

## Math 63CM Homework 7 Solutions

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### PROBLEM 1

We define  $A = DF(0)$ , and then define

$$S = \int_0^{\infty} e^{tA} e^{tA^T} dt. \quad (0.1)$$

Because the eigenvalues of  $A$  have negative real part, by Problem 2.6 in Brendle we know that  $S$  is positive-definite, and also that  $\langle Sx, Ax \rangle = -\frac{1}{2}\|x\|^2$ .

Next, because  $F$  is smooth, we observe that by Taylor expansion, for  $\|x\|$  sufficiently small, we have

$$\|F(x) - F(0) - DF(0)x\| \leq C\|x\|^2 \quad (0.2)$$

for some constant  $C > 0$  depending only on  $F$ .

Lastly, because  $S$  is symmetric, by the product rule we observe that  $\nabla L(x) = 2Sx$ .

Considering the function  $L(x) = \langle x, Sx \rangle$ , observe that its Hessian matrix is equal to  $2S$ , given by the following calculation:

$$\partial_{ij} L(x) = \partial_{ij} \sum_{k,\ell} S_{k\ell} x_k x_\ell = \begin{cases} 2S_{kk} & k = \ell \\ S_{k\ell} + S_{\ell k} = 2S_{k\ell} & k \neq \ell \end{cases} \quad (0.3)$$

Because  $S$  is positive-definite, any local extrema is a local minimum. In particular, because  $\nabla L(0) = 2S0 = 0$ , we know that  $0$  is a local minimum.

Next, recalling  $F(0) = 0$ , we observe that

$$\langle \nabla Lx, F(x) \rangle = 2\langle Sx, F(x) \rangle \quad (0.4)$$

$$= 2\langle Sx, F(0) \rangle + 2\langle Sx, DF(0)x \rangle + 2\langle Sx, (F(x) - F(0) - DF(0)x) \rangle \quad (0.5)$$

$$= -\|x\|^2 + 2\langle Sx, (F(x) - F(0) - DF(0)x) \rangle. \quad (0.6)$$

By Cauchy-Schwarz, we know

$$\|2\langle Sx, (F(x) - F(0) - DF(0)x) \rangle\| \leq 2C\|Sx\|\|x\|^2 \quad (0.7)$$

$$\leq 2C\|S\|_{\text{op}}\|x\|^3. \quad (0.8)$$

In particular, if  $\|x\|$  is sufficiently small, we know that  $-\|x\|^2 + 2\langle Sx, (F(x) - F(0) - DF(0)x) \rangle$  is still negative if  $x \neq 0$ . Asymptotic stability of  $0$  now follows from Theorem 4.12 in Brendle.

PROBLEM 2

(i). Consider the function  $F(x_1, x_2) = (x_2^2 + x_1x_2 - 2, x_1^2 + x_1x_2 - 2)$ . For any equilibrium point, we must have

$$(x_1 + x_2)^2 = 4. \quad (0.9)$$

Thus, we must have  $x_1 + x_2 = 2$  or  $x_1 + x_2 = -2$ . Moreover, we also have

$$x_1^2 = x_2^2. \quad (0.10)$$

Thus, the equilibrium points satisfy  $x_1 = 2 - x_2$  or  $x_1 = -2 - x_2$  and  $x_1 = \pm x_2$ . In particular, we have both equilibrium points as  $(1, 1)$  and  $(-1, 1)$ .

To find the hyperbolic equilibrium points, we first compute

$$DF(x_1, x_2) = \begin{pmatrix} x_2 & 2x_2 + x_1 \\ 2x_1 + x_2 & x_1 \end{pmatrix}. \quad (0.11)$$

At equilibrium points, we have

$$DF(1, 1) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad (0.12)$$

$$DF(-1, -1) = \begin{pmatrix} -1 & -5 \\ -5 & -1 \end{pmatrix}. \quad (0.13)$$

The eigenvalues of the  $DF(1, 1)$  are given by  $\lambda = 4$  and  $\lambda = -2$ . The eigenvalues of  $DF(-1, -1)$  are  $\lambda = -6$  and  $\lambda = 4$ . Thus, both are saddle points.

(iii). First,  $y(t) = x_2(t)$  and  $x(t) = x_1(t)$  throughout the following solution.

The saddle points are  $(1, 1)$  and  $(-1, -1)$ . The answers are given by

$$\text{Stable}(1, 1) = \{(x, y) : x + y = 2\}, \quad (0.14)$$

$$\text{Unstable}(1, 1) = \{(x, y) : x - y = 0, x > -1\}, \quad (0.15)$$

$$\text{Stable}(-1, -1) = \{(x, y) : x - y = 0, x < 1\}, \quad (0.16)$$

$$\text{Unstable}(-1, -1) = \{(x, y) : x + y = -2, \}. \quad (0.17)$$

We prove the first two; the latter two follow from a similar argument.

To compute  $\text{Stable}(1, 1)$ , we consider  $z(t) = x_1(t) + x_2(t)$ . We see  $z' = z^2 - 4$ , from which we deduce

$$z(t) = 2 \frac{1 + ae^{4t}}{1 - ae^{4t}}, \quad a \in \mathbb{R}. \quad (0.18)$$

Moreover, if  $(x(0), y(0)) \in \text{Stable}(1, 1)$ , then  $z(t) \rightarrow_{t \rightarrow \infty} 2$ . However, this can only happen if  $a = 0$ , so  $z(t) = 2$  for all  $t$ . This shows  $\text{Stable}(1, 1) \subset \{(x, y) : x + y = 2\}$ .

To show the reverse containment, we consider two cases.

- Suppose that  $x_2(0) = 2 + \kappa$  for some  $\kappa > 0$ . Moreover, for  $x_2(t) > 2$ , we have  $x_1(t) < 2$  and also  $x_2'(t) = x_1(t)(x_1(t) + x_2(t)) - 4 = 2x_1(t) - 4 < 0$ . In particular,  $x_2'(t)$  is monotonically decreasing, and if  $x_2(T) = 1$  for any  $T > 0$ , we have  $x_2(T + t) = 0$  for any  $t \geq 0$ . In particular, if  $\lim_{t \rightarrow \infty} x_2(t) \neq 1$ , then we have

$$\inf_{t \geq 0} x_2(t) = 1 + \delta \quad (0.19)$$

for some  $\delta > 0$ . In this case, we have  $x_2'(t) < -2\delta$  for all  $t \geq 0$ . But this clearly contradicts the infimum above.

- Suppose now that  $x_2(0) = 2 - \kappa$  for some  $\kappa > 0$ . In this case  $x_1(0) = 2 + \kappa$ , and because the equations remain the same upon swapping  $x_1, x_2$ , the above bullet point implies that  $(x_1(t), x_2(t)) \rightarrow_{t \rightarrow \infty} (1, 1)$ .

We now compute  $\text{Unstable}(1, 1)$ . To this end, we consider  $w(t) = x_1(t) - x_2(t)$ , so that  $w'(t) = -w(t)z(t)$ , so that

$$w(t) = w(0) \exp \left[ -2 \int_0^t \frac{1 + ae^{4s}}{1 - ae^{4s}} ds \right], \quad a \in \mathbb{R}. \quad (0.20)$$

We must have  $w(t) \rightarrow_{t \rightarrow -\infty} 0$ ; the only way this can happen is that  $w(0) = 0$ , since the integrand diverges within the exponential for any  $a \in \mathbb{R}$ . Thus, we have  $x_1(t) = x_2(t)$  for all  $t \leq 0$ . This shows  $\text{Unstable}(1, 1) \subset \{(x, y) : x - y = 0\}$ .

We now note that for  $(x, y)$  such that  $x = y$  with  $x \leq -1$ , if  $x = -1$  then for initial condition  $(x(0), y(0)) = (-1, -1)$  we have  $(x(t), y(t)) = (-1, -1)$  for all  $t \in \mathbb{R}$ . In particular,  $(-1, -1) \notin \text{Unstable}(1, 1)$ . Moreover, for  $x(0) < -1$ , because  $x(t) = y(t)$  for all  $t \leq 0$ , if  $(x(0), y(0)) \in \text{Unstable}(1, 1)$  we must have  $(x(T), y(T)) = (-1, -1)$  by the intermediate value theorem for some  $T < 0$ . However, we then have  $x(T + s) = -1$  and  $y(T + s) = -1$  for all  $s \leq 0$ , which contradicts  $(x(0), y(0)) \in \text{Unstable}(1, 1)$ .

To complete the proof, it remains to show that  $\{(x, y) : x - y = 0 : x > -1\} \subset \text{Unstable}(1, 1)$ . To this end, we consider two cases.

- For  $|x_1(0)| < 1$  and  $x_1(0) = x_2(0)$ , we have

$$x_1'(t) = x_2(x_1 + x_2) - 2 \quad (0.21)$$

$$= 2x_2^2 - 2 \quad (0.22)$$

$$< 0. \quad (0.23)$$

Thus, for negative times, we see that  $x_1(t)$  is monotonically increasing! A similar argument as in the first two bullet points shows that  $x_1(t) \rightarrow_{t \rightarrow -\infty} 1$ , and similarly for  $x_2(t)$ .

- For  $x_1(0) > 1$  and  $x_1(0) = x_2(0)$ , we have

$$x_1'(t) = 2x_1(t)^2 - 2 > 0. \quad (0.24)$$

In particular, for negative times, we see that  $x_1(t)$  is monotonically decreasing! A similar argument as above shows that  $x_1(t) \rightarrow_{t \rightarrow -\infty} 1$ , and similarly for  $x_2(t)$ .

### PROBLEM 3

(i). This is an immediate application of the chain rule.

(ii). The system of equations is

$$r'(t) = r(t)(1 - r(t)), \quad (0.25)$$

$$\theta'(t) = r(t)(1 - \cos \theta(t)). \quad (0.26)$$

We first show that for  $\varepsilon > 0$  sufficiently small, if  $|r(0) - 1| < \varepsilon$ , then  $r(t) \rightarrow_{t \rightarrow \infty} 1$ .

- Suppose first that  $r(t) = 1 + \varepsilon'$  for  $\varepsilon' > 0$  sufficiently small. Note that by the ODE for  $r(t)$ , we know  $r(t)$  is monotonically decreasing in  $t \geq 0$  so long as  $r(t) > 1$ . Moreover, if  $r(T) = 1$  for any finite time  $T > 0$ , then  $r(t + T) = 1$  for all  $t \geq 0$  as well. Thus, for  $\lim_{t \rightarrow \infty} r(t) \neq 1$  to happen, we must have

$$\inf_{t \geq 0} r(t) = 1 + \delta \quad (0.27)$$

for some  $\delta > 0$ . However, if this were the case, then for all  $t \geq 0$ , we have  $r'(t) = r(t)(1 - r(t)) \leq -\delta r(t)$ , which implies that  $r(t) \leq e^{-\delta t} r(0)$  by the comparison principle. Since  $r(0) > 0$  for  $\varepsilon' > 0$  sufficiently small this contradicts the infimum condition.

- Suppose now that  $r(t) = 1 - \varepsilon'$  for  $\varepsilon' > 0$  sufficiently small. In this case, if  $\varepsilon' > 0$  is sufficiently small, we see that  $r(t)$  is monotonically increasing. By the same argument above, for  $\lim_{t \rightarrow \infty} r(t) \neq 1$  to happen, we must have

$$r(0) \leq \sup_{t \geq 0} r(t) = 1 - \delta \quad (0.28)$$

for some  $\delta > 0$ . However, if this were the case, then for all  $t \geq 0$ , we have  $r'(t) = r(t)(1 - r(t)) \geq \delta r(t)$ . By the comparison principle, this implies  $r(t) \geq e^{\delta t} r(0)$  which is a contradiction since  $r(0) > 0$ .

We now show that for all  $\theta(0)$ , we have  $\theta(t) \rightarrow_{t \rightarrow \infty} 2\pi k$  for some integer  $k$ , if  $r(0) \neq 0$ . This would complete the proof.

- If  $\theta(0) = 2\pi k$  for some integer  $k$ , then by the ODE for  $\theta(t)$  we see  $\theta(t)$  is constant in time, which is what we want.
- If  $\theta(0) \in (2\pi k, 2\pi(k+1))$  for some integer  $k$ , which is the only other situation, then by the ODE for  $\theta(t)$  we see that  $\theta(t)$  is monotonically increasing in  $t \geq 0$ . By the first bullet point, for  $\lim_{t \rightarrow \infty} \theta(t) \neq 2\pi(k+1)$  to happen, we must have that

$$2\pi k + \delta < \sup_{t \geq 0} \theta(t) < 2\pi(k+1) - \delta \quad (0.29)$$

for some  $\delta > 0$ . In this situation, we have  $\theta'(t) = r(t)(1 - \cos \theta(t)) \geq C_\delta r(t)$  for all  $t \geq 0$  since  $r(t) \rightarrow 1$  as  $t \rightarrow \infty$  if  $r(0) \neq 0$ . By this same token, we have  $\theta'(t) \geq C'_\delta$  for some positive constant  $C'_\delta$ . This implies  $\theta(t) \geq \theta(0) + C'_\delta t$  for all  $t \geq 0$ , which contradicts the supremum assumption.

(iii). To show that  $(1, 0)$  is unstable, consider  $\theta(0) = \varepsilon$  for some positive and arbitrarily small  $\varepsilon > 0$ . By the previous part, we know that  $\theta(t) \rightarrow 2\pi$  as  $t \rightarrow \infty$  and that  $\theta(t)$  is increasing. In particular, for any initial condition  $\theta(0) \neq 2\pi k$  for any integer  $k$ ,  $\{\theta(t)\}_{t \geq 0}$  is the interval between  $\theta(0)$  and the first integer multiple of  $2\pi$  greater than  $\theta(t)$ , whereas stability would require that for  $\varepsilon > 0$  sufficiently small, the values  $\{\theta(t)\}_{t \geq 0}$  would be an interval of length at most  $\varepsilon' > 0$  for  $\varepsilon' > 0$  arbitrarily small.

#### PROBLEM 4

(i). The system is Hamiltonian with  $H(x, y) = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2$ . Thus, we know for all  $t \in \mathbb{R}$  that

$$(x(t), y(t)) \in \{(x, y) \in \mathbb{R}^2 : H(x, y) = H(x(0), y(0))\} \quad (0.30)$$

for any initial condition  $(x(0), y(0)) \in \mathbb{R}^2$ . However, the set  $\{(x, y) \in \mathbb{R}^2 : H(x, y) = E\}$  is compact for any fixed  $E \in \mathbb{R}$ . In particular, for any initial conditions, there exists a compact set in which  $x(t), y(t)$  lives for all  $t \in \mathbb{R}$ . This shows that global solutions must exist.

(ii). The system is Hamiltonian with  $H(x, y) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{2}y^2$ . Solutions do not necessarily exist for all time. To see this, consider any initial conditions  $x(0), y(0)$  arbitrarily large and positive. By the ODEs for  $x(t), y(t)$ , we see that  $x(t), y(t)$  remain large and positive as well. In particular, we know

$$x''(t) = -x(t) + x(t)^3 \geq \frac{1}{2}x(t)^3. \quad (0.31)$$

From this and the comparison principle, we know that  $x(t)$  must blow up in finite-time. To actually see this, consider the following two situations:

- Consider  $x''(t) \geq x'(t)^2$ . In this case, we know  $x'(t) \geq \frac{x'(0)}{1-t}$ , and thus  $x(t) \geq -C \log(1-t) + C'$  for constants  $C, C' > 0$ .
- Consider  $x''(t) \leq x'(t)^2$ , in which case  $x'(t) \geq \frac{1}{\sqrt{2}}x(t)^{\frac{3}{2}}$ . In this case, we see  $x(t) \geq C \frac{1}{(1-t)^2}$  for some constant  $C > 0$ .

In either case, we see that  $x(t)$  blows up in finite time, since it must grow at least  $-C \log(1-t) + C'$ . This is not a "contradiction" because  $H(x, y)$  does not have compact level sets.

PROBLEM 5

(i). Let  $P(x)$  denote the antiderivative to  $p(x)$ , and consider the function  $L(x, y) = P(x) + \frac{1}{2}y^2$ . We then have

$$\frac{d}{dt}L(x(t), y(t)) = P(x(t))x'(t) + y(t)y'(t) \quad (0.32)$$

$$= P(x(t))y(t) - y(t)^4 - P(x(t))y(t) \quad (0.33)$$

$$= -y(t)^4. \quad (0.34)$$

In particular, we know  $L(x(t), y(t)) \leq L(x(0), y(0))$ . We next observe that  $P(x)$  is an even-degree polynomial with positive leading-coefficient. This implies that  $x^2 + y^2 \leq CL(x, y)$  for a constant  $C > 0$ . As such, we deduce that  $0 \leq x(t)^2 + y(t)^2 \leq P(x(0)) + \frac{1}{2}y(0)^2$  for all  $t \geq 0$  for which the solution is defined, which concludes the proof.

(ii). Solutions exist for all  $t \geq 0$  because otherwise, we have  $x(t)^2 + y(t)^2 \rightarrow \infty$  as  $t \rightarrow T_+$  from the left for a finite  $T_+ > 0$ . This contradicts the previous part.

(iii). Observe that  $L(x, y)$  is a Lyapunov function with compact level sets (since  $P(x)$  is an even-degree polynomial with positive leading coefficient), and moreover by the first part, we have  $\langle \nabla L(x, y), F(x, y) \rangle \leq 0$  for  $F(x, y) = (y, -y^3 - p(x))$ . In particular, we deduce that  $\Omega \subseteq \{(x, y) : \langle \nabla L(x, y), F(x, y) \rangle = 0\} = \{(x, 0)\}$ . Thus, we obtain  $\bar{y} = 0$ .

It remains to show that  $p(\bar{x}) = 0$ . To this end, suppose not; without loss of generality, let us suppose  $p(\bar{x}) > 0$ , as the situation  $p(\bar{x}) < 0$  follows from a similar argument.

The set  $\Omega$  is positive-invariant. Thus, for all  $t \geq 0$ , under the initial condition  $(x(0), y(0)) = (\bar{x}, 0) \in \Omega$ , we have  $(x(t), y(t)) \in \Omega$ , so that  $y(t) = 0$  for all  $t \geq 0$ . However, because  $p(\bar{x}) > 0$ , we see that for some  $\varepsilon > 0$ , we have  $p(x(t)) > 0$  for all  $t \in [0, \varepsilon]$  by continuity. Thus, we deduce  $y(\varepsilon) < 0$ , which is a contradiction. Thus, we have  $p(\bar{x}) = 0$ .

(iv). The set  $\Omega$  must consist of points  $\{(x, 0)\}$  with  $x$  a root of the polynomial  $p(x)$  of which there exist finitely many. Thus,  $\Omega$  is a finite set (and non-empty, because  $L(x, y)$  has compact level sets). Moreover,  $\Omega$  is connected; thus, it must be a single point!