Math 63CM Homework 3 Solutions

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Problem 1

(i). Consider the function $y(t) = y_0 e^{at}$. Both the initial condition and ODE are readily checked.

(ii). Suppose $y_1(t), y_2(t)$ are both solutions to the ODE $y'(t) = a(t)y(t)$ with the same initial condition $y(0) = y_0$. We claim the set $\{t : y_1(t) = y_2(t)\}$ is the entire real line. To this end, because it is nonempty, it suffices to show that it is both closed and open.

- To show the closed property, observe $y_1(t), y_2(t)$ are both continuous. Now suppose we have any sequence $\{t_k\}_{k=1}^{\infty}$ with a limit t_∞ such that $y_1(t_k) = y_2(t_k)$ for all k. By taking limits and using continuity, we see $y_1(t_\infty) = y_2(t_\infty)$, proving closedness.
- To show the open property, suppose $y_1(t) = y_2(t)$; it suffices to find an open neighborhood around t for which $y_1 = y_2$ on the open neighborhood. But y_1, y_2 satisfy the same ODE everywhere with the same value at t, so local uniqueness provides us that $y_2 = y_2$ on some open neighborhood of this "new initial condition" t, which proves the openness.

We now show that if $y_0 > 0$, then $y(t) > 0$ for all $t \in \mathbb{R}$; the argument applies equally as well upon flipping signs to the problem of showing $y(t) < 0$ for all $t \in \mathbb{R}$ if $y_0 < 0$.

Suppose that $y(T) \leqslant 0$ for some $T \in \mathbb{R}$. Because $y(t)$ is continuous, for some t_0 between 0 and T we know $y(t_0) = 0$; this is consequence of the intermediate value theorem. Thus, y is a solution to $y'(t) = a(t)y(t)$ with $y(t_0) = 0$. But the zero function $z \equiv 0$ is also a solution to this ODE, which by the first half of this part implies that $y(t) = z(t) = 0$ for all $t \in \mathbb{R}$. This contradicts $y(0) = y_0 > 0$, so we deduce $y(t) > 0$ necessarily for all $t \in \mathbb{R}$.

As for why a solution exists, you can appeal to part (iii) and find the explicit solution. However, you can also do the following.

The function $F(t, x(t)) = a(t)x(t)$ is continuous in time and globally Lipschitz in the second variable. In particular, it is uniformly Lipschitz for all $t \in [-K, K]$ for any $K \in \mathbb{R}$.

For any $K > 0$, we define the function

$$F_K(t,x) = \begin{cases} F(t,x) & |t| \leqslant K \\ F(K,x) & t \geqslant K \\ F(-K,x) & t \leqslant -K \end{cases} \tag{0.1}$$

Thus, $F_K(t, x)$ is globally Lipschitz, and thus the equation $y_K'(t) = F_K(t, y_K(t))$ with initial condition $y_K(0) = y_0$ has a unique global solution. However, we know for $|t| \leqslant K$, that $y_K(t)$ solves the ODE $z'(t) = a(t)z(t)$. In particular, we have

- A family of functions $\{y_K(t)\}_{K=1}^{\infty}$ such that $y_K(t)$ solves the equation $y_K'(t) = F_K(t, y_K(t))$ with the K-independent initial condition $y_K(0) = y_0$. Moreover, we know $y_K'(t) = a(t)y_K(t)$ for $|t| \leqslant K$.
- From the above bullet point, we know that for any $K < L$, that $y_K(t)$ and $y_L(t)$ solve the same ODE with the same initial condition on $[-K, K]$. Thus, by the uniqueness proof (which applies to intervals as well as $\mathbb{R}$), we know $y_K(t) = y_L(t)$ for all $|t| \leqslant K$.

- Thus, we know that $\lim_{K\to\infty} y_K(t)$ exists for all t, and that the limiting function $y_\infty(t) = \lim_{K\to\infty} y_K(t)$ is continuously differentiable, since it coincides with continuously differentiable functions (the limit as $K \to \infty$ is constant for every t eventually). Moreover, this limit is uniform on compact sets, since for any compact interval $[-K,K]$, we know that $y_\infty(t) = y_L(t)$ for all $L \geqslant K$ and all $t \in [-K,K]$.
- Because the limit is uniform on compact sets, we know $a(s)y_K(s) \to a(s)y_\infty(s)$ uniformly on compact sets, so given the equation

$$y_K(t) \ = \ y_0 \ + \ \int_0^t a(s)y_K(s)\,\mathrm{d}s, \tag{0.2}$$

we may take limits as $K \to \infty$ to obtain

$$y_\infty(t) \ = \ y_0 \ + \ \int_0^t a(s)y_\infty(s)\,\mathrm{d}s. \tag{0.3}$$

But this implies that $y_\infty(t)$ solves the ODE $z'(t) = a(t)z(t)$ for all $t \in \mathbb{R}$ with the initial condition $z(0) = y_0$, so we have a global solution.

Roughly speaking, the point is that the factor $a(t)$ does not depend on the solution $y(t)$, the other factor of $F(t,x)$ in the equation depends on the solution in a globally Lipschitz fashion.

(iii). Suppose $y(0) = y_0 > 0$, so by part (ii) we know $y(t) > 0$ for all $t \in \mathbb{R}$. Defining $z(t) = \log y(t)$, by the chain rule we see $z'(t) = \frac{y'(t)}{y(t)}$ and $z(0) = \log y_0$. Using the ODE for $y(t)$, we deduce $z'(t) = a(t)$. Integrating, we see

$$z(T) \ = \ z(0) \ + \ \int_0^T a(t)\,\mathrm{d}t \ = \ \log y_0 \ + \ \int_0^T a(t)\,\mathrm{d}t. \tag{0.4}$$

Exponentiating and recalling the definition of $z(T)$, we see

$$y(T) \ = \ y_0 \exp\left[\int_0^T a(t)\,\mathrm{d}t\right], \tag{0.5}$$

which can be readily verified as the solution to the ODE with the appropriate initial condition. This argument works for $y_0 < 0$ upon setting $z(t) = \log[-y(t)]$, which is well-defined by part (ii) as well, and we would get the same answer.

Problem 2

(i).

- For $u_0 = 0$, we have $T_+, T_- = \infty$ because we have the solution $u(t) = 0$ for all $t \in \mathbb{R}$.
- For $u_0 \neq 0$, we have the solution $u(t) = \frac{u_0}{1-u_0 t}$.
  - We now record the candidate extremal times. In this case, if $u_0 > 0$, then $T_- = \infty$ and $T_+ = \frac{1}{u_0}$. If $u_0 < 0$, then $T_- = \frac{1}{|u_0|}$ and $T_+ = \infty$.

  It remains to show that these candidate times are the actual values of $T_\pm$ we are looking for. We consider $u_0 > 0$, since the argument for $u_0 < 0$ is the same.

  Suppose for the sake of contradiction that $T_+ > \frac{1}{u_0}$, so we have a solution $\phi(t)$ on $[-\infty, \frac{1}{u_0} + \varepsilon]$ for some $\varepsilon > 0$ sufficiently small; namely, $\phi(t)$ is continuous on this interval, and thus uniformly bounded in any sufficiently small compact neighborhood of $\frac{1}{u_0}$. However, we know $|u(t)| \to \infty$ as $t \to \frac{1}{u_0}$ from the left, so the arrive at a contradiction we need only show that $\phi(t) = u(t)$ for all $t < \frac{1}{u_0}$.

  We know $\phi(0) = u(0) = u_0$ from assumption; define S_+ as, roughly speaking, the maximal positive time less than $\frac{1}{u_0}$ so that $\phi(t) = u(t)$:

$$S_+ \ = \ \sup\{t \in [0, \frac{1}{u_0}) : \ u(t) = \phi(t)\}. \tag{0.6}$$

  It remains to show $S_+ = \frac{1}{u_0}$. To this end, suppose not. Because $S_+ < \frac{1}{u_0}$, and because both $u(t)$ and $\phi(t)$ are continuous, by definition of S_+ we can find a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to S_+$ from the left as $k \to \infty$, and $u(t_k) = \phi(t_k)$, finally allowing us to deduce $u(S_+) = \phi(S_+)$ by continuity. However, because u, ϕ solve the same ODE on any sufficiently small neighborhood of S_+, by local uniqueness, since the function $f(x) = x^2$ is locally Lipschitz, we know that there exists a small neighborhood of S_+ so that $u(t) = \phi(t)$ for all t in that neighborhood. But this includes values of $t > S_+$, which contradicts the definition of S_+ if $S_+ < \frac{1}{u_0}$. Thus, $S_+ = \frac{1}{u_0}$, which completes the proof.

From the above, we deduce $T_+, T_- = \infty$ if and only if $u_0 = 0$.

(ii). Define $F(x) = 1 + x^2$ and the initial condition $u_0 = 0$. If $u(t)$ solves the equation $u'(t) = F(u(t))$ with the initial condition $u(0) = 0$, consider the function $z(t) = \arctan u(t)$. Then, we see

$$z'(t) \;=\; \frac{u'(t)}{1 + u(t)^2} \;=\; 1 \tag{0.7}$$

and the initial condition $z(0) = \arctan u(0) = \arctan(0) = 0$ tells us $z(t) = t$.

We claim that $T_+ \leqslant \frac{\pi}{2}$. Because $u(t)$ is continuous with the initial condition $u(0) = 0$, we know $u(t) = \tan(t)$, where here the tangent function is the piece of the periodic graph that has $u(0) = 0$. However, we know $u(t) \to \infty$ as $t \to \frac{\pi}{2}$ from the left, which shows that $\frac{\pi}{2}$ is the terminal time to the right of 0. To show this is indeed the terminal time, we may use the argument from part (i).

Showing that $T_- \leqslant \frac{\pi}{2}$ follows from the same argument.

Problem 3

(i). By explicit computation, namely the chain rule, we see

$$u'(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\sin(2t) \;=\; 2\cos(2t) \;=\; v(t) \tag{0.8}$$

$$v'(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}2\cos(2t) \;=\; -4\sin(2t) \;=\; -4u(t). \tag{0.9}$$

We also have $u(0) = \sin(0) = 0$, and $v(0) = 2\cos(0) = 2$.

(ii). Let us compute the Picard iterates. We first claim that

$$u_k(t) \;=\; \sum_{\ell=0}^{k-1} \frac{(-1)^\ell 2^{2\ell+1} t^{2\ell+1}}{(2\ell+1)!}, \quad v_k(t) \;=\; 2\sum_{\ell=0}^{k} \frac{(-1)^\ell 2^{2\ell} t^{2\ell}}{(2\ell)!}, \tag{0.10}$$

where the empty sum denotes a value of 0. To this end, we proceed inductively. For $k = 0$, this is clear by assumption. Moreover, we now compute

$$u_1(t) \;=\; \int_0^t v_0(s)\,\mathrm{d}s \tag{0.11}$$

$$=\; \int_0^t 2\,\mathrm{d}s \tag{0.12}$$

$$=\; 2t, \tag{0.13}$$

and

$$v_2(t) \;=\; 2 - 4\int_0^t u_1(s)\,\mathrm{d}s \tag{0.14}$$

$$=\; 2 - 4\int_0^t 2s\,\mathrm{d}s \tag{0.15}$$

$$=\; 2 - 4t^2, \tag{0.16}$$

which both agree with the asserted claim. Suppose now the claim holds for $k \in \mathbb{Z}_{>0}$. We now compute

$$u_{k+1}(t) = \int_0^t v_k(s)\, \mathrm{d}s \tag{0.17}$$

$$= \int_0^t 2\sum_{\ell=0}^{k} \frac{(-1)^\ell 2^{2\ell} s^{2\ell}}{(2\ell)!}\, \mathrm{d}s \tag{0.18}$$

$$= \sum_{\ell=0}^{k} \frac{(-1)^\ell 2^{2\ell+1}}{(2\ell)!} \int_0^t s^{2\ell}\, \mathrm{d}s \tag{0.19}$$

$$= \sum_{\ell=0}^{k} \frac{(-1)^\ell 2^{2\ell+1} t^{2\ell+1}}{(2\ell+1)!}. \tag{0.20}$$

Similarly, we compute

$$v_{k+1} = 2 - 4\int_0^t u_k(s)\, \mathrm{d}s \tag{0.21}$$

$$= 2 - 4\int_0^t \sum_{\ell=0}^{k-1} \frac{(-1)^\ell 2^{2\ell+1} s^{2\ell+1}}{(2\ell+1)!}\, \mathrm{d}s \tag{0.22}$$

$$= 2 - 4\sum_{\ell=0}^{k-1} \frac{(-1)^\ell 2^{2\ell+1}}{(2\ell+1)!} \int_0^t s^{2\ell+1}\, \mathrm{d}s \tag{0.23}$$

$$= 2 - 4\sum_{\ell=0}^{k-1} \frac{(-1)^\ell 2^{2\ell+1} t^{2\ell+2}}{(2\ell+2)!} \tag{0.24}$$

$$= 2 - 4\sum_{\ell=1}^{k} \frac{(-1)^{\ell+1} 2^{2(\ell-1)+1} t^{2\ell}}{(2\ell)!} \tag{0.25}$$

$$= 2 - \sum_{\ell=1}^{k} \frac{(-1)^{\ell+1} 2^{2\ell+1} t^{2\ell}}{(2\ell)!} \tag{0.26}$$

$$= \sum_{\ell=0}^{k} \frac{(-1)^\ell 2^{2\ell+1} t^{2\ell}}{(2\ell)!}, \tag{0.27}$$

both of which agree with the claim.

We now claim that for $|t| < \varepsilon$ with $\varepsilon > 0$ sufficiently small, we have $u_k(t) \to \sin(2t)$ and $v_k(t) \to 2\cos(2t)$ as $k \to \infty$. To this end, observe that

$$\sin(2t) - u_k(t) = \sum_{\ell=k}^{\infty} \frac{(-1)^\ell 2^{2\ell+1} t^{2\ell+1}}{(2\ell+1)!}. \tag{0.28}$$

Taking the absolute value, applying the triangle inequality gives us

$$|\sin(2t) - u_k(t)| \leqslant \sum_{\ell=k}^{\infty} 2^{2\ell+1} |t^{2\ell+1}|. \tag{0.29}$$

Assuming $|t| < \varepsilon$ for $\varepsilon < \frac{1}{2}$, say, then the RHS is the tail of a geometric series. Thus, we may compute it exactly as

$$|\sin(2t) - u_k(t)| \leqslant \sum_{\ell=k}^{\infty} [2\varepsilon]^{2\ell+1} \tag{0.30}$$

$$= 2\varepsilon \sum_{\ell=k}^{\infty} [4\varepsilon^2]^\ell \tag{0.31}$$

$$= 2\varepsilon \frac{[4\varepsilon^2]^k}{1 - 4\varepsilon^2} \tag{0.32}$$

which clearly vanishes as $k \to \infty$ if $\varepsilon < \frac{1}{2}$.

Problem 5

(i). Define $z(t) = x_2(t) - x_1(t)$, and let $T = \sup\{t \in [0,\infty) : \inf_{s\in[0,t]} z(s) \geqslant 0\}$. Suppose for the sake of contradiction that $T < \infty$. Because $x_2(t), x_1(t)$ are continuous, we know $z(t)$ is continuous and thus $z(T) = 0$. However,

$$z'(T) \;=\; F_2(x_2(T)) - F_1(x_1(T)) \;=\; F_2(x_2(T)) - F_1(x_2(T)) \;>\; 0. \tag{0.33}$$

In particular, because $z'(T)$ is continuous, we know that it is strictly positive on some neighborhood of $T \in \mathbb{R}$. Thus, by the fundamental theorem of calculus, we have

$$z(T+\varepsilon) \;=\; z(T) \;+\; \int_0^\varepsilon z'(T+s)\,\mathrm{d}s \;=\; \int_0^\varepsilon z'(T+s)\,\mathrm{d}s \tag{0.34}$$

for any $\varepsilon > 0$ sufficiently small, since $z(T) = 0$. However, the LHS is negative by assumption if $\varepsilon > 0$ is sufficiently small, and the RHS is positive is $\varepsilon > 0$ is sufficiently small, which is a contradiction.

(ii). If $x_2^{(n)}(t)$ solves the equation $y'(t) = F_2^{(n)}(y) = F_2(y) + \frac{1}{n}$, with the initial condition $x_2^{(n)}(0) + \frac{1}{n} > x_1(0)$, then by part (i) we know that $x_2^{(n)}(t) \geqslant x_1(t)$ for all $t \in \mathbb{R}$.

We now show that $x_2^{(n)}(t) \to_{n\to\infty} x_2(t)$ uniformly in compact sets in $t \in \mathbb{R}$. To this end, define $z_n(t) = x_2^{(n)}(t) - x_2(t)$. By the fundamental theorem of calculus, and the initial conditions matching, we see

$$z_n(t) \;=\; \frac{1}{n} + \int_0^t z_n'(s)\,\mathrm{d}s \;=\; \frac{1}{n} + \int_0^t F_2(x_2^{(n)}(s)) - F_2(x_2(s))\,\mathrm{d}s \;+\; \frac{1}{n}t. \tag{0.35}$$

Taking absolute values and using the uniform Lipschitz property for F_2, for all t in some compact set K, we see

$$|z_n(t)| \;\leqslant\; \frac{1}{n} + \int_0^t \left|F_2(x_2^{(n)}(s)) - F_2(x_2(s))\right|\mathrm{d}s \;+\; \frac{1}{n}t \tag{0.36}$$

$$\leqslant\; C\int_0^t |z_n(s)|\,\mathrm{d}s \;+\; \frac{1}{n}C_K + \frac{1}{n}, \tag{0.37}$$

where C depends on the Lipschitz norm of F and C_K depends only on the compact set K. Iterating as in Problem 8 below, we see

$$|z_n(t)| \;\leqslant\; \frac{1}{n}C_k \mathrm{e}^{Ct} \tag{0.38}$$

$$\leqslant\; \frac{1}{n}C_k \mathrm{e}^{C_K'} \tag{0.39}$$

where C_K' is another constant that depends only on the compact set K. Because this upper bound is uniform in $t \in K$, taking $n \to \infty$ completes the proof.

Problem 6

We give two proofs.

Fix any $\varepsilon > 0$. For any $t \in J$, given the "weaker-looking" condition we may find a neighborhood $B_t = \{s \in J : |t-s| < \delta_t\}$ with $\delta_t > 0$ such that for all n, we have $|f_n(s) - f_n(t)| < \frac{1}{2}\varepsilon$ for all $s \in B_t$. By the triangle inequality, we deduce

$$|f_n(s) - f_n(s')| \;\leqslant\; |f_n(s) - f_n(t)| \;+\; |f_n(s') - f_n(t)| \;<\; \varepsilon \tag{0.40}$$

for all $s, s' \in B_t$.

We now consider a covering of J by the neighborhoods Q_t, where $Q_t = \{s \in J : |t-s| < \frac{1}{2}\delta_t\}$ are the neighborhoods with half the radius the corresponding B_t. Because J is compact, we have a finite subcovering, so that $J = Q_{t_1} \cup \ldots \cup Q_{t_M}$ for a finite set of points $t_1, \ldots, t_M$. Define $\delta = \min_{i=1\ldots,M} \frac{1}{2}\delta_{t_i}$.

We now prove the "stronger-looking" condition. To this end, suppose $|t-s| < \delta$, and suppose $s \in Q_{t_i}$ for some $i = 1, \ldots, M$. Clearly, we have $s \in B_{t_i}$. By the triangle inequality, we know

$$|t - t_i| \;<\; |t-s| + |s - t_i| \;<\; \delta + \frac{1}{2}\delta_{t_i} \;<\; \delta_{t_i}. \tag{0.41}$$

Thus, we have $t \in B_{t_i}$, and thus since $s \in B_{t_i}$ we have $|f_n(t) - f_n(s)| < \varepsilon$ for all n. This completes the proof.

For the second proof, we define the function $\delta : J \to \mathbb{R}$

$$\delta_t^{\varepsilon} \;=\; \sup\{r \geqslant 0 : \; |t-s| < r \;\to\; |f_n(t) - f_n(s)| < \varepsilon, \; n \in \mathbb{Z}_{>0}\}\,. \tag{0.42}$$

The weaker looking condition guarantees that $\delta_t^{\varepsilon} > 0$ for all $t \in J$ and $\varepsilon > 0$. It suffices to show that $\inf_{t \in J} \delta_t^{\varepsilon} > 0$ for all $\varepsilon > 0$.

Suppose not for the sake of contradiction. Then we have a sequence $\{t_k\}_{k=1}^{\infty}$ of points in J so that $\delta_{t_k}^{\varepsilon} \to 0$. Because J is compact, we have a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to t_\infty \in J$ and $\delta_{t_{n_k}} \to 0$, still. Moreover, for n_k sufficiently large, we know $|t_{n_k} - t_\infty| \leqslant \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon}$; notice the change in parameter to $\frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon}$. We now show that for such t_{n_k}, we have $\delta_{t_{n_k}}^{\varepsilon} \geqslant \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon}$; this would provide a contradiction, since the RHS strictly positive, whereas the LHS is assumed to vanish as $t_{n_k} \to t_\infty$.

To prove the claim, suppose $|s - t_{n_k}| < \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon}$. By the triangle inequality, we have

$$|s - t_\infty| \;<\; |s - t_{n_k}| \;+\; |t_{n_k} - t_\infty| \tag{0.43}$$

$$\;<\; \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon} + \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon} \tag{0.44}$$

$$\;=\; \delta_{t_\infty}^{\frac{1}{2}\varepsilon}. \tag{0.45}$$

By definition, we deduce $|f_n(s) - f_n(t_\infty)| < \frac{1}{2}\varepsilon$ for all n. Meanwhile, we know $|t_{n_k} - t_\infty| < \delta_{t_\infty}^{\frac{1}{2}\varepsilon}$ as was used above, so we have $|f_n(t_{n_k}) - f_n(t_\infty)| < \frac{1}{2}\varepsilon$ as well. By the triangle inequality, we have

$$|f_n(s) - f_n(t_{n_k})| \;<\; |f_n(s) - f_n(t_\infty)| + |f_n(t_\infty) - f_n(t_{n_k})| \tag{0.46}$$

$$\;<\; \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \tag{0.47}$$

$$\;=\; \varepsilon. \tag{0.48}$$

So in particular, for any s such that $|s - t_{n_k}| < \frac{1}{2}\delta_{t_\infty}^{\frac{1}{2}\varepsilon}$, we know $|f_n(s) - f_n(t_{n_k})| < \varepsilon$ for all n, which proves the claim.

Problem 7

We define the following sequence of functions:

$$f_n(x) \;=\; \begin{cases} 1 & x \in [0, \frac{1}{n}] \\ 2 - nx & x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & x \in [\frac{2}{n}, 1] \end{cases} \tag{0.49}$$

for $n \geqslant 3$, say. In words, $f_n(x)$ is the linear interpolation between the constant function 1 on $[0, \frac{1}{n}]$ and the 0 function on $[\frac{2}{n}, 1]$.

Clearly, the sequence of functions is uniformly bounded. We claim that $\{f_n\}_{n=1}^{\infty}$ cannot converge uniformly to a continuous limit. To this end, suppose that it did, and that $f_n \to g$ uniformly as $n \to \infty$ with g a continuous function.

We first claim $g(x) = 0$ for any $x \in (0, 1]$. To see this, for any such $x \in (0, 1]$, we take N sufficiently large so that $x > \frac{1}{N}$. Then we have $f_M(x) = 0$ for all $M \geqslant N$, and thus

$$g(x) \;=\; \lim_{M \to \infty} f_M(x) \;=\; 0. \tag{0.50}$$

Moreover, we claim that $g(0) = 1$; this would show that g is not continuous. This follows from the observation that $f_n(0) = 1$ for all n.

It now remains to show that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is not equicontinuous. To this end, pick any small $\varepsilon > 0$, and pick any $\delta > 0$. It suffices to show that there exists a function f_n in the sequence for which there exist two points $s, t \in [0, 1]$

such that $|t-s|<\delta$ but $|f_n(t)-f_n(s)|>\varepsilon$, since the parameters $\delta,\varepsilon>0$ were arbitrarily small. To this end, we choose N sufficiently large so that $\frac{1}{N}<\delta$. We then choose $s=\frac{1}{N}$ and $t=\frac{2}{N}$. This gives

$$|f_N(t)-f_N(s)| \;=\; 1 \;>\; \varepsilon \tag{0.51}$$

if ε is sufficiently small. However, we know $|t-s|=\frac{1}{N}<\delta$; this shows the sequence is not equicontinuous.

Problem 8

Suppose $x(t)$ is a solution to the ODE $x'(t)=F(x(t))$ with the initial condition $x(0)=x_0$, and suppose that the maximal interval on which we have the solution is $(-T_-,T_+)$ for $T_-,T_+\in(0,\infty)\cup\{+\infty\}$.

Suppose first that $T_+<\infty$. Then there exists a sequence of points $\{t_n\}_{n\to\infty}$ converging to T_+ so that $\|x(t_n)\|\to\infty$ as $n\to\infty$. On the other hand, for each t_n, we have

$$x(t_n) \;=\; x_0 \;+\; \int_0^{t_n} F(x(s))\,\mathrm{d}s. \tag{0.52}$$

Taking norms and using the triangle inequality, we have

$$\|x(t_n)\| \;\leqslant\; \|x_0\| \;+\; \int_0^{t_n} \|F(x(s))\|\,\mathrm{d}s \tag{0.53}$$

$$\leqslant\; \|x_0\| \;+\; \int_0^{t_n} C\|x(s)\|\,\mathrm{d}s. \tag{0.54}$$

Indeed, this bound is true upon replacing t_n with any $t\in[0,t_n]$. In particular, we deduce

$$\|x(t_n)\| \;\leqslant\; \|x_0\| \;+\; \int_0^{t_n} \|x_0\|\,\mathrm{d}s \;+\; \int_0^{t_n}\int_0^{s} \|x(r)\|\,\mathrm{d}r \tag{0.55}$$

and further iterating as in Problem 5 of the previous homework, we see

$$\|x(t_n)\| \;\leqslant\; \|x_0\|\sum_{k=0}^{\infty}\frac{t_n^k}{k!} \tag{0.56}$$

$$\leqslant\; \|x_0\|\mathrm{e}^{t_n} \tag{0.57}$$

$$\leqslant\; \|x_0\|\mathrm{e}^{T_+}. \tag{0.58}$$

However, this is uniformly bounded as $t_n\to T$, which contradicts the assumption that $T_+<\infty$. To show $T_-=\infty$, the same argument applies.

Problem 9

We first make the observation that because $T<\infty$, we must have $|y(t_n)|\to_{n\to\infty}+\infty$ along some sequence $\{t_n\}_{n=1}^{\infty}$ converging to T from the left. In particular, we must have $|y'(t_k)|\to_{k\to\infty}+\infty$ for possibly another sequence $\{t_k\}_{k=1}^{\infty}$ converging to T, since by the fundamental theorem of calculus, we have

$$y(T-\varepsilon) \;=\; y(0) \;+\; \int_0^{T-\varepsilon} y'(s)\,\mathrm{d}s, \tag{0.59}$$

and if $|y'(s)|$ were uniformly bounded near T, we would have $|y(t)|$ be uniformly bounded near T, which is a contradiction.

Given the ODE, we multiply by $2y'(t)$ on both sides to obtain

$$2y''(t)y'(t) \;=\; -2\nabla V(y(t))y'(t) \tag{0.60}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}[y'(t)^2] \;=\; -2\frac{\mathrm{d}}{\mathrm{d}t}V(y(t)), \tag{0.61}$$

so we have, for any small $\varepsilon > 0$,

$$y'(t_k)^2 \;=\; y'(0)^2 \;-\; 2V(y(t_k)) \;+\; 2V(y(0)). \tag{0.62}$$

Letting $k \to \infty$, the LHS diverges. Because the first and third quantities on the RHS are independent of k, we deduce that $V(y(t_k)) \to -\infty$.