Math 63CM Midterm 2 Solutions

Kevin Yang

Stanford University

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Problem 1

The solution to the ODE is given by $x(t) = \mathrm{e}^{tA}x_0$. In particular, we have

$$\begin{aligned} \|x(t)\| &\leqslant \|\mathrm{e}^{tA}x_0\| && (0.1) \\ &\leqslant \|\mathrm{e}^{tA}\|_{\mathrm{op}}\|x_0\|. && (0.2) \end{aligned}$$

Suppose A has Jordan canonical form $A = S\Lambda S^{-1}$. This implies

$$\begin{aligned} \|\mathrm{e}^{tA}\|_{\mathrm{op}} &= \|S\mathrm{e}^{t\Lambda}S^{-1}\|_{\mathrm{op}} && (0.3) \\ &\leqslant \|S\|_{\mathrm{op}}\|S^{-1}\|_{\mathrm{op}}\|\mathrm{e}^{t\Lambda}\|_{\mathrm{op}}. && (0.4) \end{aligned}$$

In particular, allowing our constant to depend on S, it remains to bound the last operator norm. To this end, we first observe that the entries of $\mathrm{e}^{t\Lambda}$ are of the form 0 or $\frac{t^k}{k!}\mathrm{e}^{t\lambda}$ for $k \in [0, n]$, and are thus bounded, in absolute value, by $C_n + C_n t^n$. In particular, we have

$$\begin{aligned} \|\mathrm{e}^{t\Lambda}\|_{\mathrm{op}} &\leqslant C_n' \sup_{ij} \left|[\mathrm{e}^{t\Lambda}]_{ij}\right| && (0.5) \\ &\leqslant C_n C_n'(1 + t^n), && (0.6) \end{aligned}$$

which completes the proof.

Problem 2

Observe A is invertible because $\mathrm{Re}(\lambda) < -1$ for all eigenvalues λ. We now observe

$$\begin{aligned} \left\| A^{-1}y + \int_0^T x(t)\,\mathrm{d}t \right\| &= \left\| A^{-1}\left(y + A\int_0^T x(t)\,\mathrm{d}t \right) \right\| && (0.7) \\ &\leqslant \|A^{-1}\|_{\mathrm{op}} \left\| y + A\int_0^T x(t)\,\mathrm{d}t \right\|. && (0.8) \end{aligned}$$

As in Problem 1, it suffices to bound the second norm. Moreover, by approximating the integral by a Riemann sum, we have

$$\begin{aligned} A\int_0^T x(t)\,\mathrm{d}t &= \int_0^T Ax(t)\,\mathrm{d}t && (0.9) \\ &= \int_0^T x'(t)\,\mathrm{d}t && (0.10) \\ &= x(T) - x(0) && (0.11) \\ &= x(T) - y. && (0.12) \end{aligned}$$

Thus, we have

$$\left\| y + A\int_0^T x(t)\,\mathrm{d}t \right\| \;=\; \|x(T)\|\,. \tag{0.13}$$

Suppose $\eta > 1$ is chosen such that $\mathrm{Re}(\lambda) < -\eta$ for all eigenvalues λ. As in Problem 1 we have

$$\begin{aligned} \|x(T)\| \;&\leqslant\; C_n(1+T^n)\mathrm{e}^{-T\eta}\|y\| && (0.14)\\ &\leqslant\; C_n(1+t^n)\mathrm{e}^{(1-\eta)T}\mathrm{e}^{-T}\|y\| && (0.15)\\ &\leqslant\; C_n''\mathrm{e}^{-T}\|y\|, && (0.16) \end{aligned}$$

since $\eta > 1$, so the polynomial grows much slower than the exponential decays. This completes the proof.

PROBLEM 3

(i). Define $z(t) = x(t+1) - x(t)$, so that $z'(t) = Az(t) + f(t+1) - f(t) = Az(t)$ since f is 1-periodic. In particular, as in Problem 2, if $\alpha > 0$ is chosen so that $\mathrm{Re}(\lambda) < -\alpha - \varepsilon$ for $\varepsilon > 0$ arbitrarily small, we have

$$\begin{aligned} \|z(t)\| \;&\leqslant\; C_n(1+t^n)\mathrm{e}^{-\varepsilon t}\mathrm{e}^{-\alpha t}\|z(0)\| && (0.17)\\ &\leqslant\; C_n'\mathrm{e}^{-\alpha t}\|z(0)\|. && (0.18) \end{aligned}$$

However, $z(0) = x(1) - y$, whose norm depends only on A and y. This completes the proof.

(ii). Here's a neat way to do this problem. Choose $f(t) = \sin(t)$, and suppose we had a 2π-periodic solution. In particular, differentiating the x_1' equation and using the x_2' equation, we obtain

$$x_1''(t) \;=\; -x_1(t) + \sin(t). \tag{0.19}$$

We multiply both sides by $\sin(t)$ and integrate on $[0, 2\pi]$. Integrating by parts and realizing the boundary terms cancel by the periodicity of x_1, we have

$$\begin{aligned} \int_0^{2\pi} \sin(t)x_1''(t)\,\mathrm{d}t \;&=\; -\int_0^{2\pi} \cos(t)x_1'(t)\,\mathrm{d}t && (0.20)\\ &=\; -\int_0^{2\pi} \sin(t)x_1(t)\,\mathrm{d}t. && (0.21) \end{aligned}$$

In particular, we have

$$-\int_0^{2\pi} \sin(t)x_1(t)\,\mathrm{d}t \;=\; \int_0^{2\pi} \sin(t)x_1(t)\,\mathrm{d}t \;+\; \int_0^{2\pi} |\sin(t)|^2\,\mathrm{d}t. \tag{0.22}$$

The LHS cancels with the first term on the RHS, so we obtain $\int_0^{2\pi} |\sin(t)|^2\,\mathrm{d}t = 0$, which is absurd.

PROBLEM 4

Consider $z(t) = \|x(t)\|^2$, so that

$$\begin{aligned} z'(t) \;&=\; x'(t)\cdot x(t) && (0.23)\\ &=\; (x(t)\cdot y)^2 - \|x(t)\|^2 + x(t)\cdot G(x(t)) && (0.24)\\ &\leqslant\; \|x(t)\|^2\|y\|^2 - \|x(t)\|^2 + \|x(t)\|^3 && (0.25)\\ &=\; z(t)(-1+\|y\|^2) + z(t)^{\frac{3}{2}}. && (0.26) \end{aligned}$$

The inequality follows from Cauchy-Schwarz, and the $\frac{3}{2}$-power is well-defined because $z(t) \geqslant 0$. By the comparison principle, because $z(t) \geqslant 0$, it suffices to show that the desired claim is true for the solution to

$$w'(t) \;=\; w(t)(-1+\|y\|^2) + w(t)^{\frac{3}{2}} \tag{0.27}$$

with an arbitrarily small but positive initial condition.

Consider the function $F(w) = w(-1 + \|y\|^2) + |w|^{\frac{3}{2}}$. Observe $F(0) = 0$, so that 0 is an equilibrium point. Moreover, its derivative at 0 is $F'(0) = -1 + \|y\|^2 < 0$ given our assumption for y. Thus, 0 is an asymptotically stable equilibrium for $F(w)$, which gives the claim.

It is false in general if we allow $\|y\| = 1$. For any n, consider the example $G(x) = |x \cdot e_1|^2$, where e_1 denotes the first standard basis vector. Finally choose $y = e_1$.

If we start with any initial condition $x(0) = re_1$ for $|r| > 0$, then we claim that $x(t) \not\to_{t\to\infty} 0$. To see this, observe that

$$x_1'(t) \;=\; x_1(t) - x_1(t) + |x_1(t)|^2 \;=\; |x_1(t)|^2. \tag{0.28}$$

In particular, if $r > 0$, then x_1 is increasing, proving the claim.

Problem 5

The characteristic polynomial is $p_A(\lambda) = (\lambda - 1)^3$, as indicated by the following calculation:

$$\begin{aligned}
p_A(\lambda) \;&=\; \det\begin{pmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-3 & -3 \\ 1 & 2 & \lambda+2 \end{pmatrix} && (0.29)\\
&=\; (\lambda-2)\det\begin{pmatrix} \lambda-3 & -3 \\ 2 & \lambda+2 \end{pmatrix} + 2\det\begin{pmatrix} -1 & -3 \\ 1 & \lambda+2 \end{pmatrix} - 3\det\begin{pmatrix} -1 & \lambda-3 \\ 1 & 2 \end{pmatrix} && (0.30)\\
&=\; (\lambda-2)(\lambda-3)(\lambda+2) + 6(\lambda-2) \;-\; 2(\lambda+2) + 6 \;+\; 6 \;+\; 3(\lambda-3) && (0.31)\\
&=\; \lambda^3 - 3\lambda^2 - 4\lambda + 12 + 6\lambda - 12 - 2\lambda - 4 + 12 + 3\lambda - 9 && (0.32)\\
&=\; \lambda^3 - 3\lambda^2 + 3\lambda - 1 && (0.33)\\
&=\; (\lambda-1)^3. && (0.34)
\end{aligned}$$

Thus, the Jordan canonical form has either 1, 2, or 3 blocks each corresponding to an eigenvalue of $\lambda = 1$. To show that $(A - \mathrm{I})^2 = 0$, it suffices to show that there must be at least two blocks, so there exist at least 2 linearly independent eigenvectors of eigenvalue $\lambda = 1$. To check this, it suffices to find the solution space for the following system of linear equations:

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x-y-z \\ -x-y-z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{0.35}$$

Solutions are given by the plane $x + y + z = 0$, which gives the result.