Math 63CM Homework 1 Solutions

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1. Problem 1

It suffices to show $\det(B-\lambda I) = \det(A-\lambda I)$ for all $\lambda \in \mathbb{C}$. Indeed, this would identify the respective roots and thus eigenvalues. To prove this, we note $I = C^{-1}IC$ and thus

$$\begin{aligned} \det(B-\lambda I) &= \det\left(C^{-1}AC - C^{-1}\lambda IC\right) && (1.1)\\ &= \det\left(C^{-1}(A-\lambda I)C\right) && (1.2)\\ &= \det(C^{-1})\det(A-\lambda I)\det(C) && (1.3)\\ &= \det(A-\lambda I) && (1.4)\end{aligned}$$

since $\det(C^{-1}) = \det(C)^{-1}$ if C is invertible.

2. Problem 2

Take $y \in f(U)$ given by $y = f(x)$. It suffices to show that there exists a neighborhood of y that is contained in $f(U)$. To this end, by the inverse function theorem, since $\det Df(x) \neq 0$, there exists a neighborhood $B(x) \subseteq U$ such that f defines a continuously differentiable bijection $f : B(x) \to f(B(x))$ whose inverse is continuously differentiable with $f(B(x))$ an open neighborhood of y. But $f(B(x))$ is the neighborhood we want, since it is open, it contains $y = f(x)$, and it is contained in $f(U)$ since $B(x) \subseteq U$.

3. Problem 3

We first prove that the annulus is open. To do this, we write it as

$$U = \left\{x \in \mathbb{R}^2 : \|x\| < 1\right\} \cap \left\{x \in \mathbb{R}^2 : \|x\| > \frac{1}{2}\right\}. \quad (3.1)$$

Both sets on the RHS are open, since one is the interior of the unit disc and the other is the complement of the closed disc of radius $2^{-\frac{1}{2}}$. Thus, their intersection is open.

Consider the example function

$$f(x,y) = \begin{pmatrix} \frac{x^2-y^2}{\sqrt{x^2+y^2}} \\ \frac{2xy}{\sqrt{x^2+y^2}} \end{pmatrix}. \quad (3.2)$$

It is easy to check that $\|f(x,y)\| = \|(x,y)\|$, i.e. the function preserves lengths, since

$$\begin{aligned} \|f(x,y)\|^2 &= \frac{(x^2-y^2)^2}{x^2+y^2} + \frac{4x^2y^2}{x^2+y^2} && (3.3)\\ &= \frac{x^4+2x^2y^2+4y^2}{x^2+y^2} && (3.4)\\ &= x^2+y^2 && (3.5)\\ &= \|(x,y)\|^2, && (3.6)\end{aligned}$$

which means it defines a map from U to itself. Moreover, if $y = 0$, then

$$f(x,0) = \begin{pmatrix} |x| \\ 0 \end{pmatrix} = f(-x,0), \tag{3.7}$$

and because $(x,0) \in U$ if and only if $(-x,0) \in U$, this function is not injective. So it remains to check that $\det Df(x,y) > 0$ for every $(x,y) \in U$; it happens to be true for every $(x,y) \neq (0,0)$; which can be checked directly. In particular, we may compute

$$Df(x) = \begin{pmatrix} \frac{2x}{[x^2+y^2]^{\frac{1}{2}}} - \frac{x(x^2-y^2)}{[x^2+y^2]^{\frac{3}{2}}} & -\frac{2y}{[x^2+y^2]^{\frac{1}{2}}} - \frac{y(x^2-y^2)}{[x^2+y^2]^{\frac{3}{2}}} \\ \frac{2y}{[x^2+y^2]^{\frac{1}{2}}} - \frac{2x^2y}{[x^2+y^2]^{\frac{3}{2}}} & \frac{2x}{[x^2+y^2]^{\frac{1}{2}}} - \frac{2xy^2}{[x^2+y^2]^{\frac{3}{2}}} \end{pmatrix}, \tag{3.8}$$

from which the determinant turns out to be 2.

Alternatively, in polar coordinates, the map is given by $(r,\theta) \mapsto (r, 2\theta)$; thus this map is linear in these coordinates so its derivative matrix in these coordinates is equal to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, or maybe $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ depending on your ordering of coordinates. So its derivative matrix has determinant 2 at all values of (r,θ).

Denoting the map in Cartesian coordinates by $f^{X,Y}$ and the map in polar coordinates by $f^{r,\theta}$, if C denotes the map given by changing from Cartesian to polar coordinates, then we have $f^{X,Y} = C^{-1} \circ f^{r,\theta} \circ C$. Applying the Chain rule, we have

$$\det Df^{X,Y} = \det\left[DC^{-1}(f^{r,\theta} \circ C)\right] \cdot \det\left[Df^{r,\theta}(C)\right] \cdot \det C. \tag{3.9}$$

The determinant of C at a point (r,θ) is equal to r; because $f^{r,\theta}$ preserves the radius r, we see $\det\left[DC^{-1}(f^{r,\theta} \circ C)\right] = r^{-1}$ when evaluated at any point (r,θ). Thus, we have

$$\det Df^{X,Y} = r^{-1} \cdot 2 \cdot r = 2, \tag{3.10}$$

which is what we want.

4. Problem 4

Since $||Ax|| \leqslant 1$ is equivalent to $||Ax||^2 \leqslant 1$ and $||Ax||^2 = [Ax] \cdot [Ax] = x \cdot A^T Ax$, we see

$$E = \left\{x \in \mathbb{R}^N : x \cdot A^T Ax \leqslant 1\right\}. \tag{4.1}$$

Because $A^T A$ is symmetric, we can diagonalize it into $A^T A = Q^T DQ$, where Q is an orthogonal matrix and D is the diagonal matrix of eigenvalues.

We now claim the eigenvalues of $A^T A$ are positive. To this end, suppose λ is an eigenvalue with eigenvector v; we have

$$v \cdot A^T Av = \lambda ||v||^2 \tag{4.2}$$

and

$$v \cdot A^T Av = ||Av||^2. \tag{4.3}$$

Thus, we see $\lambda \geqslant 0$. Because A is invertible, its transpose is invertible and thus $A^T A$ is invertible, so it does not have 0 as an eigenvalue. Thus, all eigenvalues are positive. Thus, we have

$$E = \left\{x \in \mathbb{R}^N : x \cdot A^T Ax \leqslant 1\right\} \tag{4.4}$$

$$= \left\{x \in \mathbb{R}^N : x \cdot Q^T DQx \leqslant 1\right\} \tag{4.5}$$

$$= \left\{x \in \mathbb{R}^N : \sum_{i=1}^{N} \lambda_i [Qx]_i \right\}, \tag{4.6}$$

with $\lambda_i > 0$, which is exactly the desired claim.

5. Problem 5

(i). Suppose that $x(t)$ is k-times differentiable. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \;=\; \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\int_0^t x(s)\,\mathrm{d}s \tag{5.1}$$

$$= \; x(t), \tag{5.2}$$

which thus has k-many more derivatives, and thus $x(t)$ has $k+1$-derivatives. Because $x(t)$ has 1 derivative already, induction tells us that $x(t)$ is infinitely differentiable. Moreover, we know $\frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t) = x(t)$ also by induction. Thus, for any point $t_0 \in \mathbb{R}$, we know $\frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t)|_{t=t_0} = x(t_0)$. This implies the Taylor approximations of $x(t)$ near t_0 are given by

$$T_n(t) \;=\; \sum_{k=0}^{n}\frac{x(t_0)}{k!}(t-t_0)^k. \tag{5.3}$$

Moreover, for any $t \in \mathbb{R}$, the error between $T_n(t)$ and the solution $x(t)$ is given by

$$x(t) \;-\; T_n(t) \;=\; \frac{\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}x(t)|_{t=t'}}{(n+1)!}(t-t_0)^{n+1} \tag{5.4}$$

$$= \; \frac{x(t')}{(n+1)!}(t-t_0)^{n+1}. \tag{5.5}$$

where t' is somewhere between t_0, t. Because this last quantity converges to 0 uniformly in compact sets around any t_0, we see $|x(t)-T_n(t)| \to 0$ as $n \to \infty$ uniformly in compact sets around any t_0, which shows that the Taylor approximations of $x(t)$ indeed converge to $x(t)$ itself.

(ii). The function $\widetilde{x}_s : t \mapsto x(t+s)$ solves the ODE $\widetilde{x}_s'(t) = x'(t+s) = x(t+s) = \widetilde{x}_s'(t)$, with initial condition $\widetilde{x}_s(0) = x(s)$. The function $\hat{x}_s : t \mapsto x(t)x(s)$ solves the ODE $\hat{x}_s'(t) = x'(t)x(s) = x(t)x(s) = \hat{x}_s(t)$ with the initial condition $\hat{x}_s(0) = x(s)$. Because for each fixed s, the functions $\widetilde{x}_s(t)$ and $\hat{x}_s(t)$ both solve the equation $y' = y$ with the same initial condition, and because the function $f(y) = y$ is globally Lipschitz, by global uniqueness $\hat{x}_s(t) = \widetilde{x}_s(t)$, or that $x(t+s) = x(t)x(s)$ for all $s, t \in \mathbb{R}$.

(iii). The claim is that

$$T^n x(t) \;=\; \sum_{k=0}^{n}\frac{t^k}{k!}. \tag{5.6}$$

To prove this, we proceed inductively. Clearly, we have $T^1x(t) = 1 + \int_0^t x(0)\,\mathrm{d}s = 1 + t$. Inductively, we have

$$T^{n+1}x(t) \;=\; 1 \;+\; \int_0^t T^n x(s)\,\mathrm{d}s \tag{5.7}$$

$$= \; 1 \;+\; \int_0^t \sum_{k=0}^{n}\frac{s^k}{k!}\,\mathrm{d}s \tag{5.8}$$

$$= \; 1 \;+\; \sum_{k=0}^{n}\frac{1}{k!}\int_0^t s^k\,\mathrm{d}s \tag{5.9}$$

$$= \; 1 \;+\; \sum_{k=0}^{n}\frac{1}{(k+1)!}t^{k+1} \tag{5.10}$$

$$= \; \sum_{k=0}^{n+1}\frac{t^k}{k!}, \tag{5.11}$$

which completes the proof.

6. PROBLEM 6

(i). Differentiating via the chain rule, we see

$$g'(x) \;=\; \frac{\mathrm{d}}{\mathrm{d}x}[-\pi x^2]\mathrm{e}^{-\pi x^2} \;=\; -2\pi x g(x), \tag{6.1}$$

which is the desired ODE. Checking $g(0) = 1$ is straightforward.

(ii). Differentiating directly, we see

$$\begin{aligned} u'(k) \;&=\; \frac{\mathrm{d}}{\mathrm{d}k}\int_{-\infty}^{\infty} \cos(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x & (6.2)\\ &=\; \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}k}\cos(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x & (6.3)\\ &=\; \int_{-\infty}^{\infty} -2\pi x\sin(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x & (6.4)\\ &=\; \int_{-\infty}^{\infty} \sin(2\pi kx)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-\pi x^2}\,\mathrm{d}x & (6.5)\\ &=\; -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\sin(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x & (6.6)\\ &=\; -2\pi k\int_{-\infty}^{\infty} \cos(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x. & (6.7)\end{aligned}$$

Moreover, we know $u(0) = \int_{-\infty}^{\infty}\mathrm{e}^{-\pi x^2}\,\mathrm{d}x = 1$. It remains to justify the second identity. To do this, it suffices to show that for each $k \in \mathbb{R}$, we have

$$\lim_{k\to\infty}\left[\frac{u(k+h)-u(k)}{h} - \int_{-\infty}^{\infty}(-2\pi x)\sin(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x\right] \;=\; 0. \tag{6.8}$$

Computing the inside of the limit, because all of the integrals are absolutely convergent (i.e. by comparing to the associated infinite series and apply the ratio test), we see

$$\begin{aligned} &\left|\frac{u(k+h)-u(k)}{h} - \int_{-\infty}^{\infty}(-2\pi x)\sin(2\pi kx)\mathrm{e}^{-\pi x^2}\,\mathrm{d}x\right| \\ &\quad=\; \lim_{L\to\infty}\left|\int_{-L}^{L}\left[\frac{\cos(2\pi(k+h)x)-\cos(2\pi kx)}{h} - (-2\pi x)\sin(2\pi kx)\right]\mathrm{e}^{-\pi x^2}\,\mathrm{d}x\right| & (6.9)\\ &\quad\leqslant\; \lim_{L\to\infty}\int_{-L}^{L}\left|\frac{\cos(2\pi(k+h)x)-\cos(2\pi kx)}{h} - (-2\pi x)\sin(2\pi kx)\right|\mathrm{e}^{-\pi x^2}\,\mathrm{d}x. & (6.10)\end{aligned}$$

Estimating inside, by the fundamental theorem of calculus we have

$$\begin{aligned} \left|\frac{\cos(2\pi(k+h)x)-\cos(2\pi kx)}{h} - (-2\pi x)\sin(2\pi kx)\right| \;&=\; \left|\frac{1}{h}\int_0^h(-2\pi x)\sin(2\pi(k+t)x) - (-2\pi x)\sin(2\pi kx)\,\mathrm{d}t\right| & (6.11)\\ &\leqslant\; \frac{1}{|h|}\int_0^{|h|}|(-2\pi x)\sin(2\pi(k+t)x) - (-2\pi x)\sin(2\pi kx)|\,\mathrm{d}t. & (6.12)\end{aligned}$$

Using the inequality $|f(k+t)-f(k)| \leqslant \sup_{k\in\mathbb{R}}|f'(k)|\cdot|t|$ applied to the sine function, we have

$$\begin{aligned} \frac{1}{|h|}\int_0^{|h|}|(-2\pi x)\sin(2\pi(k+t)x) - (-2\pi x)\sin(2\pi kx)|\,\mathrm{d}t \;&\leqslant\; \frac{1}{|h|}\int_0^{|h|}4\pi^2x^2\cdot|t|\,\mathrm{d}t & (6.13)\\ &=\; 2\pi^2x^2|h|. & (6.14)\end{aligned}$$

This is true FOR ALL $x \in \mathbb{R}$, so we have

$$\lim_{L\to\infty} \int_{-L}^{L} \left| \frac{\cos(2\pi(k+h)x) - \cos(2\pi kx)}{h} - (-2\pi x)\sin(2\pi kx) \right| e^{-\pi x^2}\, dx \tag{6.15}$$

$$\leqslant \lim_{L\to\infty} \int_{-L}^{L} 2\pi^2 |x|^2 |h| e^{-\pi|x|^2}\, dx \tag{6.16}$$

$$= 2\pi^2 |h| \lim_{L\to\infty} \int_{-L}^{L} |x|^2 e^{-\pi|x|^2}\, dx \tag{6.17}$$

$$\leqslant 2C\pi^2 |h| \tag{6.18}$$

for some constant $C > 0$, since the last indefinite integral converges (by a similar argument as the previous indefinite integrals). This vanishes as $h \to 0$, which completes the proof.

(iii). Consider the difference $w(k) = u(k) - g(k)$. It's easy to check via straightforward differentiation that $w(k)$ solves the equation

$$w'(k) + 2\pi k w(k) = 0, \quad w(0) = 0. \tag{6.19}$$

Observe that $w(k)$ is a continuous function, so the set $\{k \in \mathbb{R} : w(k) = 0\}$ is a closed subset of $\mathbb{R}$. Also, suppose $w(k) = 0$. Then by local existence and uniqueness (since $2\pi k w(k)$ is continuous in k and Lipschitz in $w(k)$), there exists a neighborhood $(k-\delta, k+\delta)$ with $\delta > 0$ so that $w'(k) + 2\pi k w(k) = 0$ has a unique solution with $w(k) = 0$. But this unique solution is 0, so w is equal to 0 on $(k-\delta, k+\delta)$. This implies that the set $\{k \in \mathbb{R} : w(k) = 0\}$ is open. But it's nonempty since $w(0) = 0$; because the only closed, open, nonempty subset of $\mathbb{R}$ is the whole thing, this implies that $w(k) = 0$ for all $k \in \mathbb{R}$, which implies $g(k) = u(k)$ for all $k \in \mathbb{R}$.