Spring 2018 Math 63CM: Final Exam

Answer all questions[1]. This is a two hour exam. This is a closed-book, closed-notes and closed-internet exam. Please write all your answers in (one or multiple) blue books.

Unless explicitly stated otherwise, you may quote standard theorems from lectures or the textbook, as long as you state them clearly.

1. (5 points) Write down at least three different solutions to the ODE

$$x(t) = \sqrt{|x(t)|},\ x(0) = 0.$$

   Briefly explain why this does not contradict the uniqueness part of the Picard–Lindelöf theorem regarding solutions of ODEs.

2. (5 points) We say that a matrix[2] $A \in \mathbb{C}^{n\times n}$ is *evil* if it is neither nilpotent nor diagonalizable. Prove that for $n = 2$, any power of such a matrix is evil again.

   Bonus question (Please think about this only after you have attempted all the other problems): For $n = 3$, does there exist an evil matrix whose square is not evil?

3. (5 points) Solve explicitly the system

$$x'(t) = -x(t) + (y(t))^2, \quad y'(t) = y(t).$$

   Determine the stable manifold and the unstable manifold corresponding to the equilibrium point $(0, 0)$.

4. (10 points) Consider the ODE

$$x'(t) = ax(t) - bx(t)y(t), \quad y'(t) = -cy(t) + dx(t)y(t),$$

   where a, b, c, $d \in \mathbb{R}$ are positive constants.

   (a) Show that given any solution $(x(t), y(t))$, the function $L(x(t), y(t)) = dx(t) - c\log x(t) + by(t) - a\log y(t)$ is independent of t.

   (b) Find all equilibrium points. For each equilibrium point, determine whether it is stable. For each stable equilibrium point, determine whether it is asymptotically stable.

   (c) Prove that the ODE admits at least one non-constant periodic solution.

5. (10 points) Consider the ODE

$$x'(t) = -x(t) + F(x(t), y(t)), \quad y'(t) = -y(t) + G(x(t), y(t)),$$

[1] except for the bonus question.

[2] Here, $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ matrices with complex entries.

where F, $G : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth functions with the properties that

$$F(0,0) = G(0,0) = 0$$

and there exists $C > 0$ such that

$$\sup_{(x,y)\in\mathbb{R}^2} (|F(x,y)| + |G(x,y)|) \leq C.$$

(a) Prove that there exists $R_0 > 0$ such that for all $R \geq R_0$, the set

$$A := \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq R^2\}$$

is a positively invariant set.

(b) Using part (a), or otherwise, show that given any initial data $(x_0, y_0) \in \mathbb{R}^2$, the unique solution exists for all $t \geq 0$.

(c) Assume moreover that the matrix

$$-I + \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} (0,0)$$

has positive trace and postive determinant. Prove that either the ODE has a non-constant periodic solution or that there is an equilibrium point other than $(0,0)$.