Math 63CM Homework 1 Solutions

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1. Problem 1

1.1. **i.** Suppose $f(x)$ is monotonically non-decreasing; the situation where $f(x)$ is monotonically non-increasing follows from consider $g(x) = -f(x)$ and then noting this latter function is monotonically non-decreasing, allowing us to apply the following argument.

The left and right limits are equal to

$$\ell(c) = \sup_{x \in (a,c)} f(x), \quad r(c) = \inf_{x \in (c,b)} f(x). \tag{1.1}$$

Indeed, by definition of the supremum for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a point $x_0 \in (a,c)$ such that

$$\left| f(x_0) - \sup_{x \in (a,c)} f(x) \right| = \sup_{x \in (a,c)} f(x) - f(x_0) < \varepsilon. \tag{1.2}$$

Because $f(x)$ is monotonically non-decreasing, for any $y \in (x_0, c)$, we have $f(y) \geqslant f(x_0)$ and thus

$$\left| f(y) - \sup_{x \in (a,c)} f(x) \right| = \sup_{x \in (a,c)} f(x) - f(y) \leqslant \sup_{x \in (a,c)} f(x) - f(x_0) < \varepsilon. \tag{1.3}$$

Thus, for this arbitrary $\varepsilon \in \mathbb{R}_{>0}$, we may choose $\delta = c - x_0$. Proving the infimum formula for the right-limit at $c \in (a,b)$ follows from the same considerations.

1.2. **ii.** Suppose again that $f(x)$ is monotonically non-decreasing; if $f(x)$ is monotonically non-increasing, the claim follows from applying the argument to $g(x) = -f(x)$.

Suppose for the sake of contradiction that for any given $n \in \mathbb{Z}_{>0}$, there exist infinitely many points $\{c_i\}_{i=1}^{\infty}$ in (a,b) such that $|\ell(c_i) - r(c_i)| = r(c_i) - \ell(c_i) > \frac{1}{n}$. In particular, we may find infinitely many pairs of points $\{(d_i, e_i)\}_{i=1}^{\infty}$ such that $d_i < e_i$ and $f(e_i) - f(d_i) > \frac{1}{2n}$; indeed, take e_i to be a point to the right but sufficiently close to c_i and take d_i to be a point to the left but sufficiently close to c_i.

We now fix any $K \in \mathbb{Z}_{>0}$ and relabel the indices of the points $\{(d_i, e_i)\}_{i=1}^{K}$ so to make them in increasing order, so that $e_i < d_{i+1}$ for all i; this can be done by relabeling the indices of $\{c_i\}_{i=1}^{K}$ to make these in increasing order, and because the points $\{(d_i, e_i)\}_{i=1}^{K}$ were chosen to be very close to the respective c_i, the desired ordering is achieved.

We now observe

$$f(b) - f(a) = f(b) - f(e_K) + \sum_{i=2}^{K} (f(e_i) - f(e_{i-1})) + f(e_1) - f(a). \tag{1.4}$$

Because f is monotonically non-decreasing, the RHS is lower-bounded by the sum itself. Moreover, by the same token, we know $f(e_i) - f(e_{i-1}) \geqslant f(e_i) - f(d_i)$ because of the order $d_i > e_{i-1}$. Thus,

$$f(b) - f(a) \geqslant \sum_{i=2}^{K} (f(e_i) - f(d_i)) \geqslant \frac{K}{2n}. \tag{1.5}$$

Recall $K \in \mathbb{Z}_{>0}$ was arbitrary, so taking $K \to +\infty$ implies $f(b) - f(a) = +\infty$, which is ridiculous.

1.3. **iii.** Suppose $C = \{c_\alpha\}_\alpha$ is the set of discontinuities of f. We now group them according to the "jump size":

$$C \;=\; \bigcup_{n=1}^{\infty} \left\{ c_\alpha \in C : \; |\ell(c_\alpha) - r(c_\alpha)| > \frac{1}{n} \right\}. \tag{1.6}$$

This union is definitely not necessarily a disjoint union. Nevertheless, each set within the union is finite, and the union is over a countable set of indices, which implies that the union as a set itself is countable.

2. Problem 2

In short, the answer is that the sequence of differences $(\alpha_n)_n$ is always convergent in $\mathbb{R}$, even if the metric space (X, d) is not complete; the point is that even though (X, d) is not complete, the real line $\mathbb{R}$ with its usual metric is.

If $(a_n)_n$ and $(b_n)_n$ are two Cauchy sequences in (X, d), then we first claim that $\alpha_n = d(a_n, b_n)$ is Cauchy. This would imply the above claim, since Cauchy sequences in $\mathbb{R}$ are convergent.

To prove this claim, fix any $\varepsilon \in \mathbb{R}_{>0}$. By assumption, we can find $N \in \mathbb{Z}_{>0}$ sufficiently large depending on $\varepsilon \in \mathbb{R}_{>0}$ such that $d(a_n, a_m) < \frac{1}{2}\varepsilon$ and $d(b_n, b_m) < \frac{1}{2}\varepsilon$ for all $n, m \geqslant N$. Now, by the triangle inequality for the metric on $\mathbb{R}$, we have

$$|d(a_n, b_n) - d(a_m, b_m)| \;\leqslant\; |d(a_n, b_n) - d(a_m, b_n)| + |d(a_m, b_n) - d(a_m, b_m)|. \tag{2.1}$$

We further have, by the triangle inequality for (X, d),

$$\begin{aligned} -\frac{1}{2}\varepsilon \;&<\; d(a_m, a_n) && (2.2)\\ &=\; d(a_n, b_n) - d(a_m, a_n) - d(a_n, b_n) && (2.3)\\ &\leqslant\; d(a_n, b_n) - d(a_m, b_n) && (2.4)\\ &\leqslant\; d(a_n, a_m) + d(a_m, b_n) - d(a_m, b_n) && (2.5)\\ &=\; d(a_n, a_m) && (2.6)\\ &<\; \frac{1}{2}\varepsilon, && (2.7) \end{aligned}$$

so that $|d(a_n, b_n) - d(a_m, b_n)| < \frac{1}{2}\varepsilon$; similarly, we may show $|d(a_m, b_n) - d(a_m, b_m)| < \frac{1}{2}\varepsilon$. Thus, we have

$$|d(a_n, b_n) - d(a_m, b_m)| \;<\; \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \;=\; \varepsilon \tag{2.8}$$

for all $n, m \geqslant N$. Because $\varepsilon \in \mathbb{R}_{>0}$ was arbitrary and N depended only on ε, the claim follows.

3. Problem 3

Because $f(x)$ is continuously differentiable, we may write

$$f(x) \;=\; f(a) \;+\; \int_a^x f'(t)\,\mathrm{d}t. \tag{3.1}$$

Moreover, if we decompose $f'(t) = [f'(t)]_+ + [f'(t)]_-$, where $a_+ = \max(a, 0)$ and $a_- = \min(a, 0)$, we may write

$$\begin{aligned} f(x) \;&=\; f(a) \;+\; \int_a^x \big([f'(t)]_+ + [f'(t)]_-\big)\,\mathrm{d}t && (3.2)\\ &=\; f(a) \;+\; \int_a^x [f'(t)]_+\,\mathrm{d}t \;-\; \int_a^x \big(-[f'(t)]_-\big)\,\mathrm{d}t. && (3.3) \end{aligned}$$

Indeed, if f' is continuous, then $[f'(t)]_+$ and $[f'(t)]_-$ are both piecewise continuous which allows us to define their respective integrals. Observe that the first integral in the last expression, as a function of $x \in [a, b]$, is monotone non-decreasing because its integrand is non-negative. Similarly, the last integral is also monotone non-decreasing because its integrand is non-negative as well. Since adding the $f(a)$-term to either piece does not change its monotonicity, this resolves the claim.

4. PROBLEM 4

4.1. **i.** The problem is equivalent to showing that $Ax - x = y$ has a unique solution. We may rewrite this as $(A-I)x = y$, so that the claim is equivalent to showing that $A-I$ is invertible, or equivalently that its kernel is 0. This is true because the spectrum of A does not include 1 by the assumption given.

4.2. **ii.** As in the setting of the hint, we let $y_k = \frac{1}{k^2}$ for all $k \in \mathbb{Z}_{>0}$, so in particular $y \in \ell_1$. On the other hand, we let $\lambda_k = 1 - \frac{1}{k}$, so our map is

$$[A(x)]_k = \frac{1}{k^2} + \left(1 - \frac{1}{k}\right) x_k, \quad k \in \mathbb{Z}_{>0}. \tag{4.1}$$

We first check that this map is a "weak contraction"; precisely, we see

$$\begin{aligned} d\left(A(x), A(w)\right) &= \sum_{k=1}^{\infty} \left| \frac{1}{k^2} + \left(1 - \frac{1}{k}\right) x_k - \frac{1}{k^2} - \left(1 - \frac{1}{k}\right) w_k \right| && (4.2) \\ &= \sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right) |x_k - w_k| && (4.3) \\ &< \sum_{k=1}^{\infty} |x_k - w_k| && (4.4) \\ &= d(x, w). && (4.5) \end{aligned}$$

The last inequality follows from noting $1 - \frac{1}{k} < 1$ and $|x_k - w_k| \geqslant 0$, both for all $k \in \mathbb{Z}_{>0}$.

We now check that A has no fixed point. For the sake of contradiction, suppose it did, so that we had a solution to the equation $x = A(x)$ for some $x \in \ell_1$. For each $k \in \mathbb{Z}_{>0}$, we then get

$$x_k = \frac{1}{k^2} + \left(1 - \frac{1}{k}\right) x_k. \tag{4.6}$$

Organizing terms, we see $x_k = \frac{1}{k}$; but $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$, so that $x \notin \ell_1$. This is a contradiction.