Lipschitz on $\mathbb{R}$.

(2) Consider the ODE

$$\begin{cases} \dot{y}(t) = y(t)(1-y(t))(1+y(t)) \\ y(0) = \alpha \end{cases} \tag{1.2}$$

For which values of α does the above ODE have a solution for all $t \in [0, \infty)$? For which values of α does it not?

(3) For the values of α that the ODE has a solution for all $t \in [0, \infty)$, what is $\lim_{t\to\infty} y(t)$? Does this limit depend continuously on α?

Exercise 1.3. This exercise is somewhat similar, but harder.

(1) Prove or convince yourself that the function $f(x) = \mathrm{e}^x + \alpha$ is not globally Lipschitz on $\mathbb{R}$. Prove that this function is locally Lipschitz, and convince yourself that it is globally Lipschitz on $(-\infty, C]$ for any fixed constant C (for the latter, a picture might make it clear and convincing).

(2) Consider the ODE

$$\begin{cases} \dot{y}(t) = e^{y(t)} + \alpha \\ y(0) = \beta \end{cases} \tag{1.3}$$

Suppose $\alpha > 0$. For any β, why does the above ODE not admit any solutions for all $t \in [0, \infty)$? Are there any pairs α, β for which the above ODE admits a solution for all $t \in [0, \infty)$?

2. Example of the Contraction Mapping Principle

The contraction mapping principle has applications outside differential equations, and we'll look into one here; it's called *Newton's method* or the *Newton-Raphson method*.

The set-up is as follows: given a function $f(x)$, find a zero! The strategy is given along the following algorithm:

- Pick a guess for a zero, call it y_0.
- Define $y_1 = y_0 - \frac{f(y_0)}{f'(y_0)}$.
- Inductively define $y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$.

A few points. First, to motivate the strategy, notice that if you replace the function $f(x)$ with its linear approximation at y_n, then if y_{n+1} is close to a zero of f, we would have $f(y_n) + f'(y_n)(y_{n+1} - y_n) = 0$, which gives the inductive formula for y_n. Second, what happens if $f'(y_n) = 0$, or more generally, why should this converge to anything at all? In general, it does not, but it does for a surprising number of functions. We'll look at a couple of examples now.

Exercise 2.1. Consider the function $f(x) = x^2$, and define the map $\Phi(y) = y - \frac{f(y)}{f'(y)}$.

(1) Compute $\Phi(y)$ explicitly and show that it is a contraction. Deduce that Newton's method returns the final value $y_\infty = \lim_{n\to\infty} y_n = 0$.
(2) Do the same upon replacing $f(x) = x^2$ with $f(x) = x^p$ for any $p \geqslant 1$. Why is the assumption $p \geqslant 1$ necessary *in your proof*?
(3) (A little more difficult) Do the same upon replacing $f(x) = x^2$ with $f(x) = e^x - 1$. Compute $\Phi(y)$ explicitly and show it is a contraction (it may be easier if look at only some very small interval around 0, i.e. $y \in [-\varepsilon, \varepsilon]$; check that $\Phi(y)$ maps this interval to itself, however!).

3. Useful Criterion for Convergence

Before motivating anything, we'll simply present the criterion which we claim is useful.

Exercise 3.1. Suppose $\{a_n\}_{n=1}^\infty$ is a sequence in a metric space (X, d) satisfying the following criterion:

- Every subsequence admits a further subsequence that converges to $x \in X$.

Then the entire sequence converges to $x \in X$, i.e. $\lim_{n\to\infty} a_n$ exists and is equal to $x \in X$.

The above claim may sound too good to be true; the likely reason is the subtlety in the criterion itself. Namely, each "further subsequence" must converge to the *same* element $x \in X$. Indeed, if not, then convergence certainly cannot happen.

Exercise 3.2. Convince yourself that if two subsequences of $\{a_n\}_{n=1}^\infty$ both converge but to different limits, then the original sequence cannot converge.