Math 63CM Homework 6 Solutions

February 20, 2020

Problem 1

Before we begin, we first observe that A has characteristic polynomial $p_A(\lambda) = \lambda^2 - 2a + a^2 - b^2 = (\lambda - a - b)(\lambda - a + b)$. In particular, the eigenvalues of A are given by $a+b$ and $a-b$, respectively.

In general, we note that a real symmetric matrix A has 0 as a stable equilibrium if and only if its eigenvalues are both non-positive. To see this, suppose the eigenvalues are both non-positive, and consider any $\varepsilon > 0$. Let v_1, v_2 be the corresponding eigenvectors to eigenvalues λ_1, λ_2, respectively. If $x = c_1 v_1 + c_2 v_2$, then

$$\|e^{tA}x\|^2 = |c_1|^2 e^{2t\lambda_1} + |c_2|^2 e^{2t\lambda_2}. \tag{0.1}$$

Since $\lambda_1, \lambda_2 \geqslant 0$, the RHS is bounded above by $|c_1|^2 + |c_2|^2 = \|x\|^2$. In particular, for any x with $\|x\| < \varepsilon$, we know $\|e^{tA}x\| \leqslant \varepsilon$ for all t.

On the other hand, suppose that $\lambda_1 > 0$, and consider any $\varepsilon > 0$. For any $\delta > 0$, assuming v_1 has unit norm, consider the initial condition $x(0) = \delta v_1$. Then we have

$$\|e^{tA}x(0)\|^2 = \delta^2 e^{2t\lambda_1}\|v_1\|^2 \tag{0.2}$$

$$= \delta^2 e^{2t\lambda_1}. \tag{0.3}$$

For any $\delta > 0$ fixed, we pick t sufficiently large os that $\delta e^{t\lambda_1} \geqslant 2\varepsilon$. This shows that 0 is not a stable equilibrium if $\lambda_1 > 0$ or $\lambda_2 > 0$ upon running the same argument in the latter case.

We further claim that 0 is an asymptotically stable equilibrium if and only if $\lambda_1, \lambda_2 < 0$. To this end, assume the conditions hold. We know from before that 0 is a stable equilibrium. Moreover, for any $x(0) = c_1 v_1 + c_2 v_2$ we have

$$\|e^{tA}x(0)\|^2 = |c_1|^2 e^{2t\lambda_1}\|v_1\|^2 + |c_2|^2 e^{2t\lambda_2}\|v_2\|^2. \tag{0.4}$$

Since $\lambda_1, \lambda_2 < 0$, the RHS vanishes as $t \to \infty$.

Now suppose that $\lambda_1 \geqslant 0$. First, we know $\lambda_1 = 0$ necessarily because otherwise 0 would not be a stable equilibrium. Then, we know $e^{tA}v_1 = v_1$ for all t. In particular, since $v_1 \neq 0$, $e^{tA}v_1$ does not vanish as $t \to \infty$.

(i). Note that 0 is a stable equilibrium if and only if $a - b \leqslant 0$ and $a + b \leqslant 0$. This implies that $a \leqslant b$ and $a \leqslant -b$.

(ii). Note that 0 is an asymptotically stable equilibrium if and only if $a - b < 0$ and $a + b < 0$, so that $a < b$ and $a < -b$.

(iii). Note that 0 is an unstable equilibrium if and only if either $a - b \geqslant 0$ or $a + b \geqslant 0$ (at least one of these needs to hold; in particular, both of these can hold as well). In particular, we have $a \geqslant b$ and $a \geqslant -b$.

Problem 2

Let v_1, v_2, v_3, v_4 denote a basis of unit norm eigenvectors with eigenvalues i,$-i$,$-1+i$, and $-1-i$, respectively.

(i). Consider the solution to $x'(t) = Ax(t)$ with $x(0) = v_1$. Then we know $x(t) = e^{tA}v_1 = e^{it}v_1$. This is periodic because e^{it} is periodic of period 2π.

(ii). Consider the solution to $x'(t) = Ax(t)$ with $x(0) = v_3$. Then we know $x(t) = e^{tA}v_3 = e^{-t+it}v_3$. Taking norms, we have

$$\begin{aligned} \|x(t)\| &= \|e^{-t+it}v_3\| && (0.5) \\ &= |e^{-t+it}|\|v_3\| && (0.6) \\ &= e^{-t}, && (0.7) \end{aligned}$$

which clearly vanishes as $t \to \infty$.

(iii). Consider the subset of vectors $U = \{v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 : \ c_i \neq 0\} \subset \mathbb{C}^4$. First, for any $x(0) \in U$, we observe

$$x(t) = e^{tA}x(0) = \sum_{j=1}^{4} c_j e^{t\lambda_j} v_j. \tag{0.8}$$

We claim that this is *not* periodic. In particular, from part (ii), we know $c_j e^{t\lambda_j} v_j \to 0$ as $t \to \infty$ for $j = 3, 4$. However, if it were periodic of period L, then for any $N > 0$, we would have $c_j e^{NL\lambda_j} = c_j$, since v_1, v_2, v_3, v_4 are linearly independent. Taking $N \to \infty$, we have $c_j = 0$, which contradicts the definition of U.

Further, we claim that $x(t)$ does not vanish as $t \to \infty$. To see this, we first note that it suffices to show $c_1 e^{t\lambda_1} v_1 + c_2 e^{t\lambda_2} v_2$ does not vanish. However, this sum is periodic and, by linear independence, does not vanish for any t, which proves the claim.

It remains to show that U is open and dense in $\mathbb{C}^4$. To see that it is open, we note that U^C is the union of the kernels of the projection maps $\Pi_j : \mathbb{C}^4 \to \mathbb{C}$ given by $\Pi_j(c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4) = c_j$ for $j = 1, 2, 3, 4$. These projection maps are continuous, and thus their kernels are closed. In particular, U^C is closed, so U is open.

To show denseness of $U \subseteq \mathbb{C}^4$, consider any $v \in \mathbb{C}^4$. For any $\varepsilon > 0$, we choose c_j' for $j = 1, 2, 3, 4$ such that $|c_j - c_j'| < \frac{\varepsilon}{100}$ and $c_j' \neq 0$; this is done via density of $\mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$. Then we have

$$\begin{aligned} \left\| v - \sum_{j=1}^{4} c_j' v_j \right\| &\leqslant \sum_{j=1}^{4} |c_j - c_j'| && (0.9) \\ &\leqslant \frac{\varepsilon}{25}. && (0.10) \end{aligned}$$

This shows denseness.

Problem 3

(i). Differentiating and using the ODE for $y_1(t)$, we see

$$\begin{aligned} E_1'(t) &= 2y_1(t)y_1'(t) + 2\omega_1^{-1}y_1'(t)y_1''(t) && (0.11) \\ &= 2y_1(t)y_1'(t) - 2y_1(t)y_1'(t) && (0.12) \\ &= 0. && (0.13) \end{aligned}$$

The same calculation shows $E_2'(t) = 0$.

(ii). For any θ, ϕ, define $\widetilde{\theta} = \theta + 2\pi n$ and $\widetilde{\phi} = \phi + 2\pi k$ for arbitrary integers k, n. It suffices to show that

$$\min_{\theta' \sim \theta, \phi' \sim \phi} \left|\theta' - \phi'\right| = \min_{\widetilde{\theta}' \sim \widetilde{\theta}, \widetilde{\phi}' \sim \widetilde{\phi}} \left|\widetilde{\theta}' - \widetilde{\phi}'\right|. \tag{0.14}$$

Indeed, this is true, because the sets $\{\theta', \phi' : \ \theta' \sim \theta, \phi' \sim \phi\}$ and $\{\widetilde{\theta}', \widetilde{\phi}' : \ \widetilde{\theta}' \sim \widetilde{\theta}, \widetilde{\phi}' \sim \widetilde{\phi}\}$ are equal. To see this, we may define a map from the former to the latter by $(\theta', \phi') \mapsto (\theta' + 2\pi n, \phi' + 2\pi k)$. This is well-defined and clearly a bijection.

(iii). Differentiating, we have

$$-\theta_i' \sin\theta_i = \frac{y_i'}{E_i}. \tag{0.15}$$

In particular, we see $\theta_i' = -\frac{y_i'}{E_i \sin\theta_i} = -\frac{y_i'}{E_i y_i'}\omega_i = \omega_i$. In particular, we know $\theta_i(t+T) = \theta_i(t) + \omega_i t$. This shows that

$$\begin{aligned} y_i(t + \frac{2\pi}{\omega_i}) &= E_i \cos\theta_i(t + \frac{2\pi}{\omega_i}) && (0.16)\\ &= E_i \cos(\theta_i(t) - \omega_i \frac{2\pi}{\omega_i}) && (0.17)\\ &= E_i \cos\theta_i(t) && (0.18)\\ &= y_i(t), && (0.19)\end{aligned}$$

which proves the periodicity of y_i. To show the periodicity of y_i', we see

$$\begin{aligned} y_i'(t + \frac{2\pi}{\omega_i}) &= E(t) \sin\theta_i(t + \frac{2\pi}{\omega_i}) && (0.20)\\ &= E_i \sin(\theta_i(t) - \frac{2\pi}{\omega_i}\omega_i) && (0.21)\\ &= E_i \sin\theta_i(t) && (0.22)\\ &= y_i'(t). && (0.23)\end{aligned}$$

However, the vector (y_1, y_1', y_2, y_2') is not periodic. If it were, then its period would have to be an integer multiple of $\frac{2\pi}{\omega_1}$ given the period of the first two components, and also an integer multiple of $\frac{2\pi}{\omega_2}$ given the period of the last two components. In particular, for some integer n we would have $\frac{2\pi}{\omega_1} = n\frac{2\pi}{\omega_2}$, which implies $\omega_1\omega_2^{-1}$ is rational; this contradicts our assumption.

(iii). Let us assume first that for some integer m, we have $\mathrm{d}\left(0, \frac{2\pi m\omega_1}{\omega_2}\right) < \delta$. For $T = \frac{2\pi m}{\omega_2}$, we then note

$$\begin{aligned} \mathrm{d}(\theta_1(t+T), \theta_1(t)) &= \mathrm{d}\left(-\frac{2\pi m\omega_1}{\omega_2}, 0\right) && (0.24)\\ &< \delta. && (0.25)\end{aligned}$$

Indeed, we computed $\theta_1(t+T) - \theta_1(t)$ explicitly in the previous part. On the other hand, we have

$$\begin{aligned} \mathrm{d}(\theta_2(t+T), \theta_2(t)) &= \mathrm{d}\left(-\frac{2\pi\omega_2}{\omega_2}, 0\right) && (0.26)\\ &= \mathrm{d}(2\pi m, 0) && (0.27)\\ &= 0. && (0.28)\end{aligned}$$

It now remains to show that there exists the desired integer m. To this end, we consider the sequence $\{[\frac{2\pi m\omega_1}{\omega_2}]\}_{m=1}^{\infty}$, where $[\frac{2\pi m\omega_1}{\omega_2}]$ denotes the representative of the equivalence class of $\frac{2\pi m\omega_1}{\omega_2}$ that is contained in $[0, 2\pi)$. Because $\omega_1\omega_2^{-1}$ is irrational, this sequence contains infinitely many distinct points (if not, then we have two of them equaling each other, which would show $\omega_1\omega_2^{-1}$ is rational).

Now, consider a decomposition of $[0, 2\pi)$ into finitely many disjoint, adjacent boxes of size $\frac{1}{2}\delta$. By the pigeonhole principle, there exists one of these boxes that contains two points in the aforementioned sequence. In particular, for some pair of distinct indices m, n, we have $[\frac{2\pi m\omega_1}{\omega_2}] = [\frac{2\pi n\omega_1}{\omega_2}]$. However, this implies that

$$\begin{aligned} \mathrm{d}\left(\frac{2\pi m\omega_1}{\omega_2}, \frac{2\pi n\omega_1}{\omega_2}\right) &= \mathrm{d}\left(0, \frac{2\pi(m-n)\omega_1}{\omega_2}\right) && (0.29)\\ &\leqslant \frac{1}{2}\delta. && (0.30)\end{aligned}$$

Thus, the desired integer is $m - n$.

(iv). For any $\varepsilon > 0$, there exists δ such that for any x with $\mathrm{d}(x,0) < \delta$, we have the following for all θ:

$$|\cos(\theta + x) - \cos(\theta)| \ + \ |\sin(\theta + x) - \sin(\theta)| \ < \ \frac{\varepsilon}{100\sqrt{\max(E_1, E_2)}}. \tag{0.31}$$

This is the uniform continuity of cosine and sine, which can be seen from their continuity and periodicity.

Choose T from the previous part for this $\delta > 0$. Then we have, for all t,

$$|y_i(t+T) - y_i(t)| \ = \ \sqrt{E_i}\,|\cos(\theta_i(t+T)) - \cos(\theta_i(t))| \tag{0.32}$$

$$< \ \sqrt{E_i}\frac{\varepsilon}{100\sqrt{\max(E_1, E_2)}} \tag{0.33}$$

$$< \ \frac{\varepsilon}{100}. \tag{0.34}$$

The same argument but with the sine representation shows that $|y_i'(t+T) - y_i'(t)| < \frac{\varepsilon}{100}$ for all t as well. In particular, this shows that (y_1, y_1', y_2, y_2') is almost periodic.

Problem 4

Let $\{v_1, \ldots, v_n\}$ denote a Jordan canonical form basis, so that if S is the matrix whose j-th column is v_j, we have $A = S\Lambda S^{-1}$ where Λ is in Jordan canonical form.

Suppose w is a vector so that $\limsup_{t\to\infty} \|\mathrm{e}^{tA}w\| < \infty$, and write $w = \sum_{j=1}^n c_j v_j$. We want to show that if v_j is a generalized eigenvector with eigenvalue λ having positive real part, or if v_j is a generalized eigenvector with imaginary eigenvalue λ that is not an eigenvector, then $c_j = 0$. To this end, suppose that one of these c_j is nonzero for the sake of contradiction and let the corresponding eigenvalue be denoted by λ and the index denoted by j_0. Next, for the block in Λ containing the index (j_0, j_0), let j_1 denote the row/column index of the top left entry in that block. If $\mathrm{e}^{tA}w = \sum_{j=1}^n c_j(t)v_j$, we then have

$$c_{j_1}(t) \ = \ c_{j_1}\mathrm{e}^{t\lambda} + c_{j_1+1}\mathrm{e}^{t\lambda}t + \ldots + c_{j_1+N}\mathrm{e}^{t\lambda}\frac{t^N}{N!}. \tag{0.35}$$

The assumption is that the RHS is a nonzero function of t. In particular, we see that $c_{j_1}(t)$ grows at least polynomially in t, since λ has positive real part with a specific growth rate of $c\mathrm{e}^{t\lambda}\frac{t^M}{M!}$ for some M and some nonzero constant c.

Among all the $c_j(t)$, consider those indices j for which $c_j(t)$ has the largest growth rate, i.e. for sufficiently large t, $|c_j(t)| \geqslant C|c_i(t)|$ for some constant C and all indices i; call the set of these indices $J_{\max}$. In particular, by the above, we know $|c_j(t)| \sim \mathrm{e}^{t\mathrm{Re}(\lambda)}\frac{t^M}{M!}$ for some positive M; here, $\sim$ denotes two constants which bound each other from above and below up to some constants. Thus, we know

$$\|w\| \ \sim \ \mathrm{e}^{t\mathrm{Re}(\lambda)}\frac{t^M}{M}\left\|\sum_{j\in J_{\max}} c_j' v_j\right\|, \tag{0.36}$$

where c_j' are nonzero constants. However, the t-dependent factor on the RHS diverges as $t \to \infty$ since either $\mathrm{Re}(\lambda) > 0$ or $\mathrm{Re}(\lambda) = 0$ and $M > 0$, which implies that the summation inside the norm on the RHS must vanish, which cannot happen because the v_j are a basis. This shows that

$$W \ = \ \left\{w : \limsup_{t\to\infty} \|\mathrm{e}^{tA}w\| < \infty\right\} \ \subseteq \ \bigoplus_{i=1}^{k} V^{(\lambda_i)} \oplus \bigoplus_{i=k+1}^{\ell} \ker(\lambda_i - A). \tag{0.37}$$

To show the other containment, we first observe that the LHS is a subspace by the triangle inequality and straightforward calculations. Thus, it suffices to show that $V^{(\lambda_i)} \subseteq W$ for each $i = 1, \ldots, k$ and that $\ker(\lambda_i - A) \subseteq W$ for all $i = k+1, \ldots, \ell$.

For the first containment, given $v \in V^{(\lambda_i)}$ for $i = 1, \ldots, k$, we first decompose $A = L + N$, where L is diagonalizable and acts by multiplication by λ_i on $V^{(\lambda_i)}$, and N is nilpotent. In particular, $\mathrm{e}^{tN}v$ is a vector whose entries are bounded by polynomials in t, since e^{tN} contains only finitely many powers of tN in its power series expansion. Thus, we know

$$\mathrm{e}^{tA}v \ = \ \mathrm{e}^{tL}\mathrm{e}^{tN}v \tag{0.38}$$

$$= \ \mathrm{e}^{t\lambda}\mathrm{e}^{tN}v. \tag{0.39}$$

Because $\mathrm{Re}(\lambda) < 0$, we know that the entries of the vector on the RHS are bounded above by $p(t)\mathrm{e}^{t\mathrm{Re}(\lambda)} \to 0$ as $t \to \infty$, where $p(t)$ is some polynomial. This shows that $v \in W$.

We now consider an eigenvector v of A with eigenvalue $\lambda = i\sigma$ for some $\sigma \in \mathbb{R}$. In this case, we have $\mathrm{e}^{tA}v = \mathrm{e}^{it\sigma}v$, which satisfies $\|\mathrm{e}^{tA}v\| = |\mathrm{e}^{it\sigma}|\|v\| = \|v\|$, so that $v \in W$ as well in this case.

Problem 5

We first show that S exists. To this end, we first write the Jordan canonical form $A = Q\Lambda Q^{-1}$, and obtain

$$\left([\mathrm{e}^{tA}]^T\mathrm{e}^{tA}\right)_{ij} = \sum_{k,\ell,r,p,q} \left[Q^{-1}\right]^T_{ik} \left[\mathrm{e}^{t\Lambda}\right]^T_{k\ell} [Q]^T_{\ell r} Q_{rp} \mathrm{e}^{t\Lambda}_{pq} Q^{-1}_{qj}. \tag{0.40}$$

Recall that the entries in $\mathrm{e}^{t\Lambda}$ are of the form $\frac{t^k}{k!}\mathrm{e}^{t\lambda}$ for some non-negative integer k and eigenvalue λ. By assumption, for some E we know $|\mathrm{e}^{t\lambda}| \leqslant \mathrm{e}^{-tE}$ for all λ and t. In particular, the entries in $\mathrm{e}^{t\Lambda}$ are bounded above by $P(t)\mathrm{e}^{-tE}$ for some polynomial $P(t)$. Thus, we deduce that the sum on the RHS above is bounded above by $Q(t)\mathrm{e}^{-2tE}$ for another polynomial $Q(t)$. This has a convergent integral on $[0, \infty)$, which proves that the limit S exists.

We now show that S is positive-definite. To this end, because $[\mathrm{e}^{tA}]^T\mathrm{e}^{tA}$ is symmetric for all t, we first note that S is symmetric. Now consider any nonzero vector v, and

$$v^T S v = v^T \int_0^\infty [\mathrm{e}^{tA}]^T \mathrm{e}^{tA}\, \mathrm{d}t v \tag{0.41}$$

$$= \int_0^\infty v^T [\mathrm{e}^{tA}]^T \mathrm{e}^{tA} v\, \mathrm{d}t \tag{0.42}$$

$$= \int_0^\infty \left\|\mathrm{e}^{tA}v\right\|^2 \mathrm{d}t. \tag{0.43}$$

The integrand is non-negative and continuous. In particular, to show that this integral is positive, it suffices to find $t \in [0, \infty)$ such that the integrand is strictly positive. For $t = 0$, the integrand becomes $\|v\|^2 > 0$, which completes the proof of positive-definiteness.

We finally show the desired identity. To this end, we observe $\mathrm{e}^{tA^T} = [\mathrm{e}^{tA}]^T$, and then we compute

$$v^T A^T S v = v^T A^T \int_0^\infty [\mathrm{e}^{tA}]^T \mathrm{e}^{tA}\, \mathrm{d}t v \tag{0.44}$$

$$= v^T \int_0^\infty A^T [\mathrm{e}^{tA^T}] \mathrm{e}^{tA}\, \mathrm{d}t v \tag{0.45}$$

$$= v^T \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{tA^T}] \mathrm{e}^{tA}\, \mathrm{d}t v \tag{0.46}$$

$$= v^T \left(-\mathrm{I} - \int_0^\infty \mathrm{e}^{tA^T} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA}\, \mathrm{d}t\right) v \tag{0.47}$$

$$= -\|v\|^2 - v^T \int_0^\infty \mathrm{e}^{tA^T} \mathrm{e}^{tA}\, \mathrm{d}t A v \tag{0.48}$$

$$= -\|v\|^2 - v^T S A v \tag{0.49}$$

$$= -\|v\|^2 - v^T S^T A v \tag{0.50}$$

$$= -\|v\|^2 - v^T A^T S v. \tag{0.51}$$

Thus, we have $v^T A^T S v = -\frac{1}{2}\|v\|^2$, which is the desired claim.