

Math 63CM Midterm 1 Solutions

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PROBLEM 1

(i). A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is uniformly Lipschitz on $K \subseteq \mathbb{R}^n$ if there exists a constant $C > 0$ such that for all $x, y \in K$, we have $\|f(x) - f(y)\| \leq C\|x - y\|$.

(ii). We first claim that for $F(y) = -e^y$, we have $\sup_{y \in (-\infty, a]} \left| \frac{d}{dy} F(y) \right| \leq C_a$, for a constant $C_a > 0$ depending only on a . This is the following calculation:

$$|F'(y)| = |e^y| \leq e^a, \quad (0.1)$$

since the exponential function is monotone increasing, so $C_a = e^a$ works.

Next, for any $x, y \in (-\infty, a]$, suppose without loss of generality that $x \leq y$. In this case, by the fundamental theorem of calculus we have

$$|F(y) - F(x)| \leq \int_x^y |F'(t)| dt \quad (0.2)$$

$$\leq \int_x^y e^a dt \quad (0.3)$$

$$= e^a |x - y|. \quad (0.4)$$

This proves F is uniformly Lipschitz on $(-\infty, a]$ with Lipschitz constant at most e^a .

To prove it is not uniformly Lipschitz on \mathbb{R} , for any $N > 0$, we see

$$e^{N+1} - e^N = \int_N^{N+1} e^t dt \quad (0.5)$$

$$\geq \int_N^{N+1} e^N dt \quad (0.6)$$

$$= e^N. \quad (0.7)$$

Thus, for any $C > 0$, we may choose a pair of points x, y with distance 1 apart such that $|F(x) - F(y)| \geq C$. This implies that F cannot be uniformly Lipschitz on \mathbb{R} .

(iii). Following the hint, for $s < t$, if we had a solution we would know

$$y(t) = y(s) + \int_s^t y'(r) dr \quad (0.8)$$

$$= y(s) - \int_s^t e^r dr \quad (0.9)$$

$$< y(s) \quad (0.10)$$

since e^r is strictly positive, if s, t were in its domain of definition.

We next realize that because $F(y)$ is locally Lipschitz, since it's continuously differentiable, there exists a neighborhood of 0, call it $(-\delta, \delta)$, such that the ODE $y'(t) = F(y(t))$ with $y(0) = 100$ has a unique solution.

Suppose T_+ is the maximal time for a solution, i.e. we had a solution on $(-\delta, T)$ for any $T < T_+$ but not for $T = T_+$. Because $y(t) < y(s)$ for all $s < t$ in the domain $(0, T_+)$, we know $y(t) \rightarrow -\infty$ as $t \rightarrow T_+$ from the left. In particular, there exists some $a \in \mathbb{R}$ such that $y(t) \in (-\infty, a]$ for all t sufficiently close to the left of T_+ .

In part (ii), we prove that $F(y)$ is uniformly Lipschitz on this domain with Lipschitz constant e^a . Thus, by the global existence and uniqueness theorem's proof, we may construct a solution to the ODE on a neighborhood of $T_+ - \frac{1}{3e^a}$ of size $\frac{1}{2e^a}$; moreover, this solution must agree with our original solution on $(-\delta, T_+)$ by the uniqueness component of that theorem. But this means our solution can be extended past T_+ , since $T_+ - \frac{1}{3e^a} + \frac{1}{2e^a} > T_+$. This is a contradiction, so the solution must be defined on $(-\delta, \infty)$. By taking $t_0 = \frac{1}{2}\delta$, we have the existence part of this problem.

To prove uniqueness on $[-t_0, \infty)$, consider any two solutions $y(t), z(t)$ to the ODE with initial value 100 at $t = 0$. Consider the set $G = \{t \in [-t_0, \infty) : y(t) = z(t)\}$. We aim to show this set is open, closed, and nonempty inside $[-t_0, \infty)$; this would imply that $G = [-t_0, \infty)$.

To this end, it's nonempty because $0 \in G$. It's closed because $y(t), z(t)$ are continuous. It's open because for any $t \in G$, there exists a neighborhood of t such that the ODE with the initial condition $y(t) = z(t)$ has a unique solution; this follows from local existence and uniqueness, since F is locally Lipschitz as it's continuously differentiable. But y, z are both solutions on this neighborhood because they are solutions on all of $[-t_0, \infty)$. This implies that G is open.

PROBLEM 2

Consider any $\varepsilon > 0$; we choose $\delta > 0$ such that for every $x, y \in [0, 1]$ with $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \frac{\varepsilon}{100}$ for all n .

Because $S \subset [0, 1]$ is dense, for any $\delta > 0$, there exists a finite subset $\{x_1, \dots, x_K\} \subset S$ such that for all $y \in [0, 1]$, we have $|x_i - y| < \delta$ for some x_i .

Lastly, we choose N sufficiently large so that for all $n \geq N$, we have

$$\sup_{i=1, \dots, K} |f_n(x_i) - f(x_i)| < \frac{\varepsilon}{100}, \quad (0.11)$$

where f is some function on $[0, 1]$ such that $f(s) = \lim_{n \rightarrow \infty} f_n(s)$ for all $s \in S$; note this function may not be continuous. This can be done because $x_i \in S$ for all $i = 1, \dots, K$, and we are taking a supremum over finitely many points.

We now show that f_n is uniformly Cauchy $[0, 1]$. By assumption, for any $m, n \geq N$, we have

$$\sup_{i=1, \dots, K} |f_n(x_i) - f_m(x_i)| \leq \sup_{i=1, \dots, K} |f_n(x_i) - f(x_i)| + \sup_{i=1, \dots, K} |f(x_i) - f_m(x_i)| \quad (0.12)$$

$$\leq \frac{\varepsilon}{50}. \quad (0.13)$$

Thus, for any $y \in [0, 1]$, let x_i denote any point in the aforementioned set so that $|x_i - y| < \delta$. We then know

$$|f_n(y) - f_m(y)| \leq |f_n(y) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(y)| \quad (0.14)$$

$$\leq \frac{\varepsilon}{100} + \frac{\varepsilon}{50} + \frac{\varepsilon}{100} \quad (0.15)$$

$$< \varepsilon. \quad (0.16)$$

This is true uniformly in $y \in [0, 1]$, so that $\sup_{y \in [0, 1]} |f_n(y) - f_m(y)| < \varepsilon$ for all $m, n \geq N$. Thus, f_n is uniformly Cauchy on $[0, 1]$. Because the metric space of continuous functions on $[0, 1]$ is complete, we know that it converges to continuous function f uniformly on $[0, 1]$, which completes the proof.

PROBLEM 3

(i). For any $t \in (-s, s)$, we know

$$\frac{d}{dt}H(x(t)) = \nabla H(x(t)) \cdot x'(t) \quad (0.17)$$

$$= \nabla H(x(t)) \cdot F(x(t)) \quad (0.18)$$

$$= 1. \quad (0.19)$$

Thus, for any $t \in (-s, s)$, we know

$$H(x(t)) = H(x(0)) + \int_0^t 1 \, ds, \quad (0.20)$$

so that $|H(x(t)) - H(x(0))| \leq |t|$ for all $t \in (-s, s)$. Since $x(t)$ is continuous on $t \in [-s, s]$ and H is smooth, we know $H(x(t))$ is continuous, and thus $H(x(t)) \rightarrow H(x(\pm s))$ as $t \rightarrow \pm s$. Thus, we know

$$|H(x(\pm s)) - H(x(0))| = \lim_{t \rightarrow \pm s} |H(x(t)) - H(x(0))| \leq \lim_{t \rightarrow \pm s} |t| = |s|, \quad (0.21)$$

which proves it at the boundary.

(ii). Suppose that the solution exists for $t \in (-T_-, T_+)$, and suppose $T_+ < \infty$. Then, because F is smooth on \mathbb{R}^n , we know $\|x(t)\| \rightarrow_{t \rightarrow T_+^-} \infty$. However, we know $|H(x(t))| \leq |H(x(0))| + T_+$, and thus by the assumption we know there exist a compact set $K_{T_+} \subseteq \mathbb{R}^n$ such that $x(t) \in K_{T_+}$ for all $t \in [0, T_+)$. However, this contradicts the blow-up in norm as $t \rightarrow T_+^-$, which shows $T_+ = \infty$ necessarily. The same argument shows $T_- = \infty$ necessarily as well.

PROBLEM 4

(i). Define $w(t) = z(t)e^{-Kt}$. Then because the exponential is positive, we know

$$w'(t) = e^{-Kt}z'(t) - Ke^{-Kt}z(t) \leq 0. \quad (0.22)$$

Thus, we know $w(t) \leq w(0) = z(0)$ for all $t \geq 0$. In particular, we deduce $z(t) \leq e^{Kt}z(0)$.

(ii). Define $y(t) = \|x(t; x_0) - x(t; x_1)\|^2$ and $z(t) = x(t; x_0) - x(t; x_1)$ for $x_0, x_1 \in \mathbb{R}^n$. Then

$$y'(t) = 2z'(t) \cdot z(t) \quad (0.23)$$

$$= 2[G(x(t; x_0)) - G(x(t; x_1))] \cdot z(t) \quad (0.24)$$

$$\leq 2M\|z(t)\|^2 \quad (0.25)$$

$$= 2My(t). \quad (0.26)$$

where M is the Lipschitz constant of G on \mathbb{R}^n (the second-to-last line is also using Cauchy-Schwarz). Thus, by part (i) we know, for any $t \geq 0$,

$$\|q(x_0) - q(x_1)\|^2 = y(t) \leq y(0)e^{2Mt} = \|x_0 - x_1\|^2 e^{2Mt}. \quad (0.27)$$

Taking square roots gives the Lipschitz property, since e^{Mt} is constant for every fixed t .

1. PROBLEM 5

(i). Consider $z(t) = \int_0^t y(s) \, ds$, so that $z'(t) = y(t) \leq y_0 + Kz(t)$. Further consider $w(t) = e^{-Kt}z(t)$, so that

$$w'(t) = e^{-Kt}z'(t) - Ke^{-Kt}z(t) \quad (1.1)$$

$$\leq y_0 e^{-Kt}. \quad (1.2)$$

Integrating, we see $w(t) = w(0) + \frac{y_0}{K}(1 - e^{-Kt})$ for any $t \geq 0$. Knowing that $w(0) = 0$, we deduce $z(t) \leq \frac{y_0}{K}(e^{Kt} - 1)$, so that

$$y(t) \leq y_0 + Kz(t) \leq y_0 + y_0 e^{Kt} - y_0 = y_0 e^{Kt}, \quad (1.3)$$

which is what we want.

(ii). See HW3, Problem 5 solutions; the part after "iterating" can be replaced by part (i) of this problem.