Lecture notes for Math 272, Fall 2015

Lenya Ryzhik*

November 16, 2015

Nothing here is original, everything is from the two books by B. Perthame, "Transport equations in biology" and "Parabolic equations in biology".

1 Basic models

Bernoulli (1760): an estimate of how many lives could be saved immunizations from smallpox.

The simplest ODE model comes from Maltus (1798):

$$\frac{dN}{dt} = \alpha N(t). \tag{1.1}$$

Here, α is the growth rate. This is approximately valid at an early stage of the population growth for any model of the type

$$\frac{dN}{dt} = f(N(t)),$$

with $\alpha = f'(0)$.

The next level is the logistic model introduced by P.-F. Verhulst (mid 19th century):

$$\frac{dN}{dt} = \alpha N(t)(K - N(t)). \tag{1.2}$$

Here, K is the carrying capacity of the environment. The steady states are $N = 0$ – unstable, and $N = K$ – stable:

$$\lim_{t\to\infty} N(t) = K.$$

A more realistic modification to account for the fact that at small population size there is no growth: the Allee effect (1931):

$$\frac{dN}{dt} = \alpha N(t)(1 - \frac{N(t)}{K_-})(\frac{N(t)}{K_+} - 1). \tag{1.3}$$

Now, if $N(0) < K_-$ then $N(t) \to 0$ as $t \to +\infty$, and if $N(0) > K_-$ then $N(t) \to K_+$ as $t \to +\infty$. The steady states $N = 0$ and $N = K_+$ stable – this is the bistable nonlinearity type.

*Department of Mathematics, Stanford University, Stanford CA 94305, USA; ryzhik@math.stanford.edu

The spatial versions: the Fisher-KPP equation

$$\frac{\partial n}{\partial t} - \Delta n = \alpha n(t,x)(K - n(t,x)), \tag{1.4}$$

the Allen-Cahn (also called bistable) equation

$$\frac{\partial n}{\partial t} - \Delta n = \alpha n(t,x)(1 - n(t,x))(n(t,x) - s), \tag{1.5}$$

with $0 < s < 1$. General reaction-diffusion equation:

$$\frac{\partial n}{\partial t} - \Delta n = f(n), \tag{1.6}$$

with $f(0) = 0$. Traveling wave solutions: $n(t,x) = \Phi(x - ct)$:

$$-c\Phi' - \Phi'' = f(\Phi), \quad \Phi(-\infty) = \Phi_-, \ \Phi(+\infty) = \Phi_+, \tag{1.7}$$

where $f(\Phi_-) = f(\Phi_+) = 0$. Assume without loss of generality that $\Phi_- = 1$, $\Phi_+ = 0$. Then multiply (1.7) by Φ' and integrate:

$$-c\int_{-\infty}(\Phi')^2 dx = \int_{-\infty}^{\infty} f(\Phi)\Phi' dx = -\int_0^1 f(u)du. \tag{1.8}$$

Thus, the speed c has the same sign as

$$\int_0^1 f(u)du.$$

In particular, if both $n = 0$ and $n = 1$ are stable steady states of f then the state $n = 1$ invades the state $n = 0$ (that is, $c > 0$) if the above integral is positive. Otherwise, the state $n = 0$ invades the state $n = 1$ ($c < 0$).

The Lotka-Volterra systems

The simplest Lotka-Volterra system (1926) is a 2×2 system for the population $F(t)$ of prey (food) and $P(t)$ of predators:

$$\frac{dF}{dt} = \alpha(t)F - r(t)F, \quad \frac{dP}{dt} = \alpha_1(t)P - \mu(t)P. \tag{1.9}$$

Here, α and α_1 are the growth rates of the prey and predators, respectively, r and μ are the corresponding death rates. The basic assumptions of the Lotka-Volterra models are:

$$\alpha(t) = \text{const}, \quad d(t) = \beta P(t), \quad \alpha_1(t) = \gamma F(t), \quad \mu(t) = \text{const}. \tag{1.10}$$

Here, β and γ are constants. This leads to the system

$$\frac{dF}{dt} = \alpha F - \beta FP, \quad \frac{dP}{dt} = \gamma FP - \mu P. \tag{1.11}$$

The (non-zero) equilibrium state is

$$\bar{F} = \frac{\mu}{\gamma}, \quad \bar{P} = \frac{\alpha}{\beta}.$$

Let us add external hunting or fishing: this means that $\alpha' = \alpha - \varepsilon$, $\mu' = \mu + \varepsilon$, and gives

$$\left(\frac{\bar{P}}{\bar{F}}\right)_{nofishing} = \frac{\alpha}{\mu}\frac{\gamma}{\beta}, \quad \left(\frac{\bar{P}}{\bar{F}}\right)_{fishing} = \frac{\alpha - \varepsilon}{\mu + \varepsilon}\frac{\gamma}{\beta}. \tag{1.12}$$

We conclude that fishing leads to a smaller fraction of predators among all fish. This was experimentally verified by D'Ancona during World War I in the Mediterranean when fishing went significantly down.

The long time behavior of the Lotka-Volterra system (1.11) is rather simple.

Proposition 1.1 *If $F(0) > 0$ and $P(0) > 0$, then $F(t) > 0$ and $P(t) > 0$ for all $t > 0$, and the solution $(F(t), P(t))$ is periodic in time.*

Proof. Write $F(t) = e^{\phi(t)}$ and $P(t) = e^{\psi(t)}$ (positivity of F and $P(t)$ is an exercise), then

$$\frac{d\phi}{dt} = \alpha - \beta e^{\psi} = -\frac{\partial H}{\partial \psi}, \quad \frac{d\psi}{dt} = \gamma e^{\phi} - \mu = \frac{\partial H}{\partial \phi}, \tag{1.13}$$

with

$$H(\phi, \psi) = \beta e^{\psi} - \alpha\psi + \gamma e^{\phi} - \mu\phi$$

It follows that the function $H(\phi, \psi)$ satisfies

$$H(\phi(t), \psi(t)) = H(\phi(0), \psi(0)).$$

The function H is bounded from below and satisfies

$$H(s, u) \to +\infty \text{ as } (s, u) \to \infty.$$

Therefore, its level sets are closed curves, except for one which is just the equilibrium point which is the minimum of H. In addition, on each level set (except for the equilibrium point) the velocity never vanishes. Thus, all trajectories are periodic. □

The periodic behavior of the Lotka-Volterra system is not very generic but nevertheless interesting.

The chemostat: several nutrients

A chemostat contains nutrients S_i, $i = 1, \ldots, I$, and a micro-organism which uses the nutrients to grow. The balance system is as follows:

$$\begin{aligned} \frac{dS_i}{dt} &= R[S_{0i} - S_i(t)] - S_i(t)\eta_i n(t), \quad i = 1, \ldots, I, \\ \frac{dn}{dt} &= n(t)\Big(\sum_{i=1}^{I} \eta_i S_i(t) - R). \end{aligned} \tag{1.14}$$

Here, S_{0i} is the influx of pure nutrients into the chemostat, and R is the outflow rate for the mixture of the nutrient and the micro-organism. The two quadratic terms represent the consumption of the nutrients and the growth of the micro-organism.

The steady state $(\bar{S}_i, \bar{n})$ of (1.14) is unique: it is determined by

$$\sum_{i=1}^{I} \eta_i \bar{S}_i = R, \quad R S_{0i} = \bar{S}_i (R + \eta_i \bar{n}),$$

hence $\bar{n}$ is the unique solution of

$$\sum_{i=1}^{I} \frac{\eta_i S_{0i}}{R + \eta_i \bar{n}} = 1. \tag{1.15}$$

A necessary and sufficient condition for the steady state to exist is, therefore:

$$\sum_{i=1}^{I} S_{0i} \eta_i > R. \tag{1.16}$$

In particular, if R is large – violates (1.16) – then there is no steady state. This is because a strong flow will take all micro-organisms away, and they will not have a chance to grow. The long time behavior of the solutions is given by the following.

Proposition 1.2 Proof. *Let us assume that $n(0) > 0$ and $S_i(0) > 0$ for all $i = 1, \dots, I$.*
(i) If $\sum_{i=1}^{I} S_{0i}\eta_i < R$ (no steady state exists) then $n(t) \to 0$ and $S_i(t) \to S_{0i}$ as $t \to +\infty$.
(ii) If $\sum_{i=1}^{I} S_{0i}\eta_i > R$ (a steady state exists) then $n(t) \to \bar{n}$ and $S_i(t) \to \bar{S}_i$ as $t \to +\infty$.

Proof. Let us define the total mass:

$$M(t) = n(t) + \sum_{i=1}^{I} S_i(t).$$

It satisfies a simple ODE

$$\frac{dM}{dt} = R\Big(\sum_{i=1}^{I} S_{0i} - M(t)\Big), \tag{1.17}$$

thus

$$M(t) = \sum_{i=1}^{I} S_{0i} + n_0 e^{-Rt}, \quad n_0 = M(0) - \sum_{i=1}^{I} S_{0i}. \tag{1.18}$$

Next, observe that

$$\frac{d}{dt}(S_i(t) - S_{0i}) \le -R(S_i(t) - S_{0i}), \tag{1.19}$$

hence

$$S_i(t) \le S_{0i} + (S_i(0) - S_{0i})e^{-Rt}. \tag{1.20}$$

To prove part (i) we now observe that (1.20) implies that for large t we have

$$\sum_{i=1}^{n} \eta_i S_i(t) \le \sum_{i=1}^{n} \eta_i S_{0i} + C_1 e^{-Rt} < R - \frac{\varepsilon_0}{2}, \tag{1.21}$$

where

$$\varepsilon_0 = R - \sum_{i=1}^{I} S_{0i}.$$

It follows that there exists $T > 0$ so that for all $t > T$ we have

$$\frac{dn}{dt} \le -\frac{\varepsilon_0}{2} n,$$

thus $n(t) \to 0$ as $t \to +\infty$. Going back to (1.18) and (1.20) we see that

$$\sum_{i=1}^{I} S_{0i} + \sum_{i=1}^{I} (S_i(0) - S_{0i}) e^{-Rt} \ge \sum_{i=1}^{I} S_i(t) = \sum_{i=1}^{I} S_{0i} + n_0 e^{-Rt} - n(t).$$

It follows that

$$\lim_{t\to\infty} \sum_{i=1}^{I} S_i(t) = \sum_{i=1}^{I} S_{0i}.$$

Together with (1.20), we see that $S_i(t) \to S_{0i}$ as $t \to +\infty$ for all $i = 1, \ldots, I$.

To prove (ii) we will only show that

$$\liminf_{t\to\infty} n(t) \ge \bar{n}. \tag{1.22}$$

The rest is proved similarly, as will be seen from the proof. First, we show that for any $\varepsilon > 0$ there exists a time T_ε so that if $t > T_\varepsilon$ and $n(t) < n_1 - \varepsilon$, then

$$\frac{dn(t)}{dt} > \frac{\varepsilon}{2} n(t) \sum_{i=1}^{I} \eta_i. \tag{1.23}$$

Here, we have defined

$$n_1 = \Big(\sum \eta_i\Big)^{-1} \Big(\sum \eta_i S_{0i} - R\Big). \tag{1.24}$$

Indeed, if $n(t) < n_1 - \varepsilon$, and t is sufficiently large, then, using (1.18) gives

$$\sum_{i=1}^{I} S_i(t) = \sum_{i=1}^{I} S_{0i} + n_0 e^{-Rt} - n(t) \ge \sum_{i=1}^{I} S_{0i} - n_1 + \varepsilon, \tag{1.25}$$

thus

$$\sum_{i=1}^{I} (S_{0i} - S_i(t)) \le n_1 - \varepsilon. \tag{1.26}$$

Once again, recalling (1.20), we see that if t is sufficiently large, then (1.26) implies that for each i we must have

$$S_i(t) \ge S_{0i} - n_1 + \frac{\varepsilon}{2}. \tag{1.27}$$

Therefore,

$$\sum_{i=1}^{I} \eta_i S_i(t) \ge \sum_{i=1}^{I} \eta_i S_{0i} - n_1 \sum \eta_i + \frac{\varepsilon}{2} \sum \eta_i = R + \frac{\varepsilon}{2} \sum \eta_i. \tag{1.28}$$

Now, (1.23) follows, and, as a consequence, we conclude that

$$\liminf_{t\to\infty} n(t) \ge n_1. \tag{1.29}$$

If $n_1 > \bar{n}$, then we are done. Otherwise, we go back to the equation for S_i:

$$\frac{dS_i}{dt} = R[S_{0i} - S_i(t)] - S_i(t)\eta_i n(t). \tag{1.30}$$

It follows from (1.29) that for each $\varepsilon > 0$ there exists T_ε so that for all $t > T_\varepsilon$ we have

$$\frac{dS_i}{dt} \le R[S_{0i} - S_i(t)] - S_i(t)\eta_i(n_1 - \varepsilon). \tag{1.31}$$

Hence, if at some time $t > T_\varepsilon$ we have

$$S_i(t) > 100\varepsilon + \frac{RS_{0i}}{R + \eta_i n_1},$$

then, with some positive $c > 0$ (which depends on R, n_1 and η_i) we have

$$\frac{dS_i}{dt} < -c\varepsilon S_i.$$

Therefore, we know that

$$\limsup_{t\to\infty} S_i(t) \le S_{1i} = \frac{RS_{0i}}{R + \eta_i n_1} < S_{0i}. \tag{1.32}$$

Returning to (1.18)

$$n(t) + \sum_{i=1}^{I} S_i(t) = \sum_{i=1}^{I} S_{0i} + n_0 e^{-Rt}, \tag{1.33}$$

we see that

$$\liminf_{t\to\infty} n(t) \ge \sum_{i=1}^{I}[S_{0i} - S_{1i}] = n_2 = n_1 \sum_i \frac{\eta_i S_{0i}}{R + \eta_1 n_1} > n_1. \tag{1.34}$$

The last inequality above follows from our assumption

$$\sum_{i=1}^{I} \eta_i S_{0i} > R.$$

We may now iterate the above argument, showing that

$$\liminf_{t\to\infty} n(t) \ge n_k, \tag{1.35}$$

with the sequence n_k defined iteratively as

$$n_{k+1} = n_k \sum_{i=1}^{I} \frac{\eta_i S_{0i}}{R + \eta_1 n_k} > n_k.$$

It is immediate to see that the increasing sequence n_k converges to $\bar{n}$. The upper bound

$$\limsup_{t\to+\infty} n(t) \le \bar{n}$$

is proved similarly, hence

$$\lim_{t\to+\infty} n(t) = \bar{n}.$$

Convergence of $S_i(t)$ to $\bar{S}_i$ then follows also by a similar argument (one can see its elements in the passage from n_1 to n_2 above). □

Chemostat: several micro-organisms

We have looked above at a chemostat with several nutrients and one micro-organism. Let us now consider a chemostat with a single nutrient and several micro-organisms competing for this resource. We will assume that the ability of each micro-organism to use the nutrient depends only on the nutrient concentration. Then the system for the nutrient concentration $S(t)$ and the micro-organism densities $N_i(t)$ is

$$\begin{aligned} \dot{S}(t) &= R(S_0 - S) - \sum_{j=1}^{I} \eta_j(S) N_j, \\ \dot{N}_i(t) &= N_i(\eta_j(S) - R). \end{aligned} \tag{1.36}$$

As before, R is the dilution rate of the input flow, and S_0 is the concentration of the input nutrient.

We will make the following assumptions:

$$\eta_j(S) \text{ are increasing in } S,\ \eta'(S) \ge \alpha > 0, \text{ and } \eta_j(S_0) > R \text{ for all } i, \tag{1.37}$$

and that

$$\text{all } \eta_j^{-1}(R) \text{ are different.} \tag{1.38}$$

In a steady state we must have either $N_j = 0$ or $\eta_j(S) = R$, for all j. Assumption (1.38) means that there are $I+1$ steady states: a trivial one where all $N_j = 0$ and $S = S_0$, and I steady states which have just one non-zero N_m for some m, and $S = \eta_m^{-1}(R)$, that is, they have the form

$$N = (0, \ldots, 0, \bar{N}_m, 0, \ldots, 0), \quad \bar{S} = \eta_m^{-1}(R), \quad \bar{N}_m = S_0 - \eta_m^{-1}(R). \tag{1.39}$$

That is, each steady state contains just one micro-organism. One may ask, which of the steady states is selected as the long time limit of the solution of the ODE system. Let us define i_0 is the minimizer of $\eta_i^{-1}(R)$, that is,

$$\eta_{i_0}^{-1}(R) = S^* := \min_{1\le i\le I} \eta_i^{-1}(R).$$

In other words, i_0 corresponds to the equilibrium which has the smallest amount of nutrient left, or, somewhat equivalently, this micro-organism is most efficient in consuming the nutrient (at least in the equilibrium). The next theorem shows the selection principle – this "most efficient" micro-organism will be selected in the long time limit.

Theorem 1.3 *Assume that (1.38) and (1.39) hold. Then* $N_i(t) \to 0$ *as* $t \to +\infty$ *for all* $i \neq i_0$*, and* $N_{i_0}(t) \to \bar{N}^*_{i_0} = S_0 - S^*$ *and* $S(t) \to S^*$ *as* $t \to +\infty$*.*

Proof. First, we need to establish a balance law, an analog of the mass conservation in the case of multiple nutrients and one organism. Note that

$$M(t) = S(t) + \sum_{i=1}^{I} N_i(t)$$

satisfies

$$\dot{M} = R(S_0 - M),$$

hence

$$M(t) = S_0 + (M(0) - S_0)e^{-Rt}.$$

In other words, we have a balance law

$$S(t) + \sum_{i=1}^{I} N_i(t) = S_0 + Q_0 e^{-Rt}, \quad Q_0 = S(0) - S_0 + \sum_{i=1}^{I} N_i(0). \tag{1.40}$$

Next, we show that at least some micro-organism does not die out. Note that

$$N(t) = \sum_{i=1}^{I} N_i(t)$$

satisfies

$$\dot{N}(t) \geq N(t)(\eta_{min}(S) - R), \quad \eta_{min}(S) = \min_{1 \leq i \leq I} \eta_i(S). \tag{1.41}$$

Now, if all $N_i(t) \to 0$ as $t \to +\infty$, then the balance law (1.40) implies that $S(t) \to S_0$ as $t \to +\infty$ (which makes perfect physical sense), and assumption (1.37) implies that $\eta_{min}(S_0) > R$. This, together with (1.41) contradicts the assumption that $N(t) \to 0$. It is easy to modify this argument to show that a slightly better conclusion holds:

$$\liminf_{t \to +\infty} N(t) = M_1 > 0. \tag{1.42}$$

Next, we show that $S(t)$ has a limit as $t \to +\infty$. We compute the evolution of $\dot{S}$. Differentiating (1.40) gives

$$\dot{S}(t) + \sum_{i=1}^{I} \dot{N}_i(t) = -RQ_0 e^{-Rt}. \tag{1.43}$$

Differentiating once again leads to

$$\frac{d\dot{S}}{dt} = -\frac{d}{dt}\sum_{i=1}^{I} N_i(\eta_i(S) - R) + R^2 Q_0 e^{-Rt} = -\sum_{i=1}^{I} \dot{N}_i(\eta_i(S) - R) - \sum_{i=1}^{I} N_i \eta_i'(S)\dot{S} + R^2 Q_0 e^{-Rt}$$

$$= -\sum_{i=1}^{I} N_i(\eta_i(S) - R)^2 - \dot{S}\sum_{i=1}^{I} N_i \eta_i'(S) + R^2 Q_0 e^{-Rt} \leq -\dot{S}\sum_{i=1}^{I} N_i \eta_i'(S) + R^2 Q_0 e^{-Rt}.$$

Let us, for $0 < \delta \ll 1$, multiply the above by a smooth increasing function $\chi_\delta(\dot S) \geq 0$ such that $\chi_\delta(u) = 0$ for $u < 0$ and $\chi_\delta(u) = 1$ for $u > \delta$. This gives, with the function $\Phi_\delta(u)$ such that $\Phi'_\delta(u) = \chi_\delta(u)$ and $\Phi_\delta(u) = 0$ for $u < 0$:

$$\frac{d\Phi_\delta(\dot S)}{dt} \leq -\Phi'_\delta(\dot S)\dot S \sum_{i=1}^{I} N_i \eta'_i(S) + R^2 Q_0 e^{-Rt} \Phi'_\delta(\dot S).$$

As $\Phi'_\delta(u)u \geq 0$, using assumption (1.37) we get

$$\frac{d\Phi_\delta(\dot S)}{dt} \leq -\alpha \Phi'_\delta(\dot S)\dot S N(t) + R^2 Q_0 e^{-Rt} \Phi'_\delta(\dot S) \leq -\frac{\alpha M_1}{2} \Phi'_\delta(\dot S)\dot S + R^2 Q_0 e^{-Rt} \Phi'_\delta(\dot S). \quad (1.44)$$

Note that $S(t)$ and $N(t)$ are uniformly bounded, as follows from the balance law (1.40). Therefore, $\dot S$ is also uniformly bounded, and so is $\Phi_\delta(\dot S)(t)$. Moreover, the last term in the right side is integrable in time, and

$$\int_0^\infty R^2 Q_0 e^{-Rt} \Phi'_\delta(\dot S) dt \leq R Q_0$$

simply because $\Phi'_\delta(u) = \chi_\delta(u) = 1$. Thus, integrating (1.44) in time, we conclude that there exists a constant $C > 0$, independent of $\delta \in (0,1)$ such that

$$\int_0^\infty \Phi'_\delta(\dot S(t))\dot S(t) dt \leq C < +\infty. \quad (1.45)$$

Passing to the limit $\delta \to 0$, using the Fatou lemma, we conclude that

$$\int_0^\infty (\dot S(t))_+ dt \leq C. \quad (1.46)$$

Lemma 1.4 *If a function $g \in L^\infty[0, +\infty) \cap C^1[0, +\infty)$ satisfies*

$$\int_0^{+\infty} \left(\frac{dg}{dt}\right)_+ dt < +\infty,$$

then $g(t)$ is of bounded variation and has a limit as $t \to +\infty$.

Proof. Note that for any x_1 and x_2 we have

$$g(x_2) - g(x_1) = \int_{x_1}^{x_2} (g')_+(x) dx - \int_{x_1}^{x_2} (g')_-(x) dx,$$

so that

$$\int_0^\infty \left(\frac{dg}{dt}\right)_- dt \leq 2\|g\|_{L^\infty} \int_0^\infty \left(\frac{dg}{dt}\right)_+ dt.$$

We conclude that then

$$\int_0^\infty |\dot g(t)| dt < +\infty.$$

It follows that for any $k > 0$ there exists M so that for all $x_1, x_2 > M$ we have

$$|g(x_1) - g(x_2)| \leq \int_{x_1}^{x_2} |g'(x)| dx \leq 2^{-k},$$

and the conclusion that $g(x)$ has a limit as $x \to +\infty$ is a simple consequence. □

Lemma 1.4 implies immediately that the function $S(t)$ has a limit as $t \to +\infty$. We now identify the limit as S^*. Indeed, if

$$\lim_{t \to +\infty} S(t) > S^*,$$

then

$$\lim_{t \to +\infty} \eta_{i_0}(S(t)) > R,$$

hence N_{i_0} has an exponential growth, contradicting the balance law (1.40). On the other hand, if

$$\lim_{t \to +\infty} S(t) < S^*,$$

then

$$\lim_{t \to +\infty} \eta_i(S(t)) < R,$$

for all i, meaning that all $N_i(t) \to 0$ as $t \to \infty$, contradicting the no extinction bound (1.42). Therefore, we have

$$\lim_{t \to +\infty} S(t) = S^*.$$

In that case, we have

$$N_i(t) \to 0 \text{ as } t \to \infty \text{ for all } i \neq i_0.$$

Then the balance law (1.40) implies that

$$\lim_{t \to \infty} N_{i_0}(t) = S_0 - S^*,$$

and we are done. □

Phytoplankton

Phytoplankton are several species of photosynthesizing microscopic organisms (2 to 200 micrometers). They live in the upper layers of lakes and oceans (50 to 100 meters deep) where light is sufficient to sustain them but they are 2 to 5 percent more dense than water. The question is why they do not sink. There are various explanations for that, for example, as they sink, food restriction makes them lighter, or that ocean mixing brings them up.

A simple model for the evolution of the phytoplankton looks at a single column of water, with the vertical variable z so that $z = 0$ at the surface, and $z \to +\infty$ at the (bottomless) bottom – the z axis points downward. The evolution of the population density $n(t, z)$ is governed by a reaction-diffusion PDE

$$\frac{\partial n(t,z)}{\partial z} + v_p \frac{\partial n(t,z)}{\partial z} - \kappa \frac{\partial^2 n}{\partial z^2} = f(z, S(t, z, [n]))n(t, z), \quad t > 0, \ z > 0, \tag{1.47}$$

with a non-negative initial condition $n(0,z) = n_0(z)$. Here, v_p is the vertical flow velocity of the gravitational sinking, and κ is the diffusion constant. The birth/death rate $f(z, S(t,z,[n]))$ depends on the local light and is decreasing in z – we will assume that it is positive for small z and is negative for large z. In addition, it takes into account the shading effect – it depends on the total amount of the phytoplankton above z:

$$S(t,z,[n]) = \int_0^z n(t,x)dx.$$

A typical example would be to take a monotonically decreasing function $G(s)$ such that $G(0) > 0$ and $G(+\infty) < 0$, and set

$$f(z, S(t,z,[n])) = G\Big(z[1+\sigma \int_0^z n(t,x)dx]\Big).$$

We need to supplement (1.47) by the no-flux boundary condition at $z = 0$:

$$\kappa \frac{\partial n(t,0)}{\partial z} - v_p n(t,0) = 0, \tag{1.48}$$

and $n(t,z) \to 0$ as $z \to +\infty$. The maximum principle implies that $n(t,z) > 0$ for all $t > 0$ and $z \ge 0$, as long as $n_0(z)$ is non-negative and not identically equal to zero.

Here, we will restrict ourselves to the simple case when $G(s)$ is a step function:

$$G(s) = \begin{cases} B^+ > 0, & \text{for } 0 \le s \le H_0, \\ B^- < 0, & \text{for } H_0 < s < +\infty. \end{cases} \tag{1.49}$$

This creates a discontinuity at $z = H$ in the function f, with H defined implicitly by the equation

$$H_0 = H\Big(1 + \sigma \int_0^H n(t,x)dx\Big), \tag{1.50}$$

which has a unique solution since $n(t,z) \ge 0$ for all $z \ge 0$. There is a unique stationary solution of the layer model if and only if the layer is sufficiently wide, and the "good" layer is sufficiently good, as stated in the following.

Theorem 1.5 *There exists $\bar{H}(\kappa, v_p, B^\pm)$ so that for $\kappa > v_p^2/(4B_+)$, there is a unique stationary non-negative solution $n(x) \in C^1(0,+\infty)$ if and only if $H_0 > \bar{H}$.*

Proof. We will use the following lemma.

Lemma 1.6 *There exists a unique $\bar{a} > 0$ so that the the problem*

$$\begin{aligned} &v_p n_a' - \kappa n_a'' = g_a(z) n_a(z), \\ &\kappa n_a'(0) - v_p n_a(0) = 0, \\ &n_a(\infty) = 0, \end{aligned} \tag{1.51}$$

where $g_a(z) = B^+$ for $z < a$ and $g_a(z) = B^-$ for $z > a$, has a positive solution in $C^1(0,+\infty)$ for $a = \bar{a}$.

Proof of Lemma 1.6. For $z > a$ the solution has the form $n_a(z) = ce^{-\gamma z}$, where $\gamma > 0$ is the positive root of the equation

$$\kappa\gamma^2 + v_p\gamma + B^- = 0,$$

that is,

$$\gamma = \frac{1}{2\kappa}(-v_p + s), \quad s = \sqrt{v_p^2 + 4\kappa|B^-|}.$$

It follows that n_a satisfies

$$n_a'(a) = -n_a(a). \tag{1.52}$$

Therefore, an equivalent formulation of (1.51) is a boundary value problem on the interval $(0, a)$:

$$\begin{aligned} &v_p n_a' - \kappa n_a'' = B^+ n_a(z), \\ &\kappa n_a'(0) - v_p n_a(0) = 0, \\ &n_a'(a) == -\gamma n_a(a). \end{aligned} \tag{1.53}$$

Therefore, $n_a(z)$ is the principal eigenfunction of the eigenvalue problem

$$\begin{aligned} &v_p n_a' - \kappa n_a'' = \mu_a n_a(z), \\ &\kappa n_a'(0) - v_p n_a(0) = 0, \\ &n_a'(a) = -\gamma n_a(a), \end{aligned} \tag{1.54}$$

with the principal eigenvalue

$$\mu_a = B^+. \tag{1.55}$$

It is convenient to re-write (1.54) for the function

$$w(z) = e^{-\lambda z} n_a(z),$$

with λ to be chosen. This leads to

$$v_p w' + \lambda v_p w - \kappa w'' - 2\kappa\lambda v' - \kappa\lambda^2 w = \mu w.$$

Taking $\lambda = v_p/(2\kappa)$ to eliminate the first order term gives

$$\frac{v_p^2}{2\kappa} w - \kappa w'' - \kappa\frac{v_p^2}{4\kappa^2}\lambda^2 w = \mu w,$$

or

$$-\kappa w'' = \Big(\mu - \frac{v_p^2}{4\kappa}) w, \tag{1.56}$$

with the boundary conditions

$$\kappa w'(0) = \frac{v_p}{2} w(0), \quad w'(a) = -\Big(\frac{v_p}{2\kappa} + \gamma\Big) w(a). \tag{1.57}$$

Thus, we can, once again, reformulate our problem as follows: find $a > 0$ so that the principal eigenvalue η_a (the unique eigenvalue that corresponds to a positive eigenfunction) of the eigenvalue problem

$$-\kappa w'' = \nu_a w, \tag{1.58}$$
$$\kappa w'(0) = \frac{v_p}{2} w(0), \quad w'(a) = -\Big(\frac{v_p}{2\kappa} + \gamma\Big) w(a), \tag{1.59}$$

satisfies

$$\nu_a = B^+ - \frac{v_p^2}{4\kappa}. \tag{1.60}$$

Exercise 1.7 Show that $\nu_a > 0$ for all $a > 0$, ν_a is a decreasing function in a, and satisfies

$$\lim_{a\to 0} \nu_a = +\infty, \quad \lim_{a\to+\infty} \nu_a = 0.$$

Hint: set $\kappa = 1$, and rescale the problem to the interval $[0, 1]$ – set $w(x) = \psi(x/a)$, so that

$$-\psi'' = a^2 \nu_a \psi, \tag{1.61}$$
$$\psi'(0) = \frac{v_p a}{2} \psi(0), \quad \psi'(1) = -a\Big(\frac{v_p}{2\kappa} + \gamma\Big)\psi(a). \tag{1.62}$$

Show that

$$a^2 \nu_a = \inf_{\|\phi\|_{L^2[0,1]}=1} \Big(\int_0^1 |\phi_x|^2 dx + \frac{v_p a}{2} \phi(0)^2 + a\Big(\frac{v_p}{2\kappa} + \gamma\Big)\phi(a)^2 \Big).$$

Deduce that ν_a is decreasing in a.

It follows that there exists a unique a so that ν_a satisfies (1.60), provided that

$$B^+ > \frac{v_p^2}{4\kappa},$$

and the proof of the lemma is complete. □

We return to the proof of Theorem 1.5. A stationary solution satisfies

$$v_p n' - \kappa n'' = g(z) n(z), \tag{1.63}$$
$$\kappa n'(0) - v_p n(0) = 0,$$

where $g(z) = B^+$ for $z < H_n$ and $g(z) = B^-$ for $z > H_n$, with H_n as in (1.50):

$$H_0 = H_n \Big(1 + \sigma \int_0^{H_n} n(x) dx \Big). \tag{1.64}$$

Lemma 1.6 implies that $H_n = \bar{a}$, the unique value for which Lemma claims existence of a positive solution, and $n = C n_{\bar{a}}$, where $n_{\bar{a}}$ is the solution of (1.51) normalized so that $n_{\bar{a}}(0) = 1$. The constant C is then determined from the equation

$$H_0 = \bar{a}\Big(1 + \sigma \int_0^{\bar{a}} n_{\bar{a}}(x) dx\Big), \tag{1.65}$$

which has a solution if and only if $H_0 > \bar{a}$. □

One may further show that the long time limit of the solutions of the Cauchy problem for $n(t, x)$ is the positive stationary solution we have constructed above but we will not address this issue now.

2 Adaptive dynamics

Adaptive dynamics studies two effects: (i) the selection principle, which favors population with the best adapted trait, and (ii) mutations which allow the off-spring to have a slightly different trait from the parent. Here, we look at simple ODE models and study how the selection principle arises as the long time limit of small mutations.

A simple of example of a structured population and the selection principle

We begin with a very simple example of the logistic equation modified to take into account the traits. We structure the population by a trait $x \in \mathbb{R}$ and assume that the reproduction rate depends on the trait but the death rate depends on the total population:

$$\frac{\partial n(t,x)}{\partial t} = b(x)n(t,x) - \rho(t)n(t,x), \tag{2.1}$$

where $\rho(t)$ is the total population:

$$\rho(t) = \int_{\mathbb{R}} n(t,x)dx.$$

The initial condition is $n(0,x) = n_0(x)$ such that $n_0(x) > 0$ for $x \in (x_m, x_M)$, and $n_0(x) = 0$ otherwise. Note that there no mutations in this model, and any state of the form

$$n(t,x) = b(y)\delta(x-y)$$

is a steady solution, for all $y \in \mathbb{R}$. The question is which of these states will be selected in the long time limit. We have the selection principle – the best adapted population will be selected.

Theorem 2.1 *Assume that $b(x)$ is continuous, $b(x) \geq \underline{b} > 0$ for all $x \in \mathbb{R}$, and that $b(x)$ attains its maximum over the interval $[x_m, x_M]$ at a single point $\bar{x} \in (x_m, x_M)$. Then the solution to (2.1) satisfies*

$$\lim_{t\to+\infty} \rho(t) = \bar{\rho} = b(\bar{x}), \quad n(t,x) \to b(\bar{x})\delta(x-\bar{x}), \text{ as to } \to +\infty \tag{2.2}$$

the last convergence in the sense of distributions.

Proof. In this simple case, we give a computational proof. The function

$$N(t,x) = n(t,x)\exp\Big\{\int_0^t \rho(s)ds\Big\}$$

satisfies

$$\frac{dN}{dt} = b(x)N(x),$$

hence

$$N(t,x) = n_0(x)e^{b(x)t}.$$

We also have

$$\frac{d}{dt}\Big(\exp\{\int_0^t \rho(s)ds\}\Big) = \rho(t)\exp\{\int_0^t \rho(s)ds\} = \int_{\mathbb{R}} N(t,x)dx = \int_{\mathbb{R}} n_0(x)e^{b(x)s}dx,$$

so that

$$\exp\{\int_0^t \rho(s)ds\} = \int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K, \quad K = 1 - \int_{\mathbb{R}} \frac{n_0(x)}{b(x)}dx,$$

whence

$$\int_0^t \rho(s)ds = \log\Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big),$$

and

$$\rho(t) = \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big)^{-1}\int_{\mathbb{R}} n_0(x)e^{b(x)t}dx.$$

To see what happens as $t \to +\infty$, we note that

$$\rho(t) \le \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big)^{-1} b(\bar{x})\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx \to b(\bar{x}),$$

as $t \to +\infty$. For the converse, we take $\varepsilon > 0$ and look at the set

$$I_\varepsilon = \{x : \ b(x) \ge b(\bar{x}) - \varepsilon\}.$$

Then

$$\begin{aligned}\rho(x) &\ge \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big)^{-1}\int_{I_\varepsilon} n_0(x)e^{b(x)t}dx\\ &\ge \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big)^{-1}(b(\bar{x}) - \varepsilon)\int_{I_\varepsilon} \frac{n_0(x)}{b(x)}e^{b(x)t}dx = \frac{(b(\bar{x}) - \varepsilon)}{A_\varepsilon(t)},\end{aligned}$$

where

$$\begin{aligned}A_\varepsilon(t) &= \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx + K\Big)\Big(\int_{I_\varepsilon} \frac{n_0(x)}{b(x)}e^{b(x)t}dx\Big)^{-1}\\ &= \Big(\int_{\mathbb{R}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx\Big)\Big(\int_{I_\varepsilon} \frac{n_0(x)}{b(x)}e^{b(x)t}dx\Big)^{-1} + o(1).\end{aligned}$$

Note that

$$\int_{I_\varepsilon} \frac{n_0(x)}{b(x)}e^{b(x)t}dx \ge \int_{I_{\varepsilon/2}} \frac{n_0(x)}{b(x)}e^{b(x)t}dx \ge Ce^{(b(\bar{x})-\varepsilon/2)t},$$

while

$$\int_{\mathbb{R}\setminus I_\varepsilon} \frac{n_0(x)}{b(x)}e^{b(x)t}dx \le Ce^{(b(\bar{x})-\varepsilon)t}.$$

It follows that $A_\varepsilon(t) \to 1$ as $t \to +\infty$, and therefore

$$\rho(t) \to b(\bar{x}) \text{ as } t \to +\infty. \tag{2.3}$$

Next, from the expression for $N(t,x)$ we know that

$$n(t,x) = n_0(x)e^{b(x)t}\exp\Big\{-\int_0^t \rho(s)ds\Big\}$$

It is easy to see from (2.3) that for $x \neq \bar{x}$ we have $n(t,x) \to 0$. It then follows from (2.3) that

$$n(t,x) \to b(\bar{x})\delta(x-\bar{x}) \text{ as } t \to +\infty.$$

This finishes the proof. □

A more general situation

A more general model than (2.1) may have the form

$$\frac{\partial n(t,x)}{\partial t} = b(x,\rho(t))n(t,x) - g(x,\rho(t))n(t,x), \tag{2.4}$$

so that the birth and death rates of the population with a trait $x \in \mathbb{R}$ depend both on x and the total population

$$\rho(t) = \int_{\mathbb{R}} n(t,x)dx.$$

We will assume that the functions $b(x,\rho)$ and $g(x,\rho)$ factorize:

$$b(x,\rho) = b(x)Q_b(\rho), \quad d(x,\rho) = d(x)Q_d(\rho). \tag{2.5}$$

We will assume that the functions b and d are continuous, and $Q_b, Q_d \in C^1(0,+\infty)$, and that the following standard bounds hold:

$$0 < b_m \le b(x) \le b_M, \quad 0 < d_m \le d(x) \le d_M, \text{ for all } x \in \mathbb{R}.$$

In addition, we will need some bounds that would ensure the population does not explode or disappear completely: first, there exists $0 < \rho_M$ so that

$$\alpha_M = \max_{x\in\mathbb{R}}[b(x)Q_b(\rho_M) - d(x)Q_d(\rho_M)] < 0, \tag{2.6}$$

and, second, there exists $\rho_m \in (0,\rho_M)$ such that

$$\alpha_m = \min_{x\in\mathbb{R}}[b(x)Q_b(\rho_m) - d(x)Q_d(\rho_m)] > 0. \tag{2.7}$$

Proposition 2.2 *Assume that $n_0(x) \ge 0$ and $\rho_m \le \rho(t=0) \le \rho_M$, then $\rho_m \le \rho(t) \le \rho_M$ for all $t > 0$.*

Proof. We will just show that $\rho(t) \ge \rho_m$. This is a consequence of the maximum principle. Indeed, assume that τ_0 is the first time such that $\rho(\tau_0) = \rho_m$, then

$$\frac{d\rho}{dt}\Big|_{t=\tau_0} = \int_{\mathbb{R}}[b(x)Q_b(\rho_m) - d(x)Q_d(\rho_m)]n(\tau_0,x)dx \ge \alpha_m\rho_m > 0.$$

It follows that $\rho(t) \geq \rho_m$ for all $t > 0$. $\square$

Next, we will assume that

$$Q_b'(\rho) < 0, \quad Q_d'(\rho) > 0, \tag{2.8}$$

that is, the growth rate decreases, and the death rate increases, as the population grows. It follows then that there exists a unique $\rho = \bar{\rho}$ such that

$$\max_{\mathbb{R}}[b(x)Q_b(\rho) - d(x)Q_d(\rho)] = 0. \tag{2.9}$$

We will assume that there exists a unique $\bar{x}$ such that

$$b(\bar{x})Q_b(\bar{\rho}) - d(\bar{x})Q_d(\bar{\rho}) = 0. \tag{2.10}$$

Therefore, if $\rho = \bar{\rho}$, then

$$\frac{\partial n(t,x)}{\partial t} < 0, \text{for all } x \neq \bar{x}. \tag{2.11}$$

The last assumption we need is that there exists $\delta_0 > 0$ and $R > 0$ so that for all $|\rho - \bar{\rho}_0| < \delta_0$ and all $R > 0$ we have

$$\beta_R = \max_{|x|\geq R}[b(x)Q_b(\rho) - d(x)Q_d(x)] < 0. \tag{2.12}$$

Then we still have the selection principle.

Theorem 2.3 *With the above assumptions, if $n_0(x) > 0$ and $\rho_m \leq \rho(t=0) \leq \rho_M$, then*

$$\rho(t) \to \bar{\rho}, \quad n(t,x) \to \bar{\rho}\delta(x - \bar{x}), \quad \text{as } t \to +\infty. \tag{2.13}$$

Proof. We consider a function $P(r)$ that satisfies

$$rP'(r) + P(r) = Q(r), \quad Q(r) = \frac{Q_d(r)}{Q_b(r)}. \tag{2.14}$$

Let us also define

$$L(t) = \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - P(\rho(t))\Big) n(t,x)dx. \tag{2.15}$$

Note that $L(t)$ is uniformly bounded – this follows from our assumptions and Proposition 2.2. We compute:

$$\frac{dL(t)}{dt} = \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - P'(\rho(t))\frac{d\rho}{dt}\Big) n(t,x)dx \tag{2.16}$$
$$+ \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - P(\rho(t)))(b(x)Q_b(\rho(t)) - d(x)Q_d(\rho(t))n(t,x)dx.$$

Note that

$$\int_{\mathbb{R}} P'(\rho(t))\frac{d\rho}{dt} n(t,x)dx = P'(\rho(t))\rho(t) \int_{\mathbb{R}} [b(x)Q_b(x) - d(x)Q_d(x)]n(t,x)dx.$$

Using this in (2.16), together with (2.14), and the fact that $b(x)$ and $Q_d(\rho(t))$ are uniformly bounded from below, gives

$$\begin{aligned}\frac{dL}{dt} &= \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - P(\rho(t) - P'(\rho(t))\rho(t))(b(x)Q_b(\rho(t)) - d(x)Q_d(\rho(t))n(t,x)dx \\ &= \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - \frac{Q_d(\rho(t))}{Q_b(\rho(t))}\Big)(b(x)Q_b(\rho(t)) - d(x)Q_d(\rho(t))n(t,x)dx \\ &= \int_{\mathbb{R}} d(x)Q_b(\rho(t))\Big(\frac{b(x)}{d(x)} - Q(\rho(t))\Big)^2 n(t,x)dx \geq d_m Q_b(\rho_M)D(t),\end{aligned}$$

where

$$D(t) = \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - Q(\rho(t))\Big)^2 n(t,x)dx. \tag{2.17}$$

Therefore, $L(t)$ is bounded and increasing, hence it approaches a limit as $t \to +\infty$:

$$L(t) \to L, \quad \text{as } t \to +\infty. \tag{2.18}$$

We also deduce a bound

$$\int_0^\infty D(t)dt < +\infty. \tag{2.19}$$

Let us now find

$$\begin{aligned}\frac{dD(t)}{dt} &= \int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - Q(\rho(t))\Big)^2 (b(x)Q_b(\rho(t)) - d(x)Q_d(\rho(t)))n(t,x)dx \qquad (2.20)\\ -2Q'(\rho(t)) &\int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - Q(\rho(t))\Big)n(t,x)dx \int_{\mathbb{R}} (b(y)Q_b(\rho(t)) - d(y)Q_d(\rho(t)))n(t,y)dy = I + II.\end{aligned}$$

As $\rho(t)$ is a priori bounded, we have

$$|I| \leq CD(t).$$

The second term can be bounded using the bound on ρ and the Cauchy-Schwartz inequality as

$$\begin{aligned}|II| &\leq C\Big(\int_{\mathbb{R}} \Big(\frac{b(x)}{d(x)} - Q(\rho(t))\Big)^2 n(t,x)dx\Big)^{1/2} \rho(t)^{1/2} \\ &\times\Big(\int_{\mathbb{R}} (b(y)Q_b(\rho(t)) - d(y)Q_d(\rho(t)))^2 n(t,y)dy\Big)^{1/2} \rho(t)^{1/2} \leq CD(t).\end{aligned}$$

We conclude that

$$\int_0^\infty \Big|\frac{dD(t)}{dt}\Big|dt < +\infty. \tag{2.21}$$

Therefore, $D(t)$ has a limit as $t \to +\infty$. In addition, as $D(t)$ is integrable, we conclude that

$$D(t) \to 0 \text{ as } t \to +\infty. \tag{2.22}$$

The Cauchy-Schwartz inequality implies that

$$\int_{\mathbb{R}} \Big|\frac{b(x)}{d(x)} - Q(\rho(t))\Big|n(t,x)dx \leq (D(t))^{1/2}\rho(t)^{1/2} \to 0 \text{ as } t \to +\infty. \tag{2.23}$$

Note that

$$[Q(\rho(t)) - P(\rho(t))]\rho(t) = L(t) + \int_{\mathbb{R}} \Big(Q(\rho(t)) - \frac{b(x)}{d(x)}\Big) n(t,x)dx,$$

thus (2.18) and (2.23) together imply that

$$[Q(\rho(t)) - P(\rho(t))]\rho(t) \to L, \text{ as } t \to +\infty.$$

As $Q - P$ is not locally constant:

$$r(P-Q)' + (P-Q) = -rQ' < 0,$$

it follows that $\rho(t)$ has a limit:

$$\rho(t) \to \rho^*. \tag{2.24}$$

Let us now show that $\rho^* = \bar{\rho}$. Indeed, if $\rho^* > \bar{\rho}$ then

$$\max_{x\in\mathbb{R}}[b(x)Q_b(\rho^*) - d(x)Q_d(\rho^*)] < 0,$$

which implies that $n(t,x) \to 0$, which is a contradiction since $\rho(t) \geq \rho_m$. On the other hand, if $\rho^* < \bar{\rho}$, then

$$\max_{x\in\mathbb{R}}[b(x)Q_b(\rho^*) - d(x)Q_d(\rho^*)] < 0,$$

which, in turn, implies that $\rho(t) \to +\infty$ as $t \to +\infty$ contradicting $\rho(t) \leq \rho_M$. Therefore, we have $\rho^* = \bar{\rho}$. It follows from assumption (2.12) that

$$\frac{d}{dt}\int_{|x|\geq R} n(t,x)dx \leq \beta_R \int_{|x|\geq R} n(t,x)dx, \tag{2.25}$$

thus

$$\int_{|x|\geq R} n(t,x)dx \to 0 \text{ as } t \to +\infty.$$

It follows that $n(t,x)$ has a weak limit $n^*(x)$ in the space of measures along a sequence $t_n \to +\infty$, and

$$\int_{\mathbb{R}} n^*(x)dx = \bar{\rho}.$$

Finally, we know from (2.23) that $n^*(x)$ has to be concentrated on the set where

$$\frac{b(x)}{d(x)} - \frac{Q_d(\bar{\rho})}{Q_b(\bar{\rho})} = 0,$$

which consists of one point $\bar{x}$. It follows that the limit is unique and

$$n^*(x) = \bar{\rho}\delta(x - \bar{x}).$$

The proof is complete.

Mutations

A slight variation of the previous system would be the dynamics of the form

$$\frac{\partial n(t,x)}{\partial t} = b(x)Q_b(\rho_b(t))n(t,x) - d(x)Q_d(\rho_d(t))n(t,x). \tag{2.26}$$

Here, we have set

$$\rho_b(t) = \int_{\mathbb{R}} \psi_b(x)n(t,x)dx, \quad \rho_d(t) = \int_{\mathbb{R}} \psi_d(x)n(t,x)dx.$$

The function $\psi_b(y)$ measures how much the presence of the species of the trait y helps other species to reproduce, and the function $\psi_d(y)$ measures how much stronger the competition becomes if species with the trait y are present. It is natural to assume, as before, that the functions $b(x)$ and $d(x)$ are continuous, and

$$0 < b_m \le b(x) \le b_M, \quad 0 < d_m \le d(x) \le d_M, \text{ for all } x \in \mathbb{R}. \tag{2.27}$$

The functions Q_b and Q_d are $C^1(\mathbb{R}^+)$ and

$$Q_b'(\rho) \le a_1 < 0, \quad Q_d'(\rho) \ge a_2 > 0 \quad \text{for all } \rho > 0. \tag{2.28}$$

For the birth and death rates we assume that

$$\psi_m \le \psi_d(x), \psi_b(x) \le \psi_M \text{ for all } x \in \mathbb{R}. \tag{2.29}$$

These assumptions help prevent the blow-up of the total population in a finite time. We also introduce a generalization of (2.6) and (2.7) first, there exists ρ_M such that:

$$\alpha_M = \max_{x\in\mathbb{R}} [b(x)Q_b(\psi_m \rho_M) - d(x)Q_d(\psi_m \rho_M)] < 0, \tag{2.30}$$

and, second, there exists $\rho_m \in (0, \rho_M)$ such that

$$\alpha_m = \min_{x\in\mathbb{R}} [b(x)Q_b(\psi_M \rho_m) - d(x)Q_d(\psi_M \rho_m)] > 0. \tag{2.31}$$

We may now also add the possibility of mutations – an individual with a trait x may give birth to offspring with a trait y. This would lead to the following dynamics:

$$\frac{\partial n(t,x)}{\partial t} = Q_b(\rho_b(t)) \int_{\mathbb{R}} b(y)K(x-y)n(t,y)dy - d(x)Q_d(\rho_d(t))n(t,x). \tag{2.32}$$

Here, $K(x)$ is a non-negative probability density:

$$\int_{\mathbb{R}} K(x)dx = 1.$$

Existence of the solutions

The first step is to prove existence of the solutions.

Theorem 2.4 *Assume that the non-negative initial condition $n_0(x) \in L^1(\mathbb{R})$, and*

$$\rho_m \le \rho_0 \le \rho_M.$$

Then (2.32) has a non-negative solution such that

$$n, \frac{\partial n}{\partial t} \in C(0, +\infty; L^1(\mathbb{R})),$$

and for all $t \ge 0$ we have

$$\rho_m \le \rho(t) \le \rho_M. \tag{2.33}$$

Proof. The proof is similar to that for the Cauchy-Kovalevskaya theorem.

An a priori bound. We first obtain the a priori bound (2.33) on the solution (assuming that it exists). Let us integrate (2.32). Note that

$$\int_{\mathbb{R}\times\mathbb{R}} b(y)K(x-y)n(t,y)dydx = \int_{\mathbb{R}} b(y)n(t,y)dy.$$

It follows that

$$\begin{aligned} \frac{d\rho(t)}{dt} &= Q_b(\rho_b(t)) \int_{\mathbb{R}} b(x)n(t,x)dx - Q_d(\rho_d(t)) \int_{\mathbb{R}} d(x)n(t,x)dx \\ &\le \rho(t) \max_y [Q_b(\rho_b(t))b(y) - Q_d(\rho_d(t))\dot(y)]. \end{aligned} \tag{2.34}$$

Note that

$$\rho_b(t) \ge \psi_m \rho(t),$$

thus

$$Q_b(\rho_b(t)) \le Q_b(\psi_m \rho(t)),$$

and

$$Q_d(\rho_d(t)) \ge Q_d(\psi_m \rho(t)).$$

Using this in (2.34) gives

$$\frac{d\rho(t)}{dt} \le \rho(t) \max_y [Q_b(\psi_m \rho(t))b(y) - Q_d(\psi_m \rho(t))\dot(y)].$$

Therefore, if $\rho(t) > \rho_M$ then

$$\frac{d\rho(t)}{dt} < 0,$$

and $\rho(t)$ decreases. Similarly, if $\rho(t) < \rho_m$, then

$$\frac{d\rho(t)}{dt} > 0,$$

and $\rho(t)$ increases. Hence, if initially we have $\rho_m \leq \rho_0 \leq \rho_M$, then for all $t > 0$ we still have $\rho_m \leq \rho(t) \leq \rho_M$.

Existence. We will use the fixed point theorem for the existence. Consider the Banach space

$$X = C([0,T]; L^1(\mathbb{R})), \quad \|m\|_X = \sup_{0 \leq t \leq T} \|m(t)\|_{L^1(\mathbb{R})},$$

for some $T > 0$ to be chosen. Let us choose $C_0 = 2\rho_M$ and T sufficiently small so that

$$\rho_0 + T b_M Q_b(0) C_0 \leq C_0,$$

and set

$$S = \{m \in X, \ m \geq 0, \ \|m\|_X \leq C_0\}.$$

Given a function $m \in S$, define

$$R_b(t) = \int_{\mathbb{R}} \psi_b(x) m(t,x) dx, \quad R_d(t) = \int_{\mathbb{R}} \psi_d(x) m(t,x) dx,$$

and let $n(t,x)$ be the solution of the ODE, that we solve x by x:

$$\frac{\partial n(t,x)}{\partial t} = Q_b(R_b(t)) \int_{\mathbb{R}} b(y) K(x-y) m(t,y) dy - d(x) Q_d(R_d(t)) n(t,x),$$

with the initial condition $n(0,x) = n_0(x)$. We may then define the mapping $m \to \Phi(m) = n$, and the claim is that Φ has a unique fixed point in S if we choose a good C_0 and a sufficiently small T. We need to verify two conditions: (i) Φ maps S into S, and (ii) that Φ is a contraction for T sufficiently small. If we can verify these conditions then the Banach-Picard fixed point theorem implies that Φ has a fixed point in S, which is a solution we seek. We can then iterate this argument on the intervals $[T, 2T]$, $[2T, 3T], \ldots$ Note that on each time step the solution will satisfy $\rho_m \leq \rho(t) \leq \rho_M$, hence we can restart the argument each time.

To check (i) we simply write down the solution formula:

$$\begin{aligned} n(t,x) &= n_0(x) \exp\Big(- d(x) \int_0^t Q_b(R_d(s)) ds \Big) \\ &+ \int_0^t Q_b(R_b(s)) \int_{\mathbb{R}} b(y) K(x-y) m(s,y) dy \exp\Big\{ - d(x) \int_s^t Q_d(R_d(s')) ds' \Big\} ds. \end{aligned} \tag{2.35}$$

It follows that $n \geq 0$, and we also have

$$\frac{\partial n(t,x)}{\partial t} \leq Q_b(R_b(t)) \int_{\mathbb{R}} b(y) K(x-y) m(t,y) dy, \tag{2.36}$$

so that

$$\|n(t)\|_{L^1} \leq \rho_0 + T Q_b(0) b_M C_0 \leq C_0, \tag{2.37}$$

if T is sufficiently small. Thus, Φ maps S to S.

To check that Φ is a contraction, take $m_{1,2} \in S$ then we can write

$$\frac{\partial}{\partial t}(n_1 - n_2) = Q_b(R_b^1(t)) \int_{\mathbb{R}} b(y)K(x-y)m_1(t,y)dy - d(x)Q_d(R_d^1(t))n_1(t,x)$$
$$-Q_b(R_b^2(t)) \int_{\mathbb{R}} b(y)K(x-y)m^2(t,y)dy + d(x)Q_d(R_d^2(t))n^2(t,x)$$
$$= [Q_b(R_b^1(t)) - Q_b(R_b^2(t))] \int_{\mathbb{R}} b(y)K(x-y)m_1(t,y)dy$$
$$+Q_b(R_b^2(t)) \int_{\mathbb{R}} b(y)K(x-y)[m^1(t,y) - m^2(t,y)]dy$$
$$-d(x)Q_d(R_d^1(t))(n_1(t,x) - n_2(t,x)) + d(x)[Q_d(R_d^2(t)) - Q_d(R_d^1(t))]n_2(t,x).$$

Integrating in x we obtain

$$\|n_1 - n_2\|_X \le \psi_M C_0 T b_M \|m_1 - m_2\|_X + Q_b(0) b_M T \|m_1 - m_2\|_X$$
$$+d_M Q_d(\rho_M) T \|n_1 - n_2\|_X + d_M \rho_M \psi_M T \|m_1 - m_2\|_X.$$

Therefore, if T is sufficiently small, then

$$\|n_1 - n_2\|_X \le c\|m_1 - m_2\|_X,$$

with $c < 1$. Thus, for such T the mapping $\Phi : S \to S$ is a contraction, and has a fixed point, which is the solution we seek. □

Small mutations: the asymptotic limit

We now consider the situation when mutations are small: this is modeled by taking a smooth compactly supported kernel $K(x)$ of the form

$$K_\varepsilon(x) = \frac{1}{\varepsilon} K\Big(\frac{x}{\varepsilon}\Big), \quad k(x) \ge 0, \quad \int_{\mathbb{R}} K(z)dz = 1. \tag{2.38}$$

Of course, one would not expect small mutations to have a non-trivial effect on times of the order $t \sim O(1)$, because

$$\int_{\mathbb{R}} b(y)K_\varepsilon(x-y)n(t,y)dy = \int_{\mathbb{R}} b(x-\varepsilon z)K(z)n(t,x-\varepsilon z)dz \to \int_{\mathbb{R}} b(x)K(z)n(t,x)dz = b(x)n(t,x), \tag{2.39}$$

as $\varepsilon \to 0$. That is, the model with small mutations should be well-approximated by the model (2.26) with no mutations. In order for the small mutations to have a non-trivial effect, we need to wait for times of the order $t \sim O(\varepsilon^{-1})$. Accordingly, we consider the system in the rescaled time variable:

$$\varepsilon \frac{\partial n^\varepsilon(t,x)}{\partial t} = Q_b(\rho_b^\varepsilon(t)) \int_{\mathbb{R}} b(y)K_\varepsilon(x-y)n^\varepsilon(t,y)dy - d(x)Q_d(\rho_d^\varepsilon(t))n^\varepsilon(t,x), \tag{2.40}$$

with

$$\rho_b^\varepsilon(t) = \int \psi_b(x)n^\varepsilon(t,x)dx, \quad \rho_d^\varepsilon(t) = \int \psi_d(x)n^\varepsilon(t,x)dx.$$

We will show that in the limit $\varepsilon \to 0$ there is a selection principle, so that at every time t there is only one dominant trait $\bar{x}(t)$ but $\bar{x}(t)$ itself has a non-trivial dynamics, so that typically we will have

$$n_\varepsilon(t,x) \to \bar{n}(t,x) = \bar{\rho}(t)\delta(x - \bar{x}(t)). \tag{2.41}$$

Our goal will be to understand the dynamics of $\bar{x}(t)$ and $\bar{\rho}(t)$. Such limiting population is called monomorphic. It is also possible that the limit is a sure of several Dirac masses at $\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_N(t)$, and then the population is called polymorphic.

We will assume that the initial population is nearly monomorphic:

$$n_0^\varepsilon(x) = e^{\phi_0^\varepsilon(x)/\varepsilon}, \tag{2.42}$$

with a function $\phi_0^\varepsilon(x)$ such that

$$\phi_0^\varepsilon(x) \to \phi_0(x) \le 0, \quad \text{uniformly in } \mathbb{R}, \tag{2.43}$$

and

$$\int_{\mathbb{R}} n_0^\varepsilon(x)dx \to M_0 > 0, \ \varepsilon \to 0. \tag{2.44}$$

Note that $n_0^\varepsilon(x)$ is very small where $\phi_0^\varepsilon(x) \ll -\varepsilon$, which is, approximately, the region where $\phi_0(x) < 0$. Thus, in order to ensure we have initially a nearly monomorphic population, we will assume that

$$\max_{x\in\mathbb{R}} \phi_0(x) = 0 = \phi_0(\bar{x}_0) \text{ for a unique } \bar{x}_0 \in \mathbb{R}. \tag{2.45}$$

A typical example is the Gaussian family

$$n_0^\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-|x|^2/(2\varepsilon)}, \quad \phi_0^\varepsilon(x) = -\frac{|x|^2}{2} - \frac{\varepsilon}{2}\log(2\pi\varepsilon).$$

Let us write the equation for ϕ_ε:

$$\frac{\partial \phi^\varepsilon(t,x)}{\partial t} = e^{-\phi^\varepsilon(t,x)/\varepsilon} Q_b(\rho_b^\varepsilon(t)) \int_{\mathbb{R}} b(y) K_\varepsilon(x-y) e^{\phi^\varepsilon(t,y)/\varepsilon} dy - d(x) Q_d(\rho_d^\varepsilon(t)) \tag{2.46}$$
$$= Q_b(\rho_b^\varepsilon(t)) \int_{\mathbb{R}} b(x - \varepsilon y) K(y) e^{[\phi^\varepsilon(t,x-\varepsilon y) - \phi^\varepsilon(t,x)]/\varepsilon} dy - d(x) Q_d(\rho_d^\varepsilon(t)).$$

It is convenient to assume that K is even: $K(y) = K(-y)$, then, expanding in ε we get the formal limit:

$$\frac{\partial \phi(t,x)}{\partial t} = Q_b(\rho_b(t)) b(x) \int_{\mathbb{R}} K(y) \exp\Big\{ y \frac{\partial \phi(t,x)}{\partial x} \Big\} dy - d(x) Q_d(\rho_d(t)). \tag{2.47}$$

Let us define

$$H(p) = \int_{\mathbb{R}} K(y) e^{py} dy.$$

The limiting constrained Hamilton-Jacobi problem should be understood as follows: the function $\phi(t,x)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial \phi(t,x)}{\partial t} = Q_b(\rho_b(t)) b(x) H\Big(\frac{\partial \phi(t,x)}{\partial x} \Big) - d(x) Q_d(\rho_d(t)). \tag{2.48}$$

In addition, there is a constraint:

$$\max_{x\in\mathbb{R}} \phi(t,x) = 0 \text{ for all } t \geq 0. \tag{2.49}$$

The total density $\bar{\rho}(t)$ is a Lagrange multiplier that ensures that the constraint (2.49) holds. If the maximum $\bar{x}(t)$ is unique then, thinking of

$$n(t,x) = \bar{\rho}(t)\delta(x - \bar{x}(t)),$$

we have

$$\bar{\rho}_b(t) = \psi_b(\bar{x}(t))\bar{\rho}(t), \quad \bar{\rho}_d(t) = \psi_d(\bar{x}(t))\bar{\rho}(t). \tag{2.50}$$

Therefore, the formal limit is as follows: find a function $\phi(t,x)$, and $\bar{\rho}(t)$ and $\bar{x}(t)$, so that $\phi(t,x)$ satisfies (2.48) with $\rho_b(t)$ and $\rho_d(t)$ given in terms of $\bar{\rho}(t)$ and $\bar{x}(t)$ by (2.50), the constraint (2.49) holds, and $\phi(t,x)$ attains its maximum at $\bar{x}(t)$, where

$$\phi(t,\bar{x}(t)) = 0. \tag{2.51}$$

An example of the constrained Hamilton-Jacobi problem

Let us explain the above scheme on a simple example. Let us assume that $Q_b \equiv 1$, $d \equiv 1$, $\psi_d \equiv 1$ and $Q_d(u) = u$, so that the starting problem is

$$\frac{\partial n^\varepsilon(t,x)}{\partial x} = \int_{\mathbb{R}} b(y)K_\varepsilon(x-y)n^\varepsilon(t,y)dy - \rho_\varepsilon(t)n^\varepsilon(t,x), \tag{2.52}$$

with

$$\rho^\varepsilon(t) = \int_{\mathbb{R}} n^\varepsilon(t,x)dx.$$

For short times this model reduces to the familiar simple problem

$$\frac{\partial n(t,x)}{\partial t} = b(x)n(t,x) - \rho(t)n(t,x),$$

with which we have started. The function $\phi^\varepsilon(t,x)$ satisfies

$$\frac{\partial \phi^\varepsilon(t,x)}{\partial t} = \int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\phi^\varepsilon(t,x+\varepsilon y)-\phi^\varepsilon(t,x)]/\varepsilon}dy - \rho^\varepsilon(t). \tag{2.53}$$

This gives the following constrained Hamilton-Jacobi problem (2.48):

$$\begin{aligned} &\frac{\partial \phi(t,x)}{\partial t} = b(x)H\Big(\frac{\partial \phi(t,x)}{\partial x}\Big) - \bar{\rho}(t), \\ &\max_{x\in\mathbb{R}} \phi(t,x) = 0 = \phi(t,\bar{x}(t)), \text{ for all } t \geq 0, \\ &\phi(0,x) = \phi_0(x). \end{aligned} \tag{2.54}$$

The Hamiltonian is, as before,

$$H(p) = \int_{\mathbb{R}} K(y)e^{py}dy.$$

In this simple example, we can use the following trick: set

$$R(t) = \int_0^t \bar{\rho}(s)ds, \quad \psi(t,x) = \phi(t,x) + R(t),$$

then we arrive at the unconstrained Hamilton-Jacobi equation for the function $\psi(t,x)$:

$$\begin{aligned} &\frac{\partial \psi(t,x)}{\partial t} = b(x) H\Big(\frac{\partial \psi(t,x)}{\partial x}\Big), \\ &\psi(0,x) = \phi_0(x). \end{aligned} \tag{2.55}$$

Then, after solving (2.55) we may simply set

$$R(t) = \max_{x \in \mathbb{R}} \psi(t,x),$$

enforcing the constraint on $\phi(t,x)$.

Theorem 2.5 *Under the above assumptions, assume, in addition, that*

$$\phi_0^\varepsilon(x) \le C_0^\varepsilon - |x|,$$

then the function

$$\psi^\varepsilon(t,x) = \phi^\varepsilon(t,x) + R^\varepsilon(t), \quad R^\varepsilon(t) = \int_0^t \rho^\varepsilon(t)dt, \tag{2.56}$$

satisfies

$$\psi^\varepsilon(t,x) \to \psi(t,x), \text{ locally uniformly in } x.$$

Here, $\psi(t,x)$ is the viscosity solution of the Hamilton-Jacobi equation (2.55), and

$$\phi^\varepsilon(t,x) \to \phi(t,x) = \psi(t,x) - \max_{y \in \mathbb{R}} \psi(t,y).$$

The first step toward the proof are the following propositions.

Proposition 2.6 *We have, for all $t \ge 0$ the bound*

$$\min\Big(\min_{y \in \mathbb{R}} b(y), \rho_0^\varepsilon\Big) \le \rho^\varepsilon(t) \le \max\Big(\max_{y \in \mathbb{R}} b(y), \rho_0^\varepsilon\Big). \tag{2.57}$$

Proof. Indeed, integrating (2.52) in x gives

$$\frac{d\rho^\varepsilon(t)}{dt} = \int_{\mathbb{R}} b(y) n^\varepsilon(t,x)dx - (\rho^\varepsilon(t))^2. \tag{2.58}$$

It follows that

$$\frac{d\rho^\varepsilon(t)}{dt} \le b_M \rho^\varepsilon(t) - (\rho^\varepsilon(t))^2, \quad \frac{d\rho^\varepsilon(t)}{dt} \ge b_m \rho^\varepsilon(t) - (\rho^\varepsilon(t))^2,$$

with

$$b_m = \min(b(y)), \quad b_M = \max b(y).$$

The maximum principle implies then (2.57). $\Box$

Proposition 2.7 *If the initial condition satisfies*

$$\phi_0^\varepsilon(x) \le C_0^\varepsilon - |x|,$$

then $\psi_\varepsilon(t,x)$ defined by (2.56) satisfies

$$\psi^\varepsilon(t,x) \le C_0^\varepsilon - |x| + t\Big(\max_{y\in\mathbb{R}} b(y)\Big)\Big(\max_{|p|\le 1} H(p)\Big), \tag{2.59}$$

and

$$\Big|\frac{\partial \psi_\varepsilon(t,x)}{\partial t}\Big| \le 2\Big(\max_{y\in\mathbb{R}} b(y)\Big) H(\|\nabla\phi_0^\varepsilon\|_{L^\infty}). \tag{2.60}$$

Proof. The function $\psi^\varepsilon(t,x)$ satisfies

$$\frac{\partial \psi^\varepsilon(t,x)}{\partial t} = \int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}dy. \tag{2.61}$$

The function

$$\overline{\psi}(t,x) = C_0^\varepsilon - |x| + tB, \quad B = (\max_{x\in\mathbb{R}} b(x))(\max_{|p|\le 1} H(p))$$

is a super-solution to (2.61): indeed, we have

$$\frac{\partial \overline{\psi}_\varepsilon(t,x)}{\partial t} = B, \tag{2.62}$$

and

$$\int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\overline{\psi}(t,x+\varepsilon y)-\overline{\psi}(t,x)]/\varepsilon}dy \le (\max_{x\in\mathbb{R}} b(x))\int_{\mathbb{R}} K(y)e^{(|x|-|x+\varepsilon y|)/\varepsilon}dy$$
$$\le (\max_{x\in\mathbb{R}} b(x))\int_{\mathbb{R}} K(y)e^{|y|}dy \le B.$$

Now, (2.59) follows from the maximum principle in a slightly roundabout way: set

$$m_\varepsilon(t,x) = e^{\psi_\varepsilon(t,x)}/\varepsilon, \quad \bar m(t,x) = e^{\bar\psi(t,x)/\varepsilon},$$

then

$$\frac{\partial m_\varepsilon(t,x)}{\partial t} = \int b(y)K_\varepsilon(x-y)m_\varepsilon(t,y)dy, \tag{2.63}$$

and

$$\frac{\partial \bar m(t,x)}{\partial t} \ge \int b(y)K_\varepsilon(x-y)m_\varepsilon(t,y)dy. \tag{2.64}$$

It is easy to see that (2.63) and (2.64) together with the inequality $m_\varepsilon(0,x) \le \bar m(0,x)$ imply that

$$m_\varepsilon(t,x) \ge \bar m(t,x), \tag{2.65}$$

and (2.59) follows.

Finally, to get (2.60) we define

$$\Psi^\varepsilon(t,x) = \frac{\partial \psi^\varepsilon(t,x)}{\partial t},$$

and differentiate (2.61) to get

$$\frac{\partial \Psi^\varepsilon(t,x)}{\partial t} = \frac{1}{\varepsilon}\int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}[\Psi^\varepsilon(t,x+\varepsilon y)-\Psi^\varepsilon(t,x)]dy. \tag{2.66}$$

Therefore, at the point x_0 where $\Psi^\varepsilon(t,x)$ attains its maximum we have

$$\frac{\partial \Psi^\varepsilon(t,x_0)}{\partial t} = \frac{1}{\varepsilon}\int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}[\Psi^\varepsilon(t,x_0+\varepsilon y)-\Psi^\varepsilon(t,x_0)]dy \le 0, \tag{2.67}$$

whence

$$\max_{x\in\mathbb{R}} \Psi^\varepsilon(t,x) \le \max_{x\in\mathbb{R}} \Psi^\varepsilon(t=0,x).$$

The same argument shows that

$$\min_{x\in\mathbb{R}} \Psi^\varepsilon(t,x) \ge \min_{x\in\mathbb{R}} \Psi^\varepsilon(t=0,x).$$

Finally, we use (2.61) at $t=0$ to observe that, with some intermediate point $\xi(y)$ we have

$$\begin{aligned}
|\Psi_\varepsilon(t=0,x)| &= \int_{\mathbb{R}} b(x+\varepsilon y)K(y)e^{[\phi_0^\varepsilon(x+\varepsilon y)-\phi_0^\varepsilon(x)]/\varepsilon}dy\\
&\le (\max_{y\in\mathbb{R}} b(y))\int_{\mathbb{R}} K(y)\exp\{y\frac{\partial\phi_0^\varepsilon(\xi(y))}{\partial x}\}dy \le 2(\max_{y\in\mathbb{R}} b(y))H\Big(\|\nabla\phi_0^\varepsilon\|_{L^\infty}\Big).
\end{aligned} \tag{2.68}$$

In the last step we used the following inequality: if $|f(y)|\le M$, then

$$\begin{aligned}
&\int K(y)e^{yf(y)}dy \le \int_{y<0} K(y)e^{-My}dy + \int_{y>0} K(y)e^{My}dy\\
&\le \int_{y<0} K(y)e^{-My}dy + \int_{y>0} K(y)e^{-My}dy + \int_{y>0} K(y)e^{My}dy + \int_{y<0} K(y)e^{My}dy = 2\int_{\mathbb{R}} e^{My}.
\end{aligned}$$

Proof of Theorem 2.5

First, we would like to bound the spatial derivative of ψ^ε. Fix a time $T>0$ and let

$$\Phi^\varepsilon(t,x) = \frac{\partial\psi_\varepsilon(t,x)}{\partial x}.$$

Let us write

$$\frac{\partial}{\partial t}\Big(\frac{\psi^\varepsilon(t,x)}{b(x)}\Big) = \int_{\mathbb{R}} \frac{b(x+\varepsilon y)}{b(x)}K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}dy, \tag{2.69}$$

and differentiate in x:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\Phi^\varepsilon(t,x)}{b(x)}\Big) &= \frac{1}{b(x)^2}\frac{\partial b(x)}{\partial x}\frac{\partial\psi_\varepsilon(t,x)}{\partial t} + \int_{\mathbb{R}} \frac{\partial}{\partial x}\Big(\frac{b(x+\varepsilon y)}{b(x)}\Big)K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}dy\\
&+\frac{1}{\varepsilon}\int_{\mathbb{R}} \frac{b(x+\varepsilon y)}{b(x)}K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}(\Phi^\varepsilon(t,x+\varepsilon y)-\Phi^\varepsilon(t,x))dy.
\end{aligned} \tag{2.70}$$

Note that

$$\Big|\frac{\partial}{\partial x}\frac{b(x+\varepsilon z)}{b(x)}\Big| \le C\varepsilon.$$

Again, we consider the maximal point of Φ^ε:

$$Q_\varepsilon(t) = \max_{x\in\mathbb{R}} \Phi^\varepsilon(t,x)$$

The last term in the right side of (2.70) is non-positive where Φ^ε attains its maximum and non-negative where Φ^ε attains its minimum, and the first term is bounded by Proposition 2.7. We obtain therefore

$$\frac{dQ_\varepsilon}{dt} \le C + C\varepsilon\int_{\mathbb{R}} K(y)e^{[\psi^\varepsilon(t,x+\varepsilon y)-\psi^\varepsilon(t,x)]/\varepsilon}dy \le C + C\varepsilon\int K(y)e^{|y|Q_\varepsilon(t)}dy.$$

As $K(y)$ is compactly supported, we deduce that there exists C_T so that when $\varepsilon < \varepsilon_0(T)$ we have $Q_\varepsilon(t) \le C_T$. Therefore, the family of functions $\psi_\varepsilon(t,x)$ is locally compact due to the Arzela-Ascoli theorem.

Thus, we may extract a subsequence $\varepsilon_k \to 0$, so that both $R_\varepsilon(t)$, which is Lipschitz continuous in time, and $\psi_\varepsilon(t,x)$ have local uniform limits. The limit $\psi(t,x)$ satisfies the Hamilton-Jacobi equation in the viscosity sense (this is a non-trivial step but part of the general theory of viscosity solutions). The fact that the maximum of $\phi(t,x)$ has to be equal to zero follows from the upper and lower bounds on $\rho^\varepsilon(t)$ – if the maximum were different from zero, then $\rho(t)$ would either tend to zero or grow at a rate which is unbounded in ε.

Dynamics of the dominant trait: the monomorphic population

Let us now explain how the dominant trait $\bar{x}(t)$ can be recovered from the solution of the Hamilton-Jacobi equation in the general case, as long as the population is monomorphic, that is, the function $\phi(t,x)$ attains a single maximum $\bar{x}(t)$ where

$$\phi(t,\bar{x}(t)) = 0. \tag{2.71}$$

Let us recall that $\phi(t,x)$ satisfies

$$\frac{\partial\phi(t,x)}{\partial t} = Q_b(\rho_b(t))b(x)H\Big(\frac{\partial\phi(t,x)}{\partial x}\Big) - d(x)Q_d(\rho_d(t)). \tag{2.72}$$

Note that (2.71) implies that, in addition to

$$\frac{\partial\phi(t,\bar{x}(t)}{\partial x} = 0, \tag{2.73}$$

which holds simply because $\bar{x}(t)$ is the maximum of $\phi(t,x)$, we have

$$0 = \frac{d}{dt}\phi(t,\bar{x}(t)) = \frac{\partial\phi(t,x(t))}{\partial t} + \frac{d\bar{x}(t)}{dt}\frac{\partial\phi(t,\bar{x}(t))}{\partial x} = \frac{\partial\phi(t,x(t))}{\partial t}. \tag{2.74}$$

We deduce then from (2.72) that

$$Q_b(\rho_b(t))b(\bar{x})H(0) - d(\bar{x})Q_d(\rho_d(t)) = 0. \tag{2.75}$$

We know that

$$H(0) = \int K(y)dy = 1, \tag{2.76}$$

thus

$$Q_b(\rho_b(t))b(\bar{x}) = d(\bar{x})Q_d(\rho_d(t)). \tag{2.77}$$

In order to get the evolution of $\bar{x}(t)$ let us differentiate (2.73) in t:

$$0 = \frac{d}{dt}\frac{\partial \phi(t,\bar{x}(t)}{\partial x} = \frac{\partial^2 \phi(t,\bar{x}(t))}{\partial t \partial x} + \frac{\partial^2 \phi(t,\bar{x}(t))}{\partial x^2}\frac{d\bar{x}(t)}{dt}. \tag{2.78}$$

On the other hand, differentiating (2.72) in x gives

$$\frac{\partial^2 \phi(t,\bar{x}(t))}{\partial t \partial x} = Q_b(\rho_b(t))\frac{\partial b(\bar{x})}{\partial x} H\Big(\frac{\partial \phi(t,\bar{x})}{\partial x}\Big) + b(\bar{x})H_p\Big(\frac{\partial \phi(t,\bar{x})}{\partial x}\Big)\frac{\partial^2 \phi(t,\bar{x})}{\partial x^2} \tag{2.79}$$
$$-\frac{\partial d(\bar{x})}{\partial x}Q_d(\rho_d(t)).$$

However, as $K(y)$ is even, we have $H(0) = 0$ and

$$H_p(0) = \int_{\mathbb{R}} yK(y)dy = 0,$$

thus we get

$$\frac{\partial^2 \phi(t,\bar{x}(t))}{\partial t \partial x} = Q_b(\rho_b(t))\frac{\partial b(\bar{x})}{\partial x} - \frac{\partial d(\bar{x})}{\partial x}Q_d(\rho_d(t)).$$

Using this in (2.78) leads to an evolution equation for $\bar{x}(t)$:

$$\frac{d\bar{x}(t)}{dt} = -\Big(\frac{\partial^2 \phi(t,\bar{x}(t))}{\partial x^2}\Big)^{-1}\Big[Q_b(\rho_b(t))\frac{\partial b(\bar{x}(t))}{\partial x} - \frac{\partial d(\bar{x}(t))}{\partial x}Q_d(\rho_d(t))\Big]. \tag{2.80}$$

If the population is monomorphic, that is, $\phi(t,x)$ attains a unique maximum, then

$$n(t,x) = \bar{\rho}(t)\delta(x - \bar{x}(t)) \tag{2.81}$$

and

$$\rho_b(t) = \psi_b(\bar{x}(t))\bar{\rho}(t), \quad \rho_d(t) = \psi_d(\bar{x}(t))\bar{\rho}(t). \tag{2.82}$$

We may then re-write (2.77) as an equation for $\bar{\rho}(t)$ in terms of $\bar{x}(t)$:

$$Q_b(\psi_b(\bar{x}(t)\bar{\rho}(t))b(\bar{x}(t)) = d(\bar{x}(t))Q_d(\psi_d(\bar{x}(t)\bar{\rho}(t)). \tag{2.83}$$

Then we may use (2.82) and (2.83) in (2.80) to get a closed equation for $\bar{x}(t)$ as soon as the function $\phi(t,x)$ is known.

The dimorphic case

Let us see what happens if the population is dimorphic: the density $n(t,x)$ has the form

$$n(t,x) = \bar{\rho}_1(t)\delta(x - \bar{x}_1(t)) + \bar{\rho}_2(t)\delta(x - \bar{x}_2(t)), \tag{2.84}$$

and

$$0 = \max_{x\in\mathbb{R}} \phi(t,x) = \phi(t,\bar{x}_1(t)) = \phi(t,\bar{x}_2(t)). \tag{2.85}$$

As before, we may derive (2.77) both at $\bar{x}_1(t)$ and $\bar{x}_2(t)$, so that

$$R(t) := \frac{Q_b(\rho_b(t))}{Q_d(\rho_d(t)} = \frac{d(\bar{x}_1(t))}{b(\bar{x}_1(t))} = \frac{d(\bar{x}_2(t))}{b(\bar{x}_2(t))}. \tag{2.86}$$

Thus, a necessary condition for dimorphism is that the function $s(x) = d(x)/b(x)$ is not one-to-one. If $s(x)$ has a "parabolic profile", so that for every y we can find two pre-images x_1 and x_2 so that

$$y = s(x_1) = s(x_2),$$

then $\bar{x}_1(t)$ and $\bar{x}_2(t)$ determine each other. The functions $\rho_b(t)$ and $\rho_d(t)$ are now given by

$$\begin{aligned} \rho_b(t) &= \psi_b(\bar{x}_1(t))\bar{\rho}_1(t) + \psi_b(\bar{x}_2(t))\bar{\rho}_2(t), \\ \rho_d(t) &= \psi_d(\bar{x}_1(t))\bar{\rho}_1(t) + \psi_d(\bar{x}_2(t))\bar{\rho}_2(t). \end{aligned} \tag{2.87}$$

Then, $\bar{\rho}_1(t)$ and $\bar{\rho}_2(t)$ are two Lagrange multipliers that are needed in the Hamilton-Jacobi equation to ensure that the solution $\phi(t,x)$ has exactly two maxima and it vanishes at both of them.

3 The renewal equation

The renewal equation is the simplest model to account for aging: $n(t,x)$ is the density of the population of age x. The population ages "at speed one", and offspring of age zero are born. The balance, which does not account for the death rate, is

$$\frac{\partial n(t,x)}{\partial t} + \frac{\partial n(t,x)}{\partial x} = 0, \tag{3.1}$$

together with the initial condition $n(0,x) = n_0(x)$, and the boundary condition

$$n(t, x = 0) = \int_0^\infty B(y)n(t,y)dy, \tag{3.2}$$

which accounts for the birth of zero-age offspring. The analysis here is rather simple and explicit, as we will see. A more complicated related model describes the cell division

$$\frac{\partial n(t,x)}{\partial t} + \frac{\partial n(t,x)}{\partial x} + B(x)n(t,x) = \int_x^\infty b(x,y)n(t,y)dy, \tag{3.3}$$

with the boundary condition $n(t, x = 0) = 0$. Here, x is not the cell age but its size that grows in time. This model includes the death rate $B(x)$, and allows the cells to produce new cells of an arbitrary "age" (or size) x, smaller than its current size y. We will look at it later, and for now focus on (3.1)-(3.2).

The eigenfunction

Let us first look for a special solution of the form

$$n(t,x) = e^{\lambda_0 t} N(x),$$

with $N(x) > 0$, normalized so that

$$\int_0^\infty N(x)dx = 1. \tag{3.4}$$

This gives

$$\begin{aligned} &\frac{\partial N(x)}{\partial x} + \lambda_0 N(x) = 0, \quad x \geq 0, \\ &N(0) = \int_0^\infty B(y)N(y)dy. \end{aligned} \tag{3.5}$$

The explicit solution, taking into account the normalization (3.4) is

$$N(x) = \lambda_0 e^{-\lambda_0 x}. \tag{3.6}$$

The eigenvalue λ_0 is determined then by the boundary condition:

$$\int_0^\infty B(y)e^{-\lambda_0 y}dy = 1. \tag{3.7}$$

It is easy to check that such λ_0 is unique. Eventually we will see that any solution of the time-dependent problem in the long time limit behaves as

$$n(t,x) \sim c_0 N(x) e^{\lambda_0 t}.$$

Equivalently, the long time behavior of the solutions of

$$\begin{aligned} &\frac{\partial m(t,x)}{\partial t} + \frac{\partial m(t,x)}{\partial x} + \lambda_0 m(t,x) = 0, \\ &m(t,x=0) = \int_0^\infty B(y)m(t,y)dy, \\ &m(0,x) = m_0(x), \end{aligned} \tag{3.8}$$

is a multiple of the eigenfunction:

$$m(t,x) \sim c_0 N(x).$$

Let us now look for an adjoint eigenfunction $\phi(x)$. It should be determined from the following condition: take any solution $m(t,x)$ of (3.8), then we should have

$$\int_0^\infty m(t,x)\phi(x)dx = \int_0^\infty m_0(x)\phi(x)dx, \tag{3.9}$$

that is,

$$\frac{d}{dt}\int_0^\infty m(t,x)\phi(x)dx = 0. \tag{3.10}$$

This is equivalent to

$$0 = \int_0^\infty (\frac{\partial m(t,x)}{\partial x} + \lambda_0 m(t,x))\phi(x)dx = -m(t,0)\phi(0) + \int_0^\infty (-\frac{\partial \phi(x)}{\partial x} + \lambda_0\phi(x))m(t,x)dx$$
$$= \int_0^\infty (-\frac{\partial \phi(x)}{\partial x} + \lambda_0\phi(x) - \phi(0)B(x))m(t,x)dx.$$

Therefore, the function $\phi(x) \geq 0$ should be the solution of

$$-\frac{\partial \phi(x)}{\partial x} + \lambda_0\phi(x) = \phi(0)B(x), \qquad x \geq 0, \tag{3.11}$$

normalized so that

$$\int_0^\infty \phi(y)N(y) = 1. \tag{3.12}$$

It is convenient to introduce

$$Q(x) = \frac{\phi(x)N(x)}{\phi(0)N(0)}. \tag{3.13}$$

The function $Q(x)$ satisfies

$$-\frac{\phi(0)N(0)}{N(x)}\frac{\partial Q(x)}{\partial x} + \frac{\phi(0)N(0)Q(x)}{N^2(x)}\frac{\partial N(x)}{\partial x} + \frac{\lambda_0\phi(0)N(0)Q(x)}{N(x)} = \phi(0)B(x), \tag{3.14}$$

that is,

$$-\frac{\partial Q(x)}{\partial x} = \frac{N(x)B(x)}{N(0)}, \quad x \geq 0, \tag{3.15}$$

with the boundary condition $Q(0) = 1$. It follows that

$$Q(x) = 1 - \int_0^x B(y)e^{-\lambda_0 y}dy = \int_x^\infty B(y)e^{-\lambda_0 y}dy \leq \|B\|_{L^\infty}\frac{1}{\lambda_0}e^{-\lambda_0 x}, \tag{3.16}$$

We took into account (3.7) in the second step. Thus, $\phi(x)$ can be written as

$$\phi(x) = \frac{\phi(0)N(0)}{\lambda_0}e^{\lambda_0 x}\int_x^\infty B(y)e^{-\lambda_0 y}dy. \tag{3.17}$$

Note that while $B(x)$ has a very simple expression, the function $\phi(x)$ is much less explicit. Note that

$$0 \leq Q(x) \leq 1,$$

which means that

$$0 \leq \phi(x)N(x) \leq \phi(0)N(0). \tag{3.18}$$

The upper bound in (3.16) implies that

$$\phi(x) = \frac{\phi(0)N(0)}{N(x)}Q(x) \leq \frac{\phi(0)\|B\|_{L^\infty}}{\lambda_0}. \tag{3.19}$$

If we use normalization (3.12), we get

$$\frac{1}{\phi(0)N(0)} = \int_0^\infty Q(x)dx = \int_0^\infty \int_x^\infty B(y)e^{-\lambda_0 y}dydx = \int_0^\infty yB(y)e^{-\lambda_0 y}dy. \tag{3.20}$$

This determines $\phi(0)$:

$$\frac{1}{\phi(0)} = \lambda_0 \int_0^\infty yB(y)e^{-\lambda_0 y}dy. \tag{3.21}$$

We thus have an upper bound for $\phi(x)$, from (3.19):

$$\phi(x) \le \frac{\|B\|_{L^\infty}}{\lambda_0^2}\Big(\int_0^\infty yB(y)e^{-\lambda_0 y}dy\Big)^{-1}. \tag{3.22}$$

The existence theory

Let us consider the function $m(t,x)$, solution of (3.8)

$$\begin{aligned} &\frac{\partial m(t,x)}{\partial t} + \frac{\partial m(t,x)}{\partial x} + \lambda_0 m(t,x) = 0, \\ &m(t,x=0) = \int_0^\infty B(y)m(t,y)dy, \\ &m(0,x) = m_0(x). \end{aligned} \tag{3.23}$$

We assume that

$$B(x) \ge 0 \text{ for all } x \ge 0, \quad B \in L^1 \cap L^\infty(0,+\infty), \quad \int_0^\infty B(x)dx > 1. \tag{3.24}$$

Theorem 3.1 *Assume that there exists C_0 such that*

$$|m_0(x)| \le C_0 N(x), \tag{3.25}$$

then there exists a unique weak solution to (3.23) in $C^1((0,+\infty); L^1(\mathbb{R};\phi(x)dx))$ such that

$$|m(t,x)| \le C_0 N(x). \tag{3.26}$$

In addition, if $n_0^1(x) \le n_0^2(x)$ for all $x \ge 0$, then $m_1(t,x) \le m_2(t,x)$ for all $x \ge 0$ and $t \ge 0$. Finally, we have

$$\int_0^\infty m(t,x)\phi(x)dx = \int_0^\infty n_0(x)\phi(x)dx, \tag{3.27}$$

and

$$\int_0^\infty |m(t,x)|\phi(x)dx \le \int_0^\infty |n_0(x)|\phi(x)dx. \tag{3.28}$$

Step 1. Existence for $n_0 \in L^1(\mathbb{R}_+)$. We first prove existence of the solution with the initial condition $n_0 \in L^1((0,+\infty);dx)$. We will fix $T > 0$ and use the Picard fixed point theorem in the Banach space

$$X = C([0,T]; L^1(\mathbb{R}^+;dx)), \quad \|m\|_X = \sup_{0\le t\le T} \|m(t)\|_{L^1(dx)}.$$

We will need to assume that T is sufficiently small but the size of the time T will not depend on the L^1-norm of the initial condition, so we will be abel to repeat this argument on $(T, 2T), (2T, 3T), \ldots$ getting the global in time existence. We will set the solution operator $\mathcal{S}: X \to X$ as follows. Fix $q \in X$ and let $n = \mathcal{S}q$ be the solution of

$$\begin{aligned}
&\frac{\partial n(t,x)}{\partial t} + \frac{\partial n(t,x)}{\partial x} + \lambda_0 n(t,x) = 0, \\
&n(t, x=0) = \int_0^\infty B(y) q(t,y) dy, \\
&n(0,x) = m_0(x).
\end{aligned} \tag{3.29}$$

Our task is to show that $\mathcal{S}$ is a contraction: let $q_{1,2} \in X$ and let $n_{1,2}$ be the corresponding solutions of (3.29). Setting $q = q_1 - q_2$ and $n = n_1 - n_2$ we deduce that

$$\begin{aligned}
&\frac{\partial n(t,x)}{\partial t} + \frac{\partial n(t,x)}{\partial x} + \lambda_0 n(t,x) = 0, \\
&n(t, x=0) = \int_0^\infty B(y) q(t,y) dy, \\
&n(0,x) = 0.
\end{aligned} \tag{3.30}$$

We claim that $|n(t,x)|$ is a weak solution of

$$\begin{aligned}
&\frac{\partial |n|(t,x)}{\partial t} + \frac{\partial |n|(t,x)}{\partial x} + \lambda_0 |n|(t,x) = 0, \\
&|n|(t, x=0) = \Big| \int_0^\infty B(y) q(t,y) dy \Big|. \\
&|n|(0,x) = 0.
\end{aligned} \tag{3.31}$$

To see that, consider a family of smooth functions $\chi_\varepsilon(n)$ such that $\chi_\varepsilon(n) \to |n|$ and $\chi_\varepsilon'(n) \to \operatorname{sgn}(n)$, then for each $\varepsilon > 0$ we have, multiplying (3.31) by $\chi_\delta'(n)$

$$\frac{\partial \chi_\varepsilon(n)(t,x)}{\partial t} + \frac{\partial \chi_\varepsilon(n)(t,x)}{\partial x} + \chi_\varepsilon'(n) \lambda_0 n(t,x) = 0. \tag{3.32}$$

Passing to the limit $\varepsilon \to 0$ gives (3.31). Integrating (3.31) in time and space we get

$$\int_0^\infty |n(t,x)| dx \le \int_0^t |n(s,0)| ds = \int_0^t \Big| \int_0^\infty B(y) q(s,y) dy \Big| ds \le t \|B\|_{L^\infty} \|q\|_X.$$

We see that if

$$T \|B\|_{L^\infty} \le \frac{1}{2},$$

then the map $\mathcal{S}$ is a contraction, hence it has a fixed point in X.

Step 2. The comparison principle. The comparison principle is a direct consequence of the construction since if $n_0^1(x) \ge n_0^2(x)$ for all $x \ge 0$, then for each $q \in X$ we have $\mathcal{S}_1 q(t,x) \ge \mathcal{S}_2 q(t,x)$ for all t and x. Recall that the fixed points can be constructed as the limit of the iteration process $m_{n+1} = \mathcal{S} m_n$. As the operator $\mathcal{S}$ respects the order, if we take $m_0^1 = m_0^2 = 0$, then we will have $m_n^1(t,x) \ge m_n^2(t,x)$, hence this order will be preserved in the limit for the fixed points as well. The maximum principle is a consequence of the comparison principle since $CN_0(x)$ is a steady solution, and can also be taken as the initial condition.

Step 3. Uniqueness and existence in $C([0,T];L^1(\mathbb{R}_+)$. Let us take an initial condition $n_0 \in L^1(\mathbb{R}_+;\phi(x)dx)$. The function $\phi(x)$ is bounded, hence there exists a sequence $n_k \in L^1(\mathbb{R};dx)$ such that $n_k \to n_0$ in $L^1(\mathbb{R}_+;\phi(x)dx)$. Let m_k be the corresponding solution. The function $m = (m_k - m_p)$ satisfies

$$\begin{aligned}
&\frac{\partial m(t,x)\phi(x)}{\partial t} + \frac{\partial m(t,x)\phi(x)}{\partial x} = -\phi(0)B(x)m(t,x),\\
&m(t,x=0) = \int_0^\infty B(y)m(t,y)dy,\\
&m(0,x) = n_k - n_p.
\end{aligned} \tag{3.33}$$

This implies

$$\begin{aligned}
&\frac{\partial |m(t,x)|\phi(x)}{\partial t} + \frac{\partial |m(t,x)|\phi(x)}{\partial x} = -\phi(0)B(x)|m(t,x)|,\\
&m(t,x=0) = \int_0^\infty B(y)m(t,y)dy,\\
&m(0,x) = n_k - n_p.
\end{aligned} \tag{3.34}$$

Integrating in x gives

$$\begin{aligned}
&\frac{d}{dt}\int |m(t,x)|\phi(x)dx = |m(t,0)|\phi(0) - \phi(0)\int B(x)|m(t,x)|dx\\
&\leq |m(t,0)|\phi(0) - |m(t,0)|\phi(0) \leq 0.
\end{aligned} \tag{3.35}$$

We conclude that

$$\int |m_k(t,x) - m_p(t,x)|\phi dx \leq \int |n_k(x) - n_p(x)|\phi(x)dx. \tag{3.36}$$

As the sequence n_k is Cauchy in $L^1(\mathbb{R}_+;\phi(x)dx)$, we conclude from (3.36) that the sequence $m_k(t,x)$ is Cauchy in $C([0,T];L^1(\mathbb{R}_+;\phi(x)dx)$. Hence, it converges to a limit $m(t,x)$, which is a weak solution in $C([0,T];L^1(\mathbb{R}_+;\phi(x)dx))$. In order to see uniqueness, note that (3.36) implies that if there are two solutions then they must coincide on the support of $\phi(x)$. However, the equation for $\phi(x)$ shows that the support of ϕ contains the support of B. Hence, the two solutions coincide on the support of B. This means that they satisfy the transport equation with the same boundary and initial conditions, hence they coincide.

Finally, we have already shown (3.28), and (3.27) also follows from integrating the equation.

A little bit of regularity

Let us now assume that the initial condition satisfies

$$|n_0(x)| \leq CN_0(x), \quad \Big|\frac{\partial n_0(x)}{\partial x}\Big| \leq C_1 N_0(x), \tag{3.37}$$

and, in addition, it is well prepared in the sense that

$$n_0(0) = \int B(y)n_0(y)dy. \tag{3.38}$$

Then, differentiating the equation in t we see that the time derivative

$$n_t = \frac{\partial n}{\partial t}$$

satisfies exactly the same problem as $n(t,x)$. Moreover, initially we have

$$n_t(0,x) = -\frac{\partial n_0}{\partial x} - \lambda_0 n_0(x),$$

hence

$$|n_t(0,x)| \le (\lambda_0 C_0 + C_1)N_0(x).$$

Therefore, the maximum principle implies that

$$\Big|\frac{\partial n(t,x)}{\partial t}\Big| \le (\lambda_0 C_0 + C_1)N_0(x), \tag{3.39}$$

for all $t \ge 0$. We have a similar estimate for the spatial derivative. This simply follows from the equation:

$$\Big|\frac{\partial n(t,x)}{\partial x}\Big| \le \Big|\frac{\partial n(t,x)}{\partial t}\Big| + \lambda_0|n(t,x)| \le C' N_0(x) \tag{3.40}$$

by what we have already shown.

Generalized relative entropy

We give here an example of how one can apply the generalized relative entropy method to the renewal equation. We will discuss more about this method later for other equations. Let $n(t,x)$ be the solution of

$$\begin{aligned} &\frac{\partial n(t,x)}{\partial t} + \frac{\partial n(t,x)}{\partial x} + \lambda_0 n(t,x) = 0, \\ &n(t, x = 0) = \int_0^\infty B(y)n(t,y)dy, \\ &n(0,x) = n_0(x). \end{aligned} \tag{3.41}$$

Note that the ratio

$$\zeta(x) = \frac{n(t,x)}{N(t,x)}$$

satisfies the homogeneous transport equation

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} = \frac{1}{N}\Big(\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x}\Big) - \frac{n}{N^2}\frac{\partial N}{\partial x} = -\frac{1}{N}\lambda_0 n + \frac{n}{N^2}\lambda_0 N = 0.$$

Therefore, any function $H(n(t,x)/N(t,x))$ satisfies the same transport equation:

$$\frac{\partial}{\partial t} H\Big(\frac{n(t,x)}{N(t,x)}\Big) + \frac{\partial}{\partial x} H\Big(\frac{n(t,x)}{N(t,x)}\Big) = 0. \tag{3.42}$$

Recall that the function ϕ satisfies

$$-\frac{\partial \phi}{\partial x} + \lambda_0 \phi = \phi(0) B(x). \tag{3.43}$$

Let us compute the equation for the function $\phi(x)N(x)H(n(t,x)/N(x))$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\phi(x)N(x)H\Big(\frac{n(t,x)}{N(x)}\Big)\Big) + \frac{\partial}{\partial x}\Big(\phi(x)N(x)H\Big(\frac{n(t,x)}{N(x)}\Big)\Big)\\
&= \phi(x)N(x)\frac{\partial}{\partial t}H\Big(\frac{n(t,x)}{N(x)}\Big) + \phi(x)N(x)\frac{\partial}{\partial x}H\Big(\frac{n(t,x)}{N(x)}\Big) + \frac{\partial \phi}{\partial x}N(x)H\Big(\frac{n(t,x)}{N(x)}\Big)\\
&+\phi(x)\frac{\partial N(x)}{\partial x}H\Big(\frac{n(t,x)}{N(x)}\Big) = \Big[\lambda_0\phi(x)N(x) - \phi(0)B(x)N(x) - \lambda_0\phi(x)N(x)\Big]H\Big(\frac{n(t,x)}{N(x)}\Big)\\
&= -\phi(0)B(x)N(x)H\Big(\frac{n(t,x)}{N(x)}\Big).
\end{aligned} \tag{3.44}$$

Let us define a probability measure

$$d\mu(x) = \frac{B(x)N(x)}{N(0)}dx, \qquad \int_0^\infty d\mu(x) = 1. \tag{3.45}$$

We integrate (3.44) in x:

$$\frac{d}{dt}\int \phi(x)N(x)H\Big(\frac{n(t,x)}{N(x)}\Big)dx = -\phi(0)N(0)\int H\Big(\frac{n(t,x)}{N(x)}\Big)d\mu(x) + \phi(0)N(0)H\Big(\frac{n(t,0)}{N(0)}\Big). \tag{3.46}$$

Let us now assume that the function $H(s)$ is convex and $H(0) = 0$. Then, as $d\mu(x)$ is a probability measure, we have

$$\int H(u(x))d\mu \geq H\Big(\int u(x)d\mu\Big). \tag{3.47}$$

As

$$\frac{n(t,0)}{N(0)} = \int \frac{B(x)n(t,x)}{N(0)}dx = \int \frac{n(t,x)}{N(x)}\frac{B(x)N(x)}{N(0)}dx = \int \frac{n(t,x)}{N(x)}d\mu(x),$$

we deduce that

$$H\Big(\frac{n(t,0)}{N(0)}\Big) \leq \int H\Big(\frac{n(t,x)}{N(x)}\Big)d\mu(x).$$

Using this in (3.46) gives

$$\frac{d}{dt}\int \phi(x)N(x)H\Big(\frac{n(t,x)}{N(x)}\Big)dx \leq 0 \text{ for all convex functions } H \text{ with } H(0) = 0. \tag{3.48}$$

Integrating (3.46) in time gives a bound for the entropy dissipation

$$D(t) = \int H\Big(\frac{n(t,x)}{N(x)}\Big)d\mu(x) - H\Big(\int \frac{n(t,x)}{N(x)}\Big)d\mu(x)\Big) \geq 0, \tag{3.49}$$

as

$$\phi(0)N(0)\int_0^\infty D(t)dt \leq \int \phi(x)N(x)H\Big(\frac{n_0(x)}{N(x)}\Big)dx < +\infty. \tag{3.50}$$

Long time asymptotics via the generalized entropy method

The entropy dissipation bound (3.49) says, roughly, that $D(t) \to 0$ as $t \to +\infty$, so we expect that in the long time limit $m(t, x)$, solution of

$$\begin{aligned} &\frac{\partial m(t,x)}{\partial t} + \frac{\partial m(t,x)}{\partial x} + \lambda_0 m(t,x) = 0, \\ &m(t,0) = \int_0^\infty B(y) m(t,y) dy, \\ &m(0,x) = n_0(x), \end{aligned} \tag{3.51}$$

would converge to a function $r(x)$ such that

$$\int H\Big(\frac{r(x)}{N(x)}\Big) d\mu(x) - H\Big(\int \frac{r(x)}{N(x)} d\mu(x)\Big) = 0. \tag{3.52}$$

This would imply that $r(x) = CN(x)$ is a multiple of $N(x)$. We will prove that asymptotic behavior in this section. As before, we assume that $B(x) \geq 0$, $B \in L^\infty(\mathbb{R}_+)$, and

$$1 < \int B(y) dy < +\infty. \tag{3.53}$$

Theorem 3.2 *Assume that $|n_0(x)| \leq CN(x)$, and set*

$$\alpha_0 = \int n_0(x) \phi(x) dx, \tag{3.54}$$

then

$$\int_0^\infty |m(t,x) - \alpha_0 N(x)| \phi(x) dx \to 0. \tag{3.55}$$

Proof. As we only assume that $n_0(x)$ satisfies $|n_0(x)| \leq CN(x)$, it is helpful to regularize the initial condition so that we would have, in addition, the bound on the derivative. Hence, we approximate n_0 in $L^1(\mathbb{R}_+; \phi(x)dx)$ by a sequence of smooth functions n_ε^0 such that $n_\varepsilon^0 \to n_0$ in $L^1(\mathbb{R}_+; \phi(x)dx)$ and each n_ε^0 satisfies

$$\Big|\frac{\partial n_\varepsilon^0(x)}{\partial x}\Big| \leq C_1 N(x), \tag{3.56}$$

and the compatibility condition holds

$$n_\varepsilon^0(0) = \int_0^\infty B(y) n_\varepsilon^0(y) dy. \tag{3.57}$$

Note that then $\alpha_\varepsilon^0 \to \alpha_0$ as $\varepsilon \to 0$, and, as we have shown, we have

$$\int_0^\infty |m(t,x) - m_\varepsilon(t,x)| \phi(x) dx \leq \int_0^\infty |n_0(x) - n_\varepsilon^0(x)| \phi(x) dx \to 0 \text{ as } \varepsilon \to 0. \tag{3.58}$$

Therefore, we have

$$\int_0^\infty |m(t,x)-\alpha_0 N(x)|\phi(x)dx \le \int_0^\infty |m(t,x)-m_\varepsilon(t,x)|\phi(x)dx \tag{3.59}$$
$$+\int_0^\infty |m_\varepsilon(t,x)-\alpha_\varepsilon N(x)|\phi(x)dx + |\alpha_\varepsilon-\alpha_0|\int_0^\infty N(x)\phi(x)dx.$$

The first and the last term in the right side go to zero as $\varepsilon \to 0$, uniformly in t, thus to establish our claim it suffices to show that the middle term tends to zero as $t \to +\infty$. In other words, it suffices to prove that the conclusion of the theorem holds for smooth initial data, with the assumption that the compatibility condition (3.57) holds. This is what we will now assume about $n_0(x)$.

Next, note that we may look at the difference

$$h(t,x) = m(t,x) - \alpha_0 N(x),$$

which satisfies the same problem as $m(t,x)$ but with the initial condition

$$h_0(x) = n_0(x) - \alpha_0 N(x),$$

so that

$$\int_0^\infty h_0(x)\phi(x)dx = 0. \tag{3.60}$$

Our goal will be to show that

$$\int_0^\infty |h(t,x)|\phi(x)dx \to 0 \text{ as } t \to +\infty. \tag{3.61}$$

Note that we know that

$$\frac{d}{dt}\int_0^\infty |h(t,x)|\phi(x)dx \le 0, \tag{3.62}$$

hence the limit in (3.61) exists:

$$\int_0^\infty |h(t,x)|\phi(x)dx \to L \text{ as } t \to +\infty. \tag{3.63}$$

Our goal is to show that $L = 0$.

Let us define the time shifts $h_k(t,x) = h(t+k,x)$, and the corresponding entropy dissipation

$$D_k(t) = \int H\Big(\frac{h_k(t,x)}{N(x)}\Big)d\mu(x) - H\Big(\int \frac{h_k(t,x)}{N(x)}d\mu(x)\Big) \ge 0. \tag{3.64}$$

Note that $D_k(t)$ is simply

$$D_k(t) = \int H\Big(\frac{m(t+k,x)}{N(x)}\Big)d\mu(x) - H\Big(\int \frac{m(t+k,x)}{N(x)}\Big)d\mu(x)\Big) \ge 0, \tag{3.65}$$

thus

$$I_k = \int_0^\infty D_k(t)dt = \int_k^\infty D_0(t)dt,$$

and we know from (3.50) that

$$\int_0^\infty D_0(t)dt \le \frac{1}{\phi(0)N(0)} \int \phi(x)N(x)H\Big(\frac{h_0(x)}{N(x)}\Big)dx < +\infty. \tag{3.66}$$

We conclude that

$$I_k = \int_0^\infty \Big[\int H\Big(\frac{h_k(t,x)}{N(x)}\Big)d\mu(x) - H\Big(\int \frac{h_k(t,x)}{N(x)}d\mu(x)\Big)\Big]dt \to 0 \text{ as } k \to +\infty. \tag{3.67}$$

Each h_k satisfies

$$\begin{aligned} &\frac{\partial h_k(t,x)}{\partial t} + \frac{\partial h_k(t,x)}{\partial x} + \lambda_0 h_k(t,x) = 0, \\ &h_k(t,0) = \int_0^\infty B(y)h_k(t,y)dy, \end{aligned} \tag{3.68}$$

and

$$\int_0^\infty h_k(t,y)\phi(y)dy = 0 \text{ for all } t \ge 0. \tag{3.69}$$

Using the regularity bounds on the derivatives of $h_k(t,x)$ in x and t, we may extract a subsequence, still denoted h_k so that $h_k \to g$ in $C([0,T]\times\mathbb{R}_+)$, and

$$|g(x)| \le C_0 N(x), \tag{3.70}$$

$$\int_0^\infty B(y)h_k(t,y)dy \to \int_0^\infty B(y)g(y)dy,$$

$$\int_0^\infty \phi(t,x)g(t,x)dx = 0, \tag{3.71}$$

and

$$\int_0^\infty \phi(x)|g(t,x)|dx = L, \tag{3.72}$$

with L as in (3.63). We now pass to the limit $k \to \infty$ in (3.67). Convexity of the weak limits (if u_k converges weakly to f, H is convex and $H(u_k) \to l$ then $l \ge H(f)$) then implies that

$$\begin{aligned} \int_0^\infty \int H\Big(\frac{g(t,x)}{N(x)}\Big)d\mu(x) &\le \lim_{k\to+\infty}\int_0^\infty \int H\Big(\frac{h_k(t,x)}{N(x)}\Big)d\mu(x) \\ = \lim_{k\to+\infty}\int_0^\infty H\Big(\int \frac{h_k(t,x)}{N(x)}d\mu(x)\Big)dt &= \int_0^\infty H\Big(\int \frac{g(t,x)}{N(x)}d\mu(x)\Big)dt. \end{aligned} \tag{3.73}$$

We used (3.67) in the last step. Jensen's inequality then implies that on the support of

$$d\mu(x) = \frac{B(x)N(x)}{N(0)}dx,$$

which is the same as the support of $B(x)$, we have

$$\frac{g(t,x)}{N(x)} = C(t) \text{ on the support of } B(x). \tag{3.74}$$

However, we may write an equation for this fraction:

$$\frac{\partial}{\partial t}\Big(\frac{g(t,x)}{N(x)}\Big)+\frac{\partial}{\partial x}\Big(\frac{g(t,x)}{N(x)}\Big)=0.$$

It follows that (3.74) holds not just on the support of $B(x)$ but everywhere, and that $C(t)$ is a constant function. We conclude that

$$g(x)=M_0N(x).$$

As

$$\int_0^\infty g(t,x)\phi(x)dx=0,$$

we see that $M_0=0$, and we are done.

The renewal equation with diffusion

Let us now show that the generalized relative entropy method also applies to the renewal equation with diffusion and a death rate

$$\begin{aligned}&\frac{\partial n(t,x)}{\partial t}+\frac{\partial n(t,x)}{\partial x}+d(x)n(t,x)=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial n(t,x)}{\partial x}\Big],\\&n(t,0)=\nu(0)\frac{\partial n(t,0)}{\partial x}+\int_0^\infty B(y)n(t,y)dy,\\&n(0,x)=n_0(x).\end{aligned}\tag{3.75}$$

Note that if both the death rate and the reproduction rate vanish: $d(x)=B(x)=0$, then the total population is preserved:

$$\frac{d}{dt}\int_0^\infty n(t,x)dx=0 \text{ if } d(x)=B(x)=0.$$

This explains the boundary condition in (3.75). We let $N(x)$ be the principal eigenfunction of the steady problem:

$$n(t,x)=e^{\lambda_0 t}N(x)$$

is a solution of (3.75) if

$$\begin{aligned}&\frac{\partial N(x)}{\partial x}+[d(x)+\lambda_0]N(x)=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial N(x)}{\partial x}\Big],\\&N(0)=\nu(0)\frac{\partial N(0)}{\partial x}+\int_0^\infty B(y)N(y)dy.\end{aligned}\tag{3.76}$$

The adjoint problem is now

$$\begin{aligned}&-\frac{\partial \phi(x)}{\partial x}+[d(x)+\lambda_0]\phi(x)=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial \phi(x)}{\partial x}\Big]+B(x)\phi(0),\\&\frac{\partial \phi(0)}{\partial x}=0,\quad \int_0^\infty \phi(x)N(x)dx=1.\end{aligned}\tag{3.77}$$

Leaving for the moment aside the question of the existence of $N(x)$ and $\phi(x)$, look for a relative entropy inequality. Let $m(t,x)$ be the solution of

$$\frac{\partial m(t,x)}{\partial t}+\frac{\partial m(t,x)}{\partial x}+(d(x)+\lambda_0)m(t,x)=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial m(t,x)}{\partial x}\Big], \tag{3.78}$$
$$m(t,0)=\nu(0)\frac{\partial m(t,0)}{\partial x}+\int_0^\infty B(y)m(t,y)dy,$$
$$m(0,x)=n_0(x).$$

Note that the

$$I(t)=\int_0^\infty m(t,x)\phi(x)dx$$

is conserved:

$$\frac{dI}{dt}=\int_0^\infty\Big[-\frac{\partial m(t,x)}{\partial x}-(d(x)+\lambda_0)m(t,x)+\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial m(t,x)}{\partial x}\Big)\Big]\phi(x)dx$$
$$=n(t,0)\phi(0)+\int_0^\infty m(t,x)\Big[-(d(x)+\lambda_0)\phi(x)+\frac{\partial\phi}{\partial x}\Big]dx-\phi(0)\nu(0)\frac{\partial m(t,0)}{\partial x}$$
$$-\int_0^\infty\frac{\partial m(t,x)}{\partial x}\nu(x)\frac{\partial\phi(x)}{\partial x}dx=\int_0^\infty m(t,x)\Big[\phi(0)B(x)-(d(x)+\lambda_0)\phi(x)+\frac{\partial\phi}{\partial x}\Big]dx$$
$$+m(t,0)\nu(0)\frac{\partial\phi(0)}{\partial x}+\int_0^\infty m(t,x)\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial\phi(x)}{\partial x}\Big)dx=0.$$

This explains the choice of the function $\phi(x)$, and the boudnary condition in (3.77).

In order to deduce entropy dissipation, we form the same object as before:

$$Q(t)=\int_0^\infty N(x)\phi(x)H\Big(\frac{m(t,x)}{N(x)}\Big)dx, \tag{3.79}$$

and compute $\dot{Q}(t)$. First, we need an equation for the ratio

$$\zeta(t,x)=\frac{m(t,x)}{N(x)}.$$

Note that $1/N(x)$ satisfies

$$\frac{\partial}{\partial x}\frac{1}{N(x)}-(d(x)+\lambda_0)\frac{1}{N(x)}=-\frac{1}{N^2}\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial N(x)}{\partial x}\Big)=\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)$$
$$-2\nu(x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)^2.$$

Therefore, the function $\zeta(t,x)$ satisfies

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}=\frac{1}{N(x)}\Big[\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial m(t,x)}{\partial x}\Big)-(d(x)+\lambda_0)m\Big]+m\Big[\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)$$
$$+(d(x)+\lambda_0)\frac{1}{N(x)}-2\nu(x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)^2\Big]=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial\zeta}{\partial x}\Big]$$
$$-2\nu(x)\frac{\partial m}{\partial x}\frac{\partial}{\partial x}\frac{1}{N(x)}-2\nu(x)m(t,x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)^2$$
$$=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial\zeta}{\partial x}\Big]-2\nu(x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)\frac{\partial\zeta}{\partial x}.$$

Then, for any function H we obtain

$$\frac{\partial H(\zeta)}{\partial t}+\frac{\partial H(\zeta)}{\partial x}=H'(\zeta)\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial \zeta}{\partial x}\Big]-2H'(\zeta)\nu(x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)\frac{\partial \zeta}{\partial x}$$
$$=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial H(\zeta)}{\partial x}\Big]-\nu(x)\Big(\frac{\partial \zeta}{\partial x}\Big)^2H''(\zeta)-2\nu(x)N(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)\frac{\partial H(\zeta)}{\partial x}.$$

It follows that the function $p(t,x)=N(x)H(\zeta(t,x))$ satisfies

$$\frac{\partial p}{\partial t}+\frac{\partial p}{\partial x}+(d(x)+\lambda_0)p=H(\zeta)\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial N(x)}{\partial x}\Big]+N(x)\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial H(\zeta)}{\partial x}\Big]$$
$$-N(x)\nu(x)\Big(\frac{\partial \zeta}{\partial x}\Big)^2H''(\zeta)-2\nu(x)N^2(x)\Big(\frac{\partial}{\partial x}\frac{1}{N(x)}\Big)\frac{\partial H(\zeta)}{\partial x}$$
$$=\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial p}{\partial x}\Big]-N(x)\nu(x)\Big(\frac{\partial \zeta}{\partial x}\Big)^2H''(\zeta).$$

Therefore, we have for the relative entropy $Q(t)$:

$$\frac{dQ}{dt}=\int_0^\infty \phi(x)\frac{\partial p(t,x)}{\partial t}dx$$
$$=\int_0^\infty \phi\Big[-\frac{\partial p}{\partial x}-(d(x)+\lambda_0)p+\frac{\partial}{\partial x}\Big[\nu(x)\frac{\partial p}{\partial x}\Big]-N(x)\nu(x)\Big(\frac{\partial \zeta}{\partial x}\Big)^2H''(\zeta)\Big]$$
$$=-\int_0^\infty \phi(x)N(x)\nu(x)\Big(\frac{\partial \zeta(t,x)}{\partial x}\Big)^2H''(\zeta)dx$$
$$+\phi(0)p(t,0)+\int_0^\infty p(t,x)\Big[\frac{\partial \phi(x)}{\partial x}-(d(x)+\lambda_0)\phi(x)+\frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial \phi}{\partial x}\Big)\Big]dx$$
$$-\phi(0)\nu(0)\frac{\partial p(t,0)}{\partial x}+p(t,0)\nu(0)\frac{\partial \phi(0)}{\partial x}$$
$$=-\int_0^\infty \phi(x)N(x)\nu(x)\Big(\frac{\partial \zeta(t,x)}{\partial x}\Big)^2H''(\zeta)dx$$
$$+\phi(0)\Big[p(t,0)-\nu(0)\frac{\partial p(t,0)}{\partial x}-\int_0^\infty B(x)p(t,x)dx\Big]. \tag{3.80}$$

Note that

$$p(t,0)-\nu(0)\frac{\partial p(t,x)}{\partial x}=N(0)H(\zeta(t,0))-\nu(0)H(\zeta(t,0))\frac{\partial N(0)}{\partial x} \tag{3.81}$$
$$-\nu(0)N(0)H'(\zeta(t,0))\frac{\partial \zeta(t,0)}{\partial x}=H(\zeta(t,0))\int_0^\infty B(x)N(x)dx-\nu(0)N(0)H'(\zeta(t,0))\frac{\partial \zeta(t,0)}{\partial x}.$$

Let us get the boundary condition for ζ:

$$\nu(0)\frac{\partial \zeta(t,0)}{\partial x}=\frac{1}{N(0)}\Big[m(t,0)-\int_0^\infty B(x)m(t,x)dx\Big]-\frac{m(t,0)}{N^2(0)}[N(0)-\int_0^\infty B(x)N(x)dx\Big]$$
$$=\frac{m(t,0)}{N^2(0)}\int_0^\infty B(x)N(x)dx-\frac{1}{N(0)}\int_0^\infty B(x)m(t,x)dx.$$

Using this in (3.81) gives

$$\begin{aligned}
&p(t,0)-\nu(0)\frac{\partial p(t,0)}{\partial x}-\int_0^\infty B(x)p(t,x)dx = H(\zeta(t,0))\int_0^\infty B(x)N(x)dx \\
&-H'(\zeta(t,0))\nu(0)N(0)\frac{\partial\zeta(t,0)}{\partial x}-\int_0^\infty B(x)N(x)H(\zeta(t,x))dx \\
&= H(\zeta(t,0))\int_0^\infty B(x)N(x)dx - H'(\zeta(t,0))\frac{m(t,0)}{N(0)}\int_0^\infty B(x)N(x)dx \qquad (3.82)\\
&+H'(\zeta(t,0))\int_0^\infty B(x)m(t,x)dx-\int_0^\infty B(x)N(x)H(\zeta(t,x))dx \\
&=\int_0^\infty B(x)N(x)\Big[H(\zeta(t,0))-H'(\zeta(t,0))\zeta(t,0)+H'(\zeta(t,0))\zeta(t,x)-H(\zeta(t,x))\Big]dx.
\end{aligned}$$

Note that

$$D_{ren}:=-\int_0^\infty B(x)N(x)\Big[H(\zeta(t,0))+H'(\zeta(t,0))(\zeta(t,x)-\zeta(t,0))-H(\zeta(t,x))\Big]dx\geq 0 \quad (3.83)$$

if the function $H(s)$ is convex. To summarize, we have obtained the following relative entropy dissipation inequality for any convex function $H(s)$:

$$Q(t)=\int N(x)\phi(x)H\Big(\frac{m(t,x)}{N(x)}\Big)dx \qquad (3.84)$$

satisfies

$$\frac{dQ}{dt}=-D_{diff}(t)-D_{ren}(t), \qquad (3.85)$$

with $D_{ren(t)}$ defined in (3.83), and

$$D_{diff}(t)=\int_0^\infty \phi(x)N(x)\nu(x)\Big(\frac{\partial\zeta(t,x)}{\partial x}\Big)^2 H''(\zeta)dx, \quad \zeta(t,x)=\frac{m(t,x)}{N(x)}. \qquad (3.86)$$

With the relative entropy dissipation inequality in hand, one may proceed as before to obtain the asymptotic limit of $m(t,x)$.

4 Cell motion and chemotaxis

Attraction potential

Before going into the Keller-Segel system, consider a population of bacteria in an attractive potential $V(x)$: they move in the direction of the gradient of $V(x)$ but also diffuse. The governing equation is

$$\rho_t+\nabla\cdot(\rho\nabla V)=\Delta\rho. \qquad (4.1)$$

This equation preserves the total mass:

$$\int\rho(t,x)dx=\int\rho_0(x)dx. \qquad (4.2)$$

On the other hand, multiplying by $e^{-V(x)}\rho(x)$ and integrating we get

$$\frac{1}{2}\frac{d}{dt}\int e^{-V}\rho^2(t,x)dx = \int e^{-V}\rho[\Delta\rho - \nabla\cdot(\rho\nabla V)]dx \tag{4.3}$$
$$= \int e^{-V}[-|\nabla\rho|^2 + 2\rho\nabla\rho\cdot\nabla V - \rho^2|\nabla V|^2]dx = -\int e^{-V(x)}|\nabla\rho - \rho\nabla V|^2 dx.$$

Therefore, the total mass is preserved but the L^2-norm with the weight $e^{-V(x)}$ decreases. Imagine that $V(x)$ is a positive function, concentrated near a point x_0. Then, having $\rho(t,x)$ large near x_0 does not lead to large weighted L^2-norm since $w(x)$ is very small around x_0. On the other hand, if $\rho(t,x)$ is large in a region where $V(x)$ is small, that would produce large contribution to the weighted L^2-norm. Thus, the dynamics tends to concentrate $\rho(t,x)$ in the regions where $V(x)$ is large.

Moreover, the right side of (4.3) vanishes if

$$\bar{\rho}(x) = Ce^{V(x)}, \tag{4.4}$$

which is a steady solution. Thus, we expect that in the long time limit $\rho(x)$ will approach $\bar{\rho}(x)$. If we allow $V(x)$ to vary in time and concentrate more as $\rho(t,x)$ concentrates, we may produce the blow-up in the solution. The Keller-Segel model is an example of how that can happen even with a very modest coupling. **Discuss that $V(x)$ has to grow at infinity which is not the case in the nonlinear problem.**

The Keller-Segel system

The Keller-Segel model for chemotaxis describes the evolution of the bacteria of density $\rho(t,x)$ attracted by chemoreactant of density $c(t,x)$. The idea is the bacteria would like to move in the direction of the maximal local increase of the chemoattractant density, that is, in the direction of $\nabla c(t,x)$. There is also diffusion present to reflect the random motion by the bacteria. This leads to the equation

$$\rho_t + \nabla\cdot(\rho v) = \Delta\rho, \tag{4.5}$$

with the velocity

$$v(t,x) = \chi\nabla c(t,x). \tag{4.6}$$

The parameter $\chi \geq 0$ measures the strength of the chemotactic attraction. In order to close the system we need an equation for $c(t,x)$. The Keller-Segel model postulates that the chemoattractant is emitted by the cells and diffuses:

$$\alpha\frac{\partial c}{\partial t} - \Delta c + \tau c = \rho. \tag{4.7}$$

Here, $\tau^{-1/2}$ is "the activation length" of the chemoattractant. One interesting regime is $\alpha, \tau \to 0$, leading to

$$c = (-\Delta)^{-1}\rho = k_d\int_{\mathbb{R}^n}\frac{\rho(y)}{|x-y|^{n-2}}dy, \tag{4.8}$$

in dimension $n \geq 3$. In dimension $n = 2$, which, in some sense, is the most interesting case, the Green's function $G(x) = k_2 \log|x|$ is not positive, meaning that the positivity of $c(t,x)$ can be violated, something we will need to avoid.

An interesting feature of the Keller-Segel system is that it is critical in $L^{n/2}$: solutions with smooth initial conditions with a small $L^{n/2}$ will exist for all times, but solutions with a large $L^{n/2}$ initial norm, even if the initial condition is smooth, will blow-up in a finite time – the bacteria concentration will become infinite. In particular, in dimension $n = 1$ global solutions always exist, but in $\mathbb{R}^2$ solutions with a larger initial mass (L^1 norm) will blow up.

The free energy

Let us first look at the heat equation

$$\phi_t = \Delta\phi, \tag{4.9}$$

and look at what happens for the free energy

$$E_{heat} = \int_{\mathbb{R}^n} \phi \log \phi dx. \tag{4.10}$$

We have

$$\frac{dE_{heat}}{dt} = \int (1 + \log\phi)(\Delta\phi) dx. = -\int \frac{1}{\phi}|\nabla\phi|^2 dx = -\int \phi|\nabla(\log\phi)|^2 dx. \tag{4.11}$$

Let us see what happens for equations of the form

$$\rho_t + \nabla\cdot(\rho v) = \Delta\rho, \tag{4.12}$$

with an advecting velocity $v(t,x)$. Then we have

$$\begin{aligned}\frac{dE_{heat}}{dt} &= \int (1 + \log\rho)(\Delta\rho - \nabla\cdot(\rho v)) dx = -\int \frac{1}{\rho}|\nabla\rho|^2 dx + \int (v\cdot\nabla\rho) dx\\ &= -\int \rho\Big|\nabla(\log\rho)|^2 dx + \int (v\cdot\nabla\rho) dx.\end{aligned} \tag{4.13}$$

In the Keller-Segel model we have

$$v = \chi\nabla c, \quad c = (-\Delta)^{-1}\rho.$$

Therefore, we have

$$\int \rho c_t dx = \int \rho(-\Delta)^{-1}\rho_t dx = \int [(-\Delta)^{-1}\rho]\rho_t = \int c\rho_t dx,$$

so that

$$\begin{aligned}&\frac{\chi}{2}\frac{d}{dt}\int \rho(t,x)c(t,x)dx = \chi\int c\rho_t dx = \chi\int c(\Delta\rho - \chi\nabla\cdot(\rho\nabla c))dx\\ &= -\chi\int (\nabla c\cdot\nabla\rho)dx + \chi^2\int \rho|\nabla c|^2 dx = \int (v\cdot\nabla\rho)dx + \int \rho|v|^2 dx.\end{aligned}$$

Combining this with (4.13) gives

$$\frac{d}{dt}\Big(E_{heat} - \frac{\chi}{2}\int \rho(t,x)c(t,x)dx\Big) = -\int \rho\Big|\nabla(\log\rho) - v|^2 dx \le 0. \tag{4.14}$$

We define the free energy then as

$$E_{cc} = \int (\rho\log\rho - \frac{\chi}{2}\rho c)dx. \tag{4.15}$$

Existence for small data in dimentions $n > 2$

We now show that if the initial condition is small in $L^{n/2}$ norm and $n > 2$ then solution exists for all times. We will consider the "time-independent" chemotactic law:

$$\begin{aligned} &\rho_t + \chi\nabla\cdot(\rho\nabla c) = \Delta\rho, \\ &-\Delta c + = \rho, \\ &\rho(0,x) = \rho_0(x). \end{aligned} \tag{4.16}$$

Theorem 4.1 *Let $n > 2$. There exists a constant K_n so that if $\rho_0 \in L^1(\mathbb{R}^n)$, and $\|\rho_0\|_{L^{n/2}} \le K_n/\chi$, then (4.16) admits a unique weak solution $\rho \in L^\infty(\mathbb{R}_+; L^1 \cap L^{n/2})$ such that*

$$\|\rho(t)\|_{L^{n/2}} \le \|\rho_0\|_{L^{n/2}}, \tag{4.17}$$

$$\int_0^\infty \int_{\mathbb{R}} |\rho|^{1+(d/2)}(t,x)dxdt < +\infty,$$

$$\nabla(\rho^{n/4}) \in L^2(\mathbb{R}^+ \times \mathbb{R}^n). \tag{4.18}$$

We also have the decay estimates:

$$\begin{aligned} &\|\rho(t)\|_{L^{n/2}} \le \frac{Cm_0}{t^\beta}, \quad \beta = \frac{d-2}{2}, \\ &\|\rho(t)\|_{L^p} \le \frac{Cm_0}{t^z}, \quad t \ge T(n,p), \quad z = \frac{n}{2}\Big(1 - \frac{1}{p}\Big). \end{aligned} \tag{4.19}$$

The decay estimates simply say that after a long time the solution with a small initial condition becomes so small that the nonlinear term may be neglected, and $\rho(t,x)$ behaves as a solution of the heat equation.
Proof. Multiply (4.16) by ρ^{p-1}:

$$\frac{1}{p}\frac{d}{dt}\int \rho^p(t,x)dx = \int (\Delta\rho - \chi\nabla\cdot(\rho\nabla c))\rho^{p-1}dx \tag{4.20}$$

$$= -(p-1)\int \rho^{p-2}|\nabla\rho|^2 dx + (p-1)\chi\int \rho^{p-1}(\nabla c\cdot\nabla\rho)dx$$

$$= -\frac{4(p-1)}{p^2}\int |\nabla(\rho^{p/2})|^2 dx + \chi\frac{p-1}{p}\int (\nabla c\cdot\nabla(\rho^p))dx. \tag{4.21}$$

Integrating by parts in the last term in the right side and using the equation for $c(t,x)$ we get

$$\frac{d}{dt}\int \rho^p(t,x)dx + \frac{4(p-1)}{p}\int |\nabla(\rho^{p/2})|^2 dx = \chi\frac{p-1}{p}\int \rho^{p+1}dx. \tag{4.22}$$

Let us set $u = \rho^{p/2}$, then (4.22) is

$$\frac{d}{dt}\int u^2(t,x)dx + \frac{4(p-1)}{p}\int |\nabla u|^2 dx = \chi\frac{p-1}{p}\int u^{2(p+1)/p}dx. \tag{4.23}$$

In order to bound the right side of (4.23) in terms of its left side we may now use the Gagliardo-Nirenberg inequality

$$\|u\|_{L^r(\mathbb{R}^n)} \le C\|\nabla u\|^{\theta}_{L^s(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \tag{4.24}$$

with $1 \le s < n$, $1 \le q \le \infty$, and

$$\frac{1}{r} = \frac{\theta}{s} - \frac{\theta}{n} + \frac{1-\theta}{q}. \tag{4.25}$$

Taking $s = 2$, $q = n/p$, and $r = 2(p+1)/p$ we see that, first, indeed, we need $n > 2$ and, second, θ is determined by

$$\frac{p}{2(p+1)} = \frac{\theta}{2} - \frac{\theta}{n} + \frac{p(1-\theta)}{n} = \frac{p}{n} + \theta\frac{n-2(p+1)}{2n}$$

so that

$$\theta = \frac{2n}{n-2(p+1)}\Big(\frac{p}{2(p+1)} - \frac{p}{n}\Big) = \frac{2np}{2n(p+1)} = \frac{p}{p+1}.$$

We see that

$$\int u^{2(p+1)/p}dx \le C\Big(\int |\nabla u|^2 dx\Big)^{\theta r/2}\Big(\int |u|^{n/p}dx\Big)^{(1-\theta)pr/n} = C\Big(\int |\nabla u|^2 dx\Big)\Big(\int |u|^{n/p}dx\Big)^{2/n}. \tag{4.26}$$

Going back to the function ρ this is

$$\int \rho^{p+1}dx \le C\Big(\int |\nabla(\rho^{p/2})|^2 dx\Big)\Big(\int |\rho|^{n/2}dx\Big)^{2/n}. \tag{4.27}$$

Returning to (4.22), we deduce

$$\frac{d}{dt}\int \rho^p(t,x)dx + \frac{4(p-1)}{p}\int |\nabla(\rho^{p/2})|^2 dx \le C\chi\frac{p-1}{p}\Big(\int |\nabla(\rho^{p/2})|^2 dx\Big)\Big(\int |\rho|^{n/2}dx\Big)^{2/n}. \tag{4.28}$$

To close this inequality, we choose $p = n/2$ so that

$$\frac{d}{dt}\int \rho^{n/2}(t,x)dx + \frac{4(p-1)}{p}\int |\nabla(\rho^{n/4})|^2 dx \le C\chi\frac{p-1}{p}\Big(\int |\nabla(\rho^{n/4})|^2 dx\Big)\Big(\int |\rho|^{n/2}dx\Big)^{2/n}. \tag{4.29}$$

Therefore, if at the time $t = 0$ we have

$$C\chi\frac{p-1}{p}\Big(\int |\rho_0|^{n/2}dx\Big)^{2/n} < \frac{1}{2}\frac{4(p-1)}{p}, \tag{4.30}$$

then

$$\frac{d}{dt}\int \rho^{n/2}(t,x)dx < 0,$$

for all $t > 0$, and inequality (4.30) persists:

$$C\chi\frac{p-1}{p}\Big(\int|\rho(t,x)|^{n/2}dx\Big)^{2/n} < \frac{1}{2}\frac{4(p-1)}{p}. \tag{4.31}$$

This proves the $L^{n/2}$ bound in the theorem. Integrating in time we see that if (4.30) holds, then

$$\int_0^\infty\int|\nabla(\rho^{n/4})|^2dxdt \le C < +\infty. \tag{4.32}$$

Now, going back to (4.27), with $p = n/2$, we see that

$$\int_0^\infty\int\rho^{n/2+1}dxdt \le C\int_0^\infty\Big(\int|\nabla(\rho^{p/2})|^2dx\Big)\Big(\int|\rho|^{n/2}dx\Big)^{2/n}dt \le C, \tag{4.33}$$

because of (4.31) and (4.32).

To get the decay bounds on ρ in time, we get back to (4.29) with the assumption (4.30)

$$\frac{d}{dt}\int\rho^{n/2}(t,x)dx \le -C\int|\nabla(\rho^{n/4})|^2dx. \tag{4.34}$$

We now use the Gagliardo-Nirenberg inequality

$$\int|u|^{2+4/n}dx \le C\Big(\int|\nabla u|^2dx\Big)\Big(\int|u|^2dx\Big)^{2/n}. \tag{4.35}$$

It is an example of (4.24) with

$$r = 2+\frac{4}{n},\quad s=2,\quad q=2,\quad \theta=\frac{2}{r},\quad 1-\theta=\frac{4}{nr}.$$

Taking $u = \rho^{n/4}$ in (4.35) gives

$$\int|\nabla(\rho^{n/4})|^2dx \ge C\Big(\int\rho^{n/2}\Big)^{-2/n}\Big(\int\rho^{1+n/2}dx\Big). \tag{4.36}$$

We obtain from (4.34):

$$\frac{d}{dt}\int\rho^{n/2}(t,x)dx \le -\frac{C}{\|\rho(t)\|_{L^{n/2}}}\int\rho^{1+n/2}dx. \tag{4.37}$$

However, the L^1-norm of $\rho(t)$ is preserved:

$$m_0 = \int\rho_0(x)dx = \int\rho(t,x)dx.$$

We may now interpolate:

$$\int\rho^{n/2}dx = \int\rho^a\rho^bdx \le \Big(\int\rho^{ap}dx\Big)^{1/p}\Big(\int\rho^{bq}dx\Big)^{1/q} = \Big(\int\rho dx\Big)^{1/p}\Big(\int\rho^{1+n/2}\Big)^{1/q}$$

if we choose a, b, p and q so that

$$\frac{1}{p}+\frac{1}{q}=1, \quad a+b=\frac{n}{2}, \quad ap=1, \quad bq=\frac{n}{2}+1,$$

that is,

$$q=\frac{n}{n-2}, \quad p=\frac{n}{2},$$

so that

$$\int \rho^{n/2}dx \le \Big(\int \rho dx\Big)^{2/n}\Big(\int \rho^{1+n/2}dx\Big)^{(n-2)/n},$$

and

$$\int \rho^{1+n/2}dx \ge m_0^{-2/(n-2)}\Big(\int \rho^{n/2}dx\Big)^{n/(n-2)}$$

Going back to (4.37) gives

$$\frac{d}{dt}\int \rho^{n/2}(t,x)dx \le -\frac{Cm_0^{-2/(n-2)}}{\|\rho(t)\|_{L^{n/2}}}\Big(\int \rho^{n/2}\Big)^{n/(n-2)} = -Cm_0^{-2/(n-2)}\Big(\int \rho^{n/2}dx\Big)^{k}, \tag{4.38}$$

with

$$k=\frac{n}{n-2}-\frac{2}{n}>1.$$

Let us set

$$u(t)=\int \rho^{n/2}(t,x)dx,$$

then we have

$$\frac{du}{dt} \le -Cm_0^{-2/(n-2)}u^k,$$

or

$$\frac{1}{k-1}\frac{d}{dt}(u^{-k+1}) \ge Cm_0^{-2/(n-2)},$$

so that

$$\frac{1}{u^{k-1}(t)} \ge \frac{1}{u_0^{k-1}} + C(k-1)m_0^{-2/(n-2)}t \ge C'm_0^{-2/(n-2)}t.$$

We obtain

$$u(t) \le \frac{C''m_0^{2/[(n-2)(k-1)]}}{t^{1/(k-1)}}.$$

It follows that

$$\|\rho(t)\|_{L^{n/2}} \le \frac{Cm_0^a}{t^b}, \tag{4.39}$$

with

$$a=\frac{2(k-1)}{(n-2)}\frac{2}{n}=\frac{4}{n(n-2)}\Big[\frac{n}{n-2}-\frac{2}{n}-1\Big]^{-1}=1,$$

and

$$b=\frac{1}{k-1}\frac{2}{n}=\frac{n-2}{2}.$$

Thus, (4.39) says that

$$\|\rho(t)\|_{L^{n/2}} \leq \frac{C}{t^{(n-2)/2}}\|\rho_0\|_{L^1}, \tag{4.40}$$

as claimed by the theorem. Finally, for the decay of the L^p-norms we return to (4.28):

$$\frac{d}{dt}\int \rho^p(t,x)dx + \frac{4(p-1)}{p}\int |\nabla(\rho^{p/2})|^2 dx \leq C\chi\frac{p-1}{p}\Big(\int |\nabla(\rho^{p/2})|^2 dx\Big)\Big(\int |\rho|^{n/2}dx\Big)^{2/n}. \tag{4.41}$$

Now we know that $\|\rho\|_{L^{n/2}} \to 0$ as $t \to +\infty$, hence we know that for all $t > T$, with some $T > 0$, we have $\|\rho\|_{L^{n/2}} \leq 1/2C\chi$, so that

$$\frac{d}{dt}\int \rho^p(t,x)dx \leq -C\Big(\int |\nabla(\rho^{p/2})|^2 dx\Big), \tag{4.42}$$

and we can proceed as before.

The existence part is done by the usual regularization plus the above a priori bounds for the regularized system plus removing the regularization argument.

Blow-up for large data in $n > 2$

We now show that solutions with large initial data can blow up, and first consider $n > 2$.

Theorem 4.2 *Assume $n > 2$, and set*

$$m_0 = \int \rho_0(x)dx.$$

There exists a constant $\bar{C} > 0$ so that if

$$\chi\int |x|^2\rho_0(x)dx \leq \bar{C}(\chi m_0)^{n/(n-2)} \tag{4.43}$$

then there is no global smooth solution with decay at infinity to the Keller-Segel system.

Let us first explain why (4.43) can not hold if ρ_0 has a small $L^{n/2}$-norm (which would guarantee the existence of a global smooth solution). To see that, let us fix $R > 0$, set

$$m_2 = \int |x|^2\rho_0(x),$$

and write

$$m_0 = \int \rho_0(x)dx = \int_{|x|\geq R}\rho_0(x)dx + \int_{|x|\leq R}\rho_0(x)dx \tag{4.44}$$
$$\leq \frac{1}{R^2}\int_{|x|\geq R}|x|^2\rho_0(x)dx + \Big(\int_{|x|\leq R}\rho_0^{n/2}(x)dx\Big)^{2/n} CR^{n(1-2/n)} \leq \frac{m_2}{R^2} + CR^{n-2}\|\rho_0\|_{L^{n/2}}.$$

We choose R so that the two terms in the right side coincide:

$$R = \frac{Cm_2^{1/n}}{\|\rho\|_{L^{n/2}}^{1/n}},$$

leading to

$$m_0 \leq C m_2^{1-2/n} \|\rho\|_{L^{n/2}}^{2/n}, \tag{4.45}$$

or, equivalently,

$$m_2 \geq m_0^{n/(n-2)} \|\rho\|_{L^{n/2}}^{-2/n}. \tag{4.46}$$

Therefore, condition (4.43) implies that

$$\chi \|\rho\|_{L^{n/2}}^{-2/n} \leq \bar{C} \chi^{n/(n-2)},$$

or

$$\|\rho\|_{L^{n/2}} \geq C \chi^{(1-n/(n-2))(n/2)} = \frac{C}{\chi}.$$

Thus, the scaling $1/\chi$ for the $\|\rho\|_{L^{n/2}}$ norm is the same both to ensure the global existence and to guarantee the blow-up.

Proof. We now prove the theorem. We will now denote

$$m_2(t) = \int |x|^2 \rho(t,x) dx.$$

Recall that

$$\nabla c(t,x) = -C_n \int \frac{x-y}{|x-y|^n} \rho(t,y) dy,$$

so that, multiplying the equation of $\rho(t,x)$ by $|x|^2$ and integrating, we get

$$\begin{aligned}
&\frac{dm_2}{dt} = \int |x|^2 \rho(t,x) dx - \chi \int |x|^2 \nabla \cdot (\rho(t,x) \nabla c(t,x)) dx \\
&= 2nm_0 + 2\chi \int \rho(t,x)(x \cdot \nabla c(t,x)) dx = 2nm_0 - 2\chi C_n \int \rho(t,x)\rho(t,y) \frac{(x \cdot (x-y))}{|x-y|^n} dxdy.
\end{aligned}$$

Symmetrizing the last expression, we can write

$$\int \rho(t,x)\rho(t,y) \frac{(x \cdot (x-y))}{|x-y|^n} dxdy = \frac{1}{2} \int \rho(t,x)\rho(t,y) \frac{1}{|x-y|^{n-2}} dxdy.$$

It follows that for any $R > 0$ we may write

$$\begin{aligned}
&\frac{dm_2}{dt} \leq 2nm_0 - C_n \chi \int_{|x-y| \leq R} \rho(t,x)\rho(t,y) \frac{1}{|x-y|^{n-2}} dxdy \\
&\leq 2nm_0 - \frac{C_n \chi}{R^{n-2}} \int_{|x-y| \leq R} \rho(t,x)\rho(t,y) dxdy \\
&= 2nm_0 - \frac{C_n \chi}{R^{n-2}} \Big[\int_{\mathbb{R}^n} \rho(x)\rho(y) dy - \int_{|x-y| \geq R} \rho(t,x)\rho(t,y) dxdy \Big] \\
&= 2nm_0 - \frac{C_n \chi m_0^2}{R^{n-2}} + \frac{C_n \chi}{R^{n-2}} \int_{\mathbb{R}^n} \rho(t,x)\rho(t,y) \frac{|x-y|^2}{R^2} dxdy.
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^n} \rho(t,x)\rho(t,y)|x-y|^2 dxdy \leq 2 \int_{\mathbb{R}^n} \rho(t,x)\rho(t,y)(x^2+y^2) dxdy = 4m_0 m_2.$$

Therefore, we have

$$\frac{dm_2}{dt} \leq 2nm_0 - \frac{C_n \chi m_0^2}{R^{n-2}} + \frac{4C_n \chi m_0 m_2}{R^n}.$$

Let us take R small, so that the first two terms would combine to be negative:

$$2nm_0 = \varepsilon \frac{C_n \chi m_0^2}{R^{n-2}},$$

or

$$R = (C\chi m_0)^{1/(n-2)}.$$

This would give

$$\frac{dm_2}{dt} \leq -\frac{m_0}{2} + C\chi m_0 m_2 \frac{1}{(\chi m_0)^{n/(n-2)}} = -m_0 \Big(\frac{1}{2} - \frac{C\chi m_2}{(\chi m_0)^{n/(n-2)}} \Big).$$

Therefore, as m_0 is conserved, if we have at $t = 0$

$$\chi m_2(0) \leq C(\chi m_0)^{n/(n-2)},$$

which is assumption (4.43), then $m_2(t)$ is a decreasing in time function such that

$$\frac{dm_2}{dt} \leq -\frac{m_0}{3}.$$

Hence, there is a finite time $T > 0$ so that $m_2(T)$ must be negative, which is a contradiction. Thus, there can not be a global smooth solution for which the moment $m_2(t)$ remains finite for all times.

The critical mass in dimension $n = 2$

In two dimensions, the result is much more precise: we will show that solutions of the Keler-Segel system

$$\begin{aligned} &\rho_t + \chi \nabla \cdot (\rho \nabla c) = \Delta \rho, \\ &-\Delta c = \rho, \\ &\rho(0, x) = \rho_0(x), \end{aligned} \tag{4.47}$$

with the initial mass m_0 larger than the critical mass

$$m_c = \frac{8\pi}{\chi}. \tag{4.48}$$

blow up in a finite time, while those with $m_0 < m_c$ (and a slightly stronger initial decay) will have global in time solutions. We will always assume that

$$m_2(0) = \int |x|^2 \rho_0(x) dx < +\infty, \quad m_0 = \int \rho_0(x) dx < +\infty, \tag{4.49}$$

and

$$\int \rho_0(x) |\log \rho_0(x)| dx < +\infty. \tag{4.50}$$

Here is the blow-up result.

Theorem 4.3 *In $n = 2$ assume that (4.49), (4.50) hold and $m_0 > m_c$. Then any solution to the Keeler-Segel system (4.47) becomes a singular measure in a finite time.*

And here is the global existence.

Theorem 4.4 *In $n = 2$ assume that (4.49), (4.50) hold and $m_0 < m_c$. Then there are global weak solutions to the Keller-Segel system (4.47) such that*

$$\int \rho_0(x)(|\log \rho_0(x)| + |x|^2)dx < +\infty. \tag{4.51}$$

Proof of the blow-up

Th proof of the blow-up itself is not difficult, the main difficulty is to show that the solution becomes a singular measure at the blow-up time. Note that in $n = 2$ we still have the formula

$$\nabla c(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \rho(t, y) dy, \tag{4.52}$$

hence the second moment

$$m_2(t) = \int_{\mathbb{R}^2} |x|^2 \rho(t, x) dx$$

satisfies

$$\frac{dm_2}{dt} = \int |x|^2 [\Delta\rho - \chi\nabla \cdot (\rho\nabla c)]dx = 4m_0 + 2\chi \int \rho(t, x)(x \cdot \nabla c(t, x))dx \tag{4.53}$$
$$= 4m_0 - \frac{\chi}{\pi} \int \frac{(x \cdot (x - y))}{|x - y|^2} \rho(t, x)\rho(t, y)dxdy$$
$$= 4m_0 - \frac{\chi}{2\pi} \int \frac{(x \cdot (x - y)) + (y \cdot (y - x))}{|x - y|^2} \rho(t, x)\rho(t, y)dxdy = 4m_0 - \frac{\chi}{2\pi} m_0^2.$$

As m_0 is conserved in time, we conclude that

$$m_2(t) = m_2(0) - 4m_0 \Big(1 - \frac{\chi}{8\pi} m_0\Big) t. \tag{4.54}$$

Therefore, if $m_0 > 8\pi/\chi$ we will have $m_2(T) < 0$ in a finite time, which is a contradiction, and global in time solution can not exist. This, however, does not answer the question of what happens at the blow-up time, and we need to address this claim in the theorem separately.

The L^1 weak solutions

Multiplying (4.47) by a test function $\psi \in C_c^\infty(\mathbb{R}^2)$ and integrating by parts we get

$$\frac{d}{dt} \int \psi(x)\rho(t, x)dx = \int \Delta\psi(x)\rho(t, x)dx - \frac{\chi}{2\pi} \int \Big((x - y) \cdot \nabla\psi(x)\Big) \frac{\rho(t, x)\rho(t, y)}{|x - y|^2} dxdy. \tag{4.55}$$

In order to deal with the singularity in the right side we symmetrize:

$$\begin{aligned}
&\int \Big((x-y)\cdot\nabla\psi(x)\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy\\
&=\frac{1}{2}\int \Big((x-y)\cdot\nabla\psi(x)\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy+\frac{1}{2}\int \Big((y-x)\cdot\nabla\psi(y)\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy\\
&=\frac{1}{2}\int \Big((x-y)\cdot\nabla(\psi(x)-\nabla\psi(y))\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy. \qquad (4.56)
\end{aligned}$$

The term in the right side of (4.56) is not singular because of the difference $\nabla\psi(x)-\nabla\psi(y)$. Therefore, we say that $\rho(t,x)\in L^\infty((0,T);L^1(\mathbb{R}^2))$ is a weak solution provided that

$$\frac{d}{dt}\int\psi(x)\rho(t,x)dx=\int\Delta\psi(x)\rho(t,x)dx-\frac{\chi}{4\pi}\int\Big((x-y)\cdot(\nabla\psi(x)-\nabla\psi(y))\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy. \qquad (4.57)$$

The weak solutions conserve mass. To see that, we use a test function of the form $\psi_R(x)=\psi(x/|R|)$, with $\psi(x)$ a smooth function such that $\psi(r)=1$ for $0\le|r|\le 1/2$ and $\psi(r)=0$ for $r>1$. Then we have

$$\Big|\int\rho(t,x)\Delta\psi(x)dx\Big|\le\frac{C}{R^2}\int\rho(t,x)dx=\frac{C}{R^2}\|\rho(t)\|_{L^1}, \qquad (4.58)$$

and

$$\Big|\int\Big((x-y)\cdot(\nabla\psi(x)-\nabla\psi(y))\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy\Big|\le\frac{C}{R^2}\int\rho(t,x)\rho(t,y)dxdy=\frac{C}{R^2}\|\rho(t)\|_{L^1}^2. \qquad (4.59)$$

These estimates imply conservation of mass: by the monotone convergence theorem we have

$$\int_{\mathbb{R}^2}\rho(t,x)dx=\lim_{R\to\infty}\int\psi_R(x)\rho(t,x)dx=\lim_{R\to\infty}\int\psi_R(x)\rho_0(x)dx=\int_{\mathbb{R}^2}\rho_0(x)dx. \qquad (4.60)$$

Let us assume that, in addition to $\rho_0\in L^1(\mathbb{R}^2)$, we also know that

$$m_2(0)=\int|x|^2\rho_0(x)dx<+\infty.$$

In order to see what happens to

$$m_2(t)=\int|x|^2\rho(t,x)dx,$$

let us take a sequence of functions $\phi_R(x)=|x|^2\psi_R(x)$, multiply the equation by $\phi_R(x)$ and integrate:

$$\frac{d}{dt}\int\phi_R(x)\rho(t,x)dx=\int\Delta\phi_R(x)\rho(t,x)dx-\frac{\chi}{4\pi}\int\Big((x-y)\cdot(\nabla\phi_R(x)-\nabla\phi_R(y))\Big)\frac{\rho(t,x)\rho(t,y)}{|x-y|^2}dxdy. \qquad (4.61)$$

Note that

$$\Delta\phi_R(x)=4\psi\Big(\frac{x}{R}\Big)+4\frac{x}{R}\cdot\nabla\psi\Big(\frac{x}{R}\Big)+\frac{x^2}{R^2}\Delta\psi\Big(\frac{x}{R}\Big),$$

hence

$$|\Delta\phi_R(x)| \le C,$$

and, similarly, we can bound any second derivative of ϕ_R:

$$\Big|\frac{\partial^2\phi_R(x)}{\partial x_j \partial x_k}\Big| \le C. \tag{4.62}$$

We also have

$$\Big|\Big((x-y)\cdot(\nabla\phi_R(x)-\nabla\phi_R(y))\Big)\Big| \le \frac{|\nabla\phi_R(x)-\nabla\phi_R(y)|}{|x-y|} \le C \tag{4.63}$$

because of (4.62). We conclude that

$$\Big|\frac{d}{dt}\int \phi_R(x)\rho(t,x)dx\Big| \le Cm_0 + m_0^2, \tag{4.64}$$

so that

$$\Big|\int \phi_R(x)\rho(t,x)dx\Big| \le C. \tag{4.65}$$

Moreover, we may now write

$$\begin{aligned}\int \phi_R(x)\rho(t,x)dx &= \int \phi_R(x)\rho_0(x)dx + \int_0^t\int \Delta\phi_R(x)\rho(s,x)dxds \\ &-\frac{\chi}{4\pi}\int_0^t\int \Big((x-y)\cdot(\nabla\phi_R(x)-\nabla\phi_R(y))\Big)\frac{\rho(s,x)\rho(s,y)}{|x-y|^2}dxdyds.\end{aligned} \tag{4.66}$$

As long as $\rho(t,x)$ remains an L^1-function, we may use the Lebesgue dominated convergence theorem, together with the expression for $\Delta\phi_R$ and

$$\nabla\phi_R(x) = 2x\psi\Big(\frac{x}{R}\Big) + \frac{|x|^2}{R}\nabla\psi\Big(\frac{x}{R}\Big),$$

to conclude that, as long $\rho(t,x)$ remains in $L^\infty((0,t);L^1(\mathbb{R}^2))$, we have

$$\int |x|^2\rho(t,x)dx = \int |x|^2\rho_0(x)dx + 4\int_0^t\int \rho(s,x)dxds - \frac{\chi}{2\pi}\int_0^t\int \rho(s,x)\rho(s,y)dxdyds. \tag{4.67}$$

Therefore, $\rho(t,x)$ can not remain in $L^\infty((0,t);L^1(\mathbb{R}^2))$ for $t > T_*$ such that

$$\int |x|^2\rho_0(x)dx + 4m_0T_* - \frac{\chi}{2\pi}m_0^2T_* = 0. \tag{4.68}$$

This shows that $\rho(t,x)$ must become a singular measure – recall that its total mass remains bounded.

Existence for subcritical mass in $n = 2$

We now prove Theorem 4.4. We will need to use the following logarithmic Hardy-Littlewood-Sobolev inequality. Let $f \geq 0$ be such that $f \log f \in L^1(\mathbb{R}^n)$, and set

$$M = \int f(x)dx.$$

Then we have

$$\int_{\mathbb{R}^n} f \log f dx + \frac{n}{M} \int f(x)f(y) \log |x-y| dxdy \geq M[\log M - C_n], \tag{4.69}$$

with

$$C_2 = 1 + \log \pi.$$

In dimension $n = 2$ we can understand it as follows. Consider the Poisson equation

$$-\Delta c = f,$$

so that

$$c(x) = -\frac{1}{2\pi} \int \log(x-y) f(y) dy.$$

Then if $f \in L^1(\mathbb{R}^2)$ we would not automatically have $c \in L^\infty(\mathbb{R}^2)$, nor $\nabla c \in L^2(\mathbb{R}^2)$, because of the logarithmic divergences. However, the lemma asserts that if $f \log f \in L^1(\mathbb{R}^2)$, then we have

$$\int |\nabla c(x)|^2 dx = \int c(x) f(x) dx = -\frac{1}{2\pi} \int \log |x-y| f(x) d(y) dxdy < +\infty,$$

as follows from (4.69).

We will use the free energy (4.15):

$$E_{cc}(t) = \int (\rho \log \rho - \frac{\chi}{2} \rho c) dx = \int \rho(t,x) \log \rho(t,x) dx + \frac{\chi}{4\pi} \int \rho(t,x) \rho(t,x) \log |x-y| dxdy. \tag{4.70}$$

Recall that

$$E_{cc}(t) \leq E_{cc}(0).$$

Hence, the logarithmic Hardy-Littlewood-Sobolev inequality implies that

$$\begin{aligned} \int \rho(t,x) \log \rho(t,x) dx &\leq E_{cc}(0) - \frac{\chi}{4\pi} \int \rho(t,x) \rho(t,x) \log |x-y| dxdy \qquad (4.71) \\ &\leq E_{cc}(0) + \frac{\chi}{4\pi} \frac{m_0}{2} \int \rho(t,x) \log \rho(t,x) dx - \frac{\chi}{4\pi} \frac{m_0}{2} m_0 (\log m_0 - C_2). \end{aligned}$$

Thus, if $m_0 < m_c = 8\pi/\chi$, we get a bound

$$\int_{\mathbb{R}^2} \rho(t,x) \log \rho(t,x) dx \leq \Big(1 - \frac{m_0}{m_c}\Big)^{-1} [E_{cc}(0) - C(m_0)]. \tag{4.72}$$

In addition, (4.67) gives us

$$\int |x|^2\rho(t,x)dx = \int |x|^2\rho_0(x)dx + 4m_0t - \frac{4m_0^2}{m_c}t. \tag{4.73}$$

We may define

$$A = \left\{x:\ 0 < \rho(t,x) < e^{-|x|^2}\right\}, \quad B = \left\{x:\ e^{-|x|^2} \le \rho(t,x) < 1\right\}$$

and write

$$\int_{|\rho|\le 1} \rho(t,x)|\log\rho(t,x)|dx = \int_A \rho(t,x)|\log\rho(t,x)|dx + \int_B \rho(t,x)|\log\rho(t,x)|dx = I + II. \tag{4.74}$$

For the first term we have, as $u|\log u|$ as an increasing function for small $u > 0$:

$$I \le \int_A |x|^2 e^{-|x|^2}dx \le C,$$

and for the second

$$II \le \int_B |x|^2\rho(t,x)dx \le C(t),$$

by (4.73). We conclude that, as $|u| = u + 2u_-$:

$$\int_{\mathbb{R}^2} \rho(t,x)|\log\rho(t,x)|dx = \int_{\mathbb{R}^2} \rho(t,x)\log\rho(t,x)dx + 2\int_{|\rho|\le 1} \rho(t,x)|\log\rho(t,x)|dx \le C(t). \tag{4.75}$$

This strengthens the assumption $\rho \in L^\infty((0,T);L^1(\mathbb{R}^2))$ to the a priori bound

$$\rho\log\rho \in L^\infty((0,T);L^1(\mathbb{R}^2)) \text{ for all } T > 0.$$

This gives then a uniform L^2-bound for ∇c, and provides enough compactness to prove existence of weak solutions.

5 Traveling waves

What is a traveling wave

Let us start with a simple ODE

$$\frac{du}{dt} = u(1-u), \quad u(0) = u_0. \tag{5.1}$$

This ODE has a stable steady state $u \equiv 1$ and an unstable steady state $u \equiv 0$. For any $u_0 \in (0,1)$ we have $u(t) \to 1$ as $t \to +\infty$. One may think of $u(t)$ as a population density, with $u = 1$ being the saturation level. Let us now assume that $u(t,x)$ also depends on a spatial variable, and the species can move in space in a diffusive way. Then the equation becomes

$$\frac{\partial u}{\partial t} = \kappa\Delta u + f(u), \quad u(0,x) = u_0(x). \tag{5.2}$$

Here, κ is the diffusion coefficient, and we have replaced $u(1-u)$ by a general nonlinearity $f(u)$. We will assume that $u \equiv 0$ and $u \equiv 1$ are steady states, meaning that $f(0) = f(1) = 0$. Let us first assume that x is one-dimensional. We are interested in initial conditions which look like fronts separating the two states. This means that $u_0(x)$ looks as follows:

$$u_0(x) \to 1 \text{ as } x \to -\infty,\ u_0(x) \to 0 \text{ as } x \to +\infty. \tag{5.3}$$

The question is whether the solution of (5.2) will converge in the limit to a steady state separating $u \approx 1$ and $u \approx 0$, or one of these states would invade the other, so that as $t \to +\infty$ we would have

$$u(t,x) \to 1 \text{ as } t \to +\infty, \text{ for any } x \in \mathbb{R} \text{ fixed}, \tag{5.4}$$

or

$$u(t,x) \to 0 \text{ as } t \to +\infty, \text{ for any } x \in \mathbb{R} \text{ fixed}. \tag{5.5}$$

In the former case we say that 1 invades 0, and that 0 invades 1 in the latter. There is a special class of solutions which invade at a constant speed c, they have the form

$$u(t,x) = U(x-ct), \tag{5.6}$$

and satisfy the boundary condition

$$U(x) \to 1 \text{ as } x \to -\infty, \text{ and } U(x) \to 0 \text{ as } x \to +\infty. \tag{5.7}$$

Thus, 1 invades 0 if $c > 0$, and 0 invades 1 if $c < 0$. The traveling wave satisfies the ODE

$$-cU' = U'' + f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0. \tag{5.8}$$

First, integrating the equation we get

$$c = \int_{\infty}^{\infty} f(U(x))dx. \tag{5.9}$$

Next, multiplying the equation by U' and integrating, gives

$$c\int_{-\infty}^{\infty} (U'(x))^2 dx = -\int_{-\infty}^{\infty} f(U(x))U'(x)dx = \int_0^1 f(s)ds. \tag{5.10}$$

That is, the sign of the speed c is determined by the sign of the integral in the right side above. In other words, if we set

$$F(s) = \int_0^s f(z)dz, \tag{5.11}$$

then the state 1 invades 0 if $F(1) > 0$, and 0 invades 1 if $F(1) < 0$.

There are three main classes of nonlinearities we will consider. All of them will satisfy $f(0) = f(1) = 0$. We say that $f(u)$ is of the Fisher-KPP class if it looks like $u(1-u)$. This means that

$$f(0) = f(1) = 0,\ f(u) > 0 \text{ and } f(u) \le f'(0)u \text{ for all } u \in (0,1). \tag{5.12}$$

The nonlinearity $f(u)$ is of the ignition class if there exists $\theta_0 \in (0,1)$ so that

$$f(u) = 0 \text{ for all } u \in [0,\theta_0],\ f(u) > 0 \text{ for } u \in (\theta_0, 1). \tag{5.13}$$

Finally, $f(u)$ is a bistable nonlinearity if there exists $\theta_0 \in (0,1)0$ so that

$$f(u) < 0 \text{ for all } u \in (0,\theta_0),\ f(u) > 0 \text{ for } u \in (\theta_0, 1). \tag{5.14}$$

Our goal will be to show the following: (1) in the ignition and bistable cases there exists a unique speed $c_* > 0$ for which a traveling wave exists, and the wave is unique up to a translation; (2) in the KPP case there exists $c_* > 0$ so that for any $c \geq c_*$ a traveling wave $U_c(x)$ exists, and for each $c \geq c_*$ fixed, the wave is unique up to a translation.

Explicit examples

The ignition case

Before we consider the general nonlinearities, we consider explicit examples, where traveling waves can be computed more or less explicitly. We begin with the ignition case, taking a very simple discontinuous $f(u)$: $f(u) = 0$ for $0 \leq u \leq \theta$ and $f(u) = \mu(1-u)$ for $\theta < u \leq 1$. Consider the traveling wave equation, setting $\kappa = 1$ for simplicity:

$$-cU' = U'' + f(U). \tag{5.15}$$

We know from (5.10) that $c > 0$. Let us fix the translation requiring that $U(0) = \theta$. Then for $x > 0$ the function $U(x)$ has the form

$$U(x) = \theta e^{-cx}, \quad x > 0, \tag{5.16}$$

while for $x < 0$ we have

$$-cU' = U'' + \mu(1-U). \tag{5.17}$$

We may write $U = 1 - w$, with $w(x)$ solving

$$cw' = w'' + \mu w, \quad x < 0. \tag{5.18}$$

That is,

$$w(x) = Ae^{\lambda x} + Be^{\lambda' x},$$

with

$$\lambda(c) = \frac{1}{2}[-c + \sqrt{c^2 + 4\mu}] > 0, \quad \lambda' = \frac{1}{2}[-c - \sqrt{c^2+4\mu}] < 0.$$

As the function $w(x)$ is bounded, and $x < 0$ we must have $B = 0$, and, as $w(0) = 1 - \theta$, we have

$$U(x) = 1 - (1-\theta)e^{\lambda x}, \quad x \leq 0.$$

Matching $U'(0^+)$ and $U'(0^-)$ gives

$$c\theta = (1-\theta)\lambda(c). \tag{5.19}$$

Note that

$$\frac{d\lambda(c)}{dc} = \frac{1}{2}[-1 + \frac{c}{\sqrt{c^2+4\mu}}] < 0,$$

hence (5.19) has a unique solution $c_* > 0$, and the traveling speed is unique.

The bistable case

Next, we consider an explicit bistable example. Let us define a discontinuos $f(u)$ by $f(u) = -\nu u$ for $0 \le u \le \theta$, and $f(u) = \mu(1-u)$ for $\theta < u \le 1$. Once again, we fix the translation by the normalization $U(0) = \theta$. For $x > 0$ we have

$$-cU' = U'' - \nu U, \quad U(0) = \theta, \quad U(+\infty) = 0. \tag{5.20}$$

It follows that

$$U(x) = \theta e^{-\lambda_r x}, \quad \lambda_r = \frac{1}{2}[c + \sqrt{c^2 + 4\nu}], \quad x > 0.$$

For $x < 0$ we have

$$-cU' = U'' + (1-\mu)U, \quad U(0) = \theta, \quad U(+\infty) = 0. \tag{5.21}$$

Setting again $U = 1 - w$ gives

$$w(x) = Ae^{\lambda_l x}, \quad \lambda_l = \frac{1}{2}[-c + \sqrt{c^2 + 4\mu}], \quad x < 0.$$

Matching at $x = 0$ implies that $A = 1 - \theta$, and

$$\theta\lambda_r(c) = (1-\theta)\lambda_l(c). \tag{5.22}$$

Next, note that $\lambda_r'(c) > 0$ and $\lambda_l'(c) < 0$. Moreover, we have, as $c \to -\infty$:

$$\lambda_r(c) = \frac{1}{2}\Big[c - c\sqrt{1 + \frac{4\nu}{c^2}}\Big] \sim \frac{1}{2}(c - c(1 + \frac{2\nu}{c^2})) \sim \frac{\nu}{c}, \quad c \to -\infty,$$

and

$$\lambda_l = \frac{1}{2}\Big[-c + \sqrt{c^2 + 4\mu}\Big] \sim |c| \text{ as } c \to -\infty,$$

and, as $c \to +\infty$:

$$\lambda_r(c) = \frac{1}{2}\Big[c + c\sqrt{1 + \frac{4\nu}{c^2}}\Big] \sim c \text{ as } c \to +\infty,$$

and

$$\lambda_l = \frac{1}{2}\Big[-c + \sqrt{c^2 + 4\mu}\Big] \sim \frac{\mu}{c} \text{ as } c \to -\infty.$$

It follows that for any $\theta \in (0,1)$ there is exactly one solution to (5.22), and the traveling wave speed c_* is unique.

The Fisher-KPP case

We now consider an explicit Fisher-KPP example. We take $\theta \in (0,1)$ and set $f(u) = \mu(1-\theta)u$ for $0 \le u \le \theta$m, and $f(u) = \mu\theta(1-u)$ for $\theta \le u \le 1$. Once again, we fix $u(0) = \theta$. Now, we have

$$-cU' = U'' + \mu\theta(1-u), \quad x < 0,$$

hence

$$U(x) = 1 - (1-\theta)e^{\lambda_l(c)x}, \quad \lambda_l(c) = \frac{1}{2}\Big[-c + \sqrt{c^2 + 4\mu\theta}\Big].$$

For $x > 0$ we have

$$-cU' = U'' + \mu(1-\theta)U, \quad x > 0. \tag{5.23}$$

The difference here, relative to the ignition and bistable examples is that we now have two decaying exponentials as solutions of (5.23):

$$U(x) = \theta e^{-\gamma_2(c)x} + a(e^{-\gamma_1(c)x} - e^{-\gamma_2(c)x}), \tag{5.24}$$

with

$$\gamma_1(c) = \frac{1}{2}[c - \sqrt{c^2 - 4\mu(1-\theta)}], \quad \gamma_2(c) = \frac{1}{2}[c + \sqrt{c^2 - 4\mu(1-\theta)}].$$

As we require positive solutions, we must have $c \geq c_*$, with

$$c_* = \sqrt{4\mu(1-\theta)}. \tag{5.25}$$

Note that the case $c = c_*$ has to be treated separately since then $\gamma_1(c_*)\gamma_2(c_*)$. Let us match the derivatives at $x = 0$:

$$-\lambda_l(c)(1-\theta) = -\gamma_2(c)\theta + a(\gamma_2(c) - \gamma_1(c)), \tag{5.26}$$

or

$$-\frac{(1-\theta)}{2}\Big[-c + \sqrt{c^2 + 4\mu\theta}\Big] = -\frac{\theta}{2}[c + \sqrt{c^2 - 4\mu(1-\theta)}] + a\sqrt{c^2 - 4\mu(1-\theta)},$$

or

$$2a\sqrt{c^2 - 4\mu(1-\theta)} = c - (1-\theta)\sqrt{c^2 + 4\mu\theta} + \theta\sqrt{c^2 - 4\mu(1-\theta)}. \tag{5.27}$$

For the positivity of $U(x)$ we need $a \geq 0$, hence we need the right side of (5.27) to be non-negative. This is true if

$$\begin{aligned} &c^2 + \theta^2(c^2 - 4\mu(1-\theta)) + 2c\theta\sqrt{c^2 - 4\mu(1-\theta)} \geq (1 - 2\theta + \theta^2)(c^2 + 4\mu\theta) \\ &= c^2 + 4\mu\theta - 2\theta c^2 - 8\theta^2\mu + \theta^2 c^2 + 4\theta^3\mu \end{aligned}$$

or

$$\begin{aligned} &2c\theta\sqrt{c^2 - 4\mu(1-\theta)} \geq c^2 + 4\mu\theta - 2\theta c^2 - 8\theta^2\mu + \theta^2 c^2 + 4\theta^3\mu - c^2 - c^2\theta^2 + 4\mu\theta^2 + 4\mu\theta^3 \\ &= 4\mu\theta - 2\theta c^2 - 4\theta^2\mu + 8\mu\theta^3. \end{aligned}$$

But if $c > c_*$ then the right side satisfies

$$4\mu\theta - 2\theta c^2 - 4\theta^2\mu + 8\mu\theta^3 \leq 4\mu\theta - 8\theta\mu(1-\theta) - 4\theta^2\mu + 8\mu\theta^3 = -4\mu\theta + 4\mu\theta^2 + 8\mu\theta^3$$

Solution should exist if $c > c_*$. The case $c = c_*$ is an exercise.

Thus, in the KPP case, the speed of a traveling wave is not unique, and actually waves exist for all $c \geq c_*$.

Existence of one dimensional travelling waves

Following the historical order of development we begin with showing that travelling wave solutions of

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{4} f(T). \tag{5.28}$$

exist when the nonlinearity $f(T)$ is of the KPP type. The factor 1/4 is introduced for convenience. We will assume that f is normalized so that $f'(0) = 1$, then the KPP condition $f'(T) \leq f'(0)T$ implies that $f(T) \leq T$.

The function $U(x)$ satisfies an ODE

$$-cU' = U'' + \frac{1}{4} f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0. \tag{5.29}$$

Introduce $V = -U'$ so that (5.29) becomes

$$\begin{aligned} \frac{dU}{dx} &= -V \\ \frac{dV}{dx} &= -cV + \frac{1}{4} f(U). \end{aligned} \tag{5.30}$$

This system has two equilibria: $(U, V) = (0, 0)$ and $(U, V) = (1, 0)$. A travelling wave corresponds to a heteroclinic orbit of (5.30) that connects the second equilibrium $(1, 0)$ at $x \to -\infty$, to the first, $(0, 0)$, at $x \to +\infty$. Linearization around $(0, 0)$ gives

$$\frac{d}{dx}\begin{pmatrix} U \\ V \end{pmatrix} = A_0 \begin{pmatrix} U \\ V \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4} f'(0) & -c \end{pmatrix}.$$

The eigenvalues of A_0 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4} f'(0) = 0$$

and are both real and negative if $c^2 \geq f'(0) = 1$. Therefore for a positive travelling wave $U(x - ct)$ to exist we need $c \geq 1$ so that $(0, 0)$ is a stable point. The linearization around $(1, 0)$ gives

$$\frac{d}{dx}\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} = A_1 \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4} f'(1) & -c \end{pmatrix}.$$

The eigenvalues of A_1 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4} f'(1) = 0$$

so that they have a different sign: $\lambda_1 > 0$, $\lambda_2 < 0$, and $(1, 0)$ is a saddle. Note that the unstable direction $(1, -\lambda_1)$ corresponding to $\lambda_1 > 0$ lies in the second and fourth quadrants.

Let us look at the the triangle D formed by the lines $l_1 = \{V = \gamma U\}$, $l_2 = \{V = \alpha(1-U)\}$ and the interval $l_3 = \{[0, 1]\}$ on the U-axis. We check that with an appropriate choice of γ and α all trajectories of (5.30) point into D on the boundary ∂D if $c \geq 1$. That means that the unstable manifold of $(1, 0)$ has to end at $(0, 0)$ since it may not cross the boundary of the triangle. That is, U and V stay positive along a heteroclinic orbit that starts at $(1, 0)$ and ends at $(0, 0)$ – this is a monotonic positive travelling wave we want to exist. In particular that will show that travelling waves exist for all $c \geq c_* = 1$.

Along l_3 we have $V = 0$ and

$$\frac{dU}{dx} = 0, \quad \frac{dV}{dx} = \frac{1}{4}f(U) > 0$$

so that trajectories point upward, that is, into D. Along l_1 we have $\frac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\gamma U}.$$

That means that the trajectory points into D if the slope $\frac{dV}{dU} \geq \gamma$ along l_1. This is true if

$$c - \frac{f(U)}{4\gamma U} \geq \gamma$$

for all $U \in [0, 1]$. This is equivalent to

$$c\gamma - \gamma^2 \geq \frac{f(U)}{4U}. \tag{5.31}$$

We have $\frac{f(U)}{U} \leq 1$ and hence (5.31) holds provided that

$$c\gamma - \gamma^2 \geq \frac{1}{4}. \tag{5.32}$$

Such $\gamma > 0$ exists if $c \geq 1$. Let us check that with this choice of c and γ all trajectories point into D also along the segment l_2. Indeed we have along l_2: $\frac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\alpha(1-U)}.$$

That means that the trajectory points into D if the slope $\frac{dV}{dU} \geq -\alpha$ along l_1. This is true if

$$c - \frac{f(U)}{4\alpha(1-U)} \geq -\alpha$$

for all $U \in [0, 1]$, or

$$c\alpha + \alpha^2 \geq \frac{f(U)}{4(1-U)}.$$

This is true for instance if $\alpha \geq \inf_{0 \leq U \leq 1} \frac{f(U)}{4(1-U)}$.

Therefore a travelling front exists provided that $c \geq 1$. However, we have also shown that no travelling front exists for $c < 1$. Thus we have proved the following theorem.

Theorem 5.1 *A travelling front solution of (5.29) with a KPP-type nonlinearity exists for all $c \geq 1$.*

Remark 5.2 We note that the travelling waves that propagate with the speeds $c > c_* = 1$ are in a sense superfluous. More precisely, their existence is unrelated to diffusion: let $U_0(x)$ be solution of

$$\frac{dU_0}{dx} = -\frac{1}{4}f(U_0), \quad U_0(0) = 1/2. \tag{5.33}$$

Such solution exists and satisfies the boundary conditions

$$U_0(x) \to 1 \text{ as } x \to -\infty \text{ and } U_0(x) \to 0 \text{ as } x \to +\infty. \tag{5.34}$$

Then given any $c > 0$ the function $T(t,x) = U_0\left(\frac{x}{c} - t\right)$ is a traveling wave solution of

$$\frac{\partial T}{\partial t} = 0 \cdot \frac{\partial^2 T}{\partial x^2} + \frac{1}{4}f(T).$$

Thus these travelling waves exist even at zero diffusion coefficient and are therefore not quite "reaction-diffusion" waves.

The shortcoming of having non-physical waves is absent in the case of ignition nonlinearity.

Theorem 5.3 *Let $f(T)$ be of ignition type. Then there exists a unique $c = c_*$ so that a travelling wave solution of (5.28) of the form $U(x-ct)$ exists.*

The proof is based on the ODE methods similar to those in the KPP case, and is left as an exercise for the reader. We note, however, that in the ignition case spurious waves at zero diffusivity do not exist: solutions of (5.33) do not satisfy the boundary conditions (5.34). This explains qualitatively uniqueness of the travelling front speed. We also remark that if we fix the travelling wave so that $U(0) = \theta_0$ then the travelling wave is given by $U(x) = \theta_0 \exp(-cx)$ for $x \geq 0$. Hence we have to find c so that in the variables $(U, V = -U')$ the stable manifold of the point $(1,0)$ in (5.30) would pass through the point $(\theta_0, c\theta_0)$. Not surprisingly such c is unique.

The front-like initial data

We look now at the behavior of solutions of (5.28) with a general initial data $T_0(x) = T(0,x)$ such that $T_0(x) = 1$ for $x \leq x_0$, $T_0(x) = 0$ for $x \geq x_1$ and $0 \leq T_0(x) \leq 1$. The main result is that such initial data propagates with the speed $c_* = 1$ of the slowest travelling front in the KPP case and with the speed of the unique travelling wave in the ignition case. More precisely, we have the following.

Theorem 5.4 *Let $T(t,x)$ be solution of (5.28) with the initial data $T_0(x)$ as above. Then given any $x \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} T(t, x+ct) = \begin{cases} 0, & \text{if } c > c_*, \\ 1, & \text{if } c < c_*. \end{cases} \tag{5.35}$$

Here $c_* = 1$ is the minimal speed in the KPP case and the unique travelling front speed in the ignition case.

This means qualitatively that T moves with the speed c_*. More precise statements on the convergence to a travelling front in the ignition case may be obtained but we will not go into details.

We will prove (5.35) only in the ignition case. The idea is to use the travelling wave solution $U(x-ct)$ to construct a super-solution and a sub-solution. Let $U(x)$ be the travelling wave, solution of

$$-c_*U' = U'' + f(U), \quad U(0) = \theta_0.$$

We dropped the factor 1/4 in front of $f(U)$ as it is useful only in the KPP case. We look for a sub-solution for T of the form

$$\psi_l(t,x) = U(x - c_*t + x_1 + \xi_1(t)) - q_1(t,x).$$

The functions $\xi_1(t)$ and $q_1(t,x,z)$ are to be chosen so as to make ψ_l be a sub-solution. The shift x_1 will be then used to make sure that initially we have $\psi_l(0,x) \le T_0(x)$. In order for ψ_l to be a sub-solution we need

$$G[\psi_l] = \frac{\partial \psi_l}{\partial t} - \frac{\partial^2 \psi_l}{\partial x^2} - f(\psi_l) \le 0.$$

We have

$$G[\psi_l] = \dot{\xi}_1 U' - \frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + f(U) - f(U - q_1).$$

With an appropriate choice of x_1, that is, by shifting U sufficiently to the left we may ensure that $T_0(x) \ge U(x) - q_{10}(x)$ with $0 \le q_{10}(x) \le (1-\theta_0)/2$ and $q_{10}(x) \in L^1(\mathbb{R})$. Then we choose $q_1(t,x)$ to be the solution of

$$\frac{\partial q_1}{\partial t} = \frac{\partial^2 q_1}{\partial x^2}, \quad q_1(0,x) = q_{10}(x) \tag{5.36}$$

so that we have

$$\|q_1(t)\|_\infty \le \frac{C}{\sqrt{t}} \|q_{10}\|_{L^1(D)} \tag{5.37}$$

for $t \ge 1$.

We may find $\delta > 0$ so that if $U \in (1-\delta, 1)$ and $q_1 \in (0, (1-\theta_0)/2)$ then $f(U) \le f(U - q_1)$. Hence we have in this range of U:

$$G[\psi_l] \le \dot{\xi}_1 U' \le 0 \tag{5.38}$$

provided that $\dot{\xi}_1 \ge 0$. Furthermore, if δ is sufficiently small and $U \in (0,\delta)$ then $f(U) = f(U-\delta) = 0$ and hence in this range of U we have (5.38) with the equality sign on the left. Finally, if $U \in (\delta, 1-\delta)$ then $|f(U) - f(U-q_1)| \le K|q|$ and $U' \le -\beta$ with positive constants K and β that depend on $\delta > 0$. Hence $G[\psi_l] \le 0$ everywhere provided that

$$\dot{\xi}_1(t) \ge \frac{K\|q_1(t)\|_\infty}{\beta}. \tag{5.39}$$

Thus we may choose

$$\xi_1(t) = C\sqrt{t}. \tag{5.40}$$

Therefore we obtain a lower bound for T:

$$T(t,x) \geq U(x - c_* t + C\sqrt{t}) - q_1(t,x). \tag{5.41}$$

In order to obtain an upper bound we set $\psi_u = U(x - c_* t - x_2 - \xi_2(t)) + q_2(t,x)$ and look for $\xi_2(t)$ and $q_2(t,x)$ so that $G[\psi_u] \geq 0$. The constant x_2 is chosen so that

$$T_0(x) \leq U(x - x_2) + q_2(0,x)$$

with $q_2(0,x) \in L^1(\mathbb{R})$ and $0 \leq q_2(0,x) \leq \theta_0/2$, as with $q_1(0,x)$. The function $q_2(t,x)$ is then chosen to satisfy the same heat equation (5.36) as q_1. Hence it obeys the same time decay bounds as q_1. With the above choice of q_2 we have

$$G(\psi_u) = -\dot{\xi}_2 U' + f(U) - f(U + q_2).$$

Once again, we consider three regions of values for U. First, if $1-\delta \leq U \leq 1$ with a sufficiently small $\delta > 0$ then $f(U) - f(U + q_2) \geq 0$, as $q_2 \geq 0$. Hence $G[\psi_u] \geq 0$ in this region provided that $\dot{\xi}_2 \geq 0$. Second, as $q_2 \leq \theta_0/2$ we have $f(U) = f(U + q_2) = 0$ if $0 \leq U \leq \delta$ with a sufficiently small $\delta > 0$. Hence $G[\psi_u] \geq 0$ in that region under the same condition $\dot{\xi}_2 \geq 0$. Finally, if $U \in (\delta, 1-\delta)$ then $U' \leq -\beta$ with $\beta > 0$ and $|f(U) - f(U + q_2)| \leq K\|q_2\|_\infty$. That means that $G[\psi_u] \geq 0$ if we choose ξ_2 so that

$$\dot{\xi}_2 \geq \frac{K\|q_2\|_\infty}{\beta}.$$

Therefore we may choose

$$\xi_2(t) = C\sqrt{t},$$

as with $\xi_1(t)$. Thus we obtain upper and lower bounds

$$U(x - c_* t + \xi_1(t) + x_1) - q_1(t,x) \leq T(t,x) \leq U(x - c_* t - \xi_2(t) - x_2) + q_2(t,x) \tag{5.42}$$

that imply in particular that

$$U(x - c_* t + C_0[1 + \sqrt{t}]) - \frac{C_0}{\sqrt{t}} \leq T(t,x) \leq U(x - c_* t - C_0[1 + \sqrt{t}]) + \frac{C_0}{\sqrt{t}} \tag{5.43}$$

with a constant C_0 determined by the initial conditions. Now, if we take $x = x_0 + ct$ with $c < c_*$ and use the lower bound in (5.43) we get $T(t, x_0 + ct) \to 1$ as $t \to \infty$. On the other hand, if we take $x = x_0 + ct$ with $c > c_*$ and use the upper bound we obtain $T(t, x_0 + ct) \to 0$ as $t \to \infty$.

One may obtain the upper bound in the KPP result simply by replacing $F(T)$ by T. The lower bound is obtained from the ignition case by cutting of a KPP-type $f(T)$ at a point $\theta_0 > 0$ and then letting $\theta_0 \to 0$. We omit the details. □

We confess that a better effort using the spectral methods and functional analysis shows that in the ignition case solution actually converges to a travelling wave exponentially fast in time. However, this requires a different technique that we do not go into here.

Compactly supported initial data

When the initial data T_0 has compact support, two scenarios are possible in the ignition case. First, if the support is too small, solution may become extinct:

$$T(t,x) \to 0 \text{ as } t \to \infty, \text{ uniformly on } \mathbb{R}. \tag{5.44}$$

On the other hand, if the support is sufficiently large, two fronts will form: one on the right, another on the left that will propagate to the left and to the right so that eventually

$$T(t,x) \to 1 \text{ as } t \to \infty, \text{ uniformly on compact sets.} \tag{5.45}$$

Extinction is impossible in the KPP case: solution with an arbitrarily small initial support satisfies (5.45). The difference between the KPP and ignition case is best seen from the linearization at small T: in the ignition case we get the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

that has solutions decaying in time. In the KPP case the linearization is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f'(0)T$$

that has solutions growing in time.

We consider solutions of

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f(T) \tag{5.46}$$

with an initial data of the form

$$T_0(x) = \begin{cases} 1, & |x| \le x_0, \\ 0, & |x| \ge x_1, \end{cases} \tag{5.47}$$

and with $0 \le T_0 \le 1$. The first result concerns extinction.

Theorem 5.5 *Let f be of the ignition type. There exists $l > 0$ so that if $x_1 \le l$ then solution becomes extinct, that is, (5.44) holds.*

Proof. The proof is simple: the comparison principle implies that $T(t,x) \le \Psi(t,x)$, solution of

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + M\Psi, \quad \Psi(0,x) = T_0(x)$$

with the constant M chosen so that $f(T) \le MT$. However, Ψ is given explicitly by

$$\Psi(t,x) = \frac{e^t}{\sqrt{2\pi t}} \int e^{-(x-y)^2/4t} T_0(y)dy \le \frac{e^t}{\sqrt{2\pi t}} \|T_0\|_{L^1} \le \frac{e^t}{\sqrt{2\pi t}} |x_1|.$$

Hence $T(t=1,x) \le \Psi(t=1,x) \le \theta_0$ provided that $|x_1| \le C\theta_0$. However, if $T(t=1,x) \le \theta_0$ then uniquness of the solution of the Cauchy problem for (5.46) implies that $T(t,x)$ satisfies the heat equation for $t > 1$:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}.$$

Then the limit behavior (5.44) follows from the fact that $\|T(t=1)\|_{L^1} < \infty$. $\square$

On the other hand, if $T_0 = 1$ on a sufficiently large set, then the flame propagates: (5.45) holds.

Theorem 5.6 *If f is of the ignition type then there exists $L > 0$ so that if $x_0 \geq L$ then (5.45) holds. If f is of the KPP type then (5.45) holds provided that $x_1 > 0$.*

Proof. We will consider the ignition case but the KPP case is similar. The proof is in two steps. First we will find $d > 0$ and a sub-solution $\phi(x)$ on the interval $[-d, d]$ so that

$$\phi''(x) + f(\phi) \geq 0, \quad \phi(-d) = \phi(d) = 0 \tag{5.48}$$

and $0 \leq \phi \leq 1$. We will then consider a special solution $\Phi(t, x)$ of (5.46) with the initial data $\Phi(t, x) = \phi(x)$. It will turn out to be monotonically increasing in time and thus converge to a limit as $t \to +\infty$. It will only remain to identify the limit as $\Phi_\infty \equiv 1$.

Choose θ_1, θ_2 so that $\theta_2 > \theta_1 > \theta_0$ and define $f_1(T)$ by

$$f_1(T) = \begin{cases} 0, & T \leq \theta_1 \\ \dfrac{f(\theta_2)(T - \theta_1)}{\theta_2 - \theta_1}, & \theta_1 \leq T \leq \theta_2 \\ f(T), & \theta_2 \leq T \leq 1 \end{cases}$$

The function $f(T)$ is Lipschitz continuous, and hence we may choose θ_1 and θ_2 so that $f_1(T) \leq f(T)$. Therefore if ϕ satisfies

$$\phi'' + f_1(\phi) = 0 \tag{5.49}$$

then ϕ satisfies (5.48). We are going to exhibit an explicit solution $\phi(x)$ of (5.49) with the "initial" conditions

$$\phi(0) = \theta_2, \quad \frac{d\phi}{dx}(0) = 0.$$

Indeed, $\phi(x)$ is given explicitly by

$$\phi(r) = \theta_1 + (\theta_2 - \theta_1)\cos\left(x\sqrt{\alpha}\right), \quad \alpha = \frac{f(\theta_2)}{\theta_2 - \theta_1} \text{ for } x \leq R_1 = \frac{\pi}{2\sqrt{\alpha}}. \tag{5.50}$$

Furthermore, we have

$$\phi(x) = B(d - |x|), \text{ for } R_1 \leq |x| \tag{5.51}$$

with B and d determined by matching (5.50) and (5.51) at $x = R_1$: $B = (\theta_2 - \theta_1)\sqrt{\alpha}$ and $d = R_1 + \dfrac{\theta_1}{B}$. The function $\phi(x)$ satisfies (5.48). We choose x_0 sufficiently large so that $T_0(x) \geq \phi(x)$. We now let $\Phi(t, x)$ be solution of (5.46) with the initial data $\Phi(0, x) = \phi(x)$, as we discussed above. Then

$$T(t, x) \geq \Phi(t, x) \geq \phi(x), \tag{5.52}$$

as the first inequality follows from the comparison principle, while the second from the fact that $\phi(x)$ is a sub-solution. This implies that $v(t, x) = \frac{\partial \Phi}{\partial t} > 0$ for $t > 0$. Indeed, given any $h > 0$, let $v_h(t, x) = \Phi(t + h, x) - \Phi(t, x)$. Then $v_h(0, x) > 0$ is positive as implied by the second inequality in (5.52). The function v_h satisfies an equation of the form

$$\frac{\partial v_h}{\partial t} = \frac{\partial^2 v_h}{\partial x^2} + \frac{f(\Phi(t + h, x)) - f(\Phi(t, x))}{\Phi(t + h, x) - \Phi(t, x)} v_h.$$

Hence the maximum principle implies that $v_h(t,x) > 0$ for all $t > 0$, and all $h > 0$. Thus the function $\Phi(t,x)$ is monotonically increasing in time and has a limit $\bar\Phi(x)$ as $t \to +\infty$. The limit function has to be a non-negative solution of

$$\bar\Phi_{xx} + f(\bar\Phi) = 0, \quad x \in \mathbb{R}, \tag{5.53}$$

such that $\bar\Phi(x) \ge \phi(x)$, for $x \in [-d,d]$. We may now define the shifts

$$\phi_h(x) = \phi(x+h), \quad x \in (-d-h, d-h),$$

which all satisfy

$$0 = \phi_h'' + f_1(\phi_h) \le \phi_h'' + f(\phi_h), \quad -d-h < \phi_h(x) < d-h.$$

We know that for h small we have $\bar\Phi(x) > \phi_h(x)$ for all $x \in (-d-h, d-h)$. If we let h_0 be the supremum of all h such that this inequality holds, we deduce that $\bar\Phi(x) \ge \phi_{\bar h}(x)$ for all $x \in (-d-h, d-h)$ and there exists x_0 such that $\bar\Phi(x_0) = \phi_{\bar h}(x_0)$. This contradicts the maximum principle, hence $\bar\Phi(x) > \phi_h(x)$ for all h. We conclude that $\bar\Phi(x) \ge \theta_2$ everywhere. This and (5.53) implies that $\bar\Phi$ may not attain a local minimum in $\mathbb{R}$, hence either it is equal to a constant T_0, or it is monotonic on two sides of the point x_0 where it attains its maximum, and thus has limits as $x \to \pm\infty$. In the former case the constant $T_0 \ge \theta_2$ has to be such that $f(T_0) = 0$ and hence $\bar\Phi = 1$. In the latter case there exist two sequences $x_n \to -\infty$ and $y_n \to +\infty$ so that $\bar\Phi_x(x_n) \to 0$ and $\bar\Phi_x(y_n) \to 0$ as $n \to +\infty$. Integrating (5.53) between x_n and y_n we get

$$\bar\Phi_x(y_n) - \bar\Phi_x(x_n) + \int_{x_n}^{y_n} f(\bar\Phi)dx = 0.$$

Passing to the limit $n \to +\infty$ we obtain $f(\bar\Phi) \equiv 0$, hence $\bar\Phi \equiv 1$. □

It is not difficult to construct a sub-solution that propagates to the left and right with the speed c_*. That means that not only (5.45) holds for $x_1 > l$ but also the front expands at the speed of the travelling front.

6 Explosion, extinction and diffusion

Here, we investigate the interaction of a local nonlinearity and diffusion, a simpler setting than for the non-local Keller-Segel system. Solution of a simple ODE

$$\dot z(t) = z^m, \quad z(0) = z_0 > 0, \tag{6.1}$$

is explicit:

$$z(t) = \frac{z_0}{(1-(m-1)tz_0^{m-1})^{1/(m-1)}}.$$

It blows up in a finite time for all $m > 1$. On the other hand, solution of the same nonlinear ODE with a negative sign:

$$\dot z(t) = -z^m, \quad z(0) = z_0 > 0, \tag{6.2}$$

is also explicit:

$$z(t) = \frac{z_0}{(1+(m-1)tz_0^{m-1})^{1/(m-1)}},$$

and it becomes extinct in a finite time:

$$z(t) \to 0 \text{ as } t \to +\infty.$$

Here, we ask how the presence of diffusion can affect these phenomena. Intuitively, diffusion spreads the solution and tries to "bring it down". This is an obvious competition with local growth in (6.1), so one may expect that the addition of diffusion can prevent the blow-up. On the other hand, for (6.2) one may expect that diffusion will combine with the decay to speed up the convergence of the solution to zero – this is true but depends on the way how we measure the convergence to zero.

Blow-up in the semilinear heat equation: using the maximum principle

We will consider the following problem in a bounded domain:

$$\begin{aligned} & u_t = \Delta u + u^2, \quad t \geq 0, \quad x \in \Omega, \\ & u = 0 \text{ on } \partial\Omega, \\ & u(0,x) = u_0(x) \geq 0. \end{aligned} \tag{6.3}$$

In order to understand what to expect, let ϕ be the principal eigenfucntion of the Dirichlet Laplacian in Ω:

$$\begin{aligned} & -\Delta\phi = \lambda_1\phi, \quad \phi(x) > 0, \quad x \in \Omega, \\ & \phi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.4}$$

Let us multiply (6.3) and integrate by parts:

$$\frac{d}{dt}\int_\Omega u(t,x)\phi(x)dx = \int_\Omega [\Delta u(t,x) + u(t,x)^2]\phi(x)dx = \int_\Omega [-\lambda_1 u(t,x)\phi(x) + u^2(t,x)\phi(x)]dx. \tag{6.5}$$

The Cauchy-Schwartz inequality implies

$$\Big(\int_\Omega u(t,x)\phi(x)dx\Big)^2 dx \leq \int_\Omega u^2(t,x)\phi(x)dx \int \phi(x)dx. \tag{6.6}$$

Using this in (6.5) gives

$$\frac{d}{dt}\int_\Omega u(t,x)\phi(x)dx \geq -\lambda_1 \int_\Omega u(t,x)\phi(x)dx + \frac{1}{a}\Big(\int_\Omega u(t,x)\phi(x)dx\Big)^2 dx, \tag{6.7}$$

with

$$a = \int_\Omega \phi(x)dx.$$

Thus,

$$Z(t) = \int_\Omega u(t,x)\phi(x)dx$$

satisfies

$$\frac{dZ(t)}{dt} \geq -\lambda_1 Z(t) + \frac{1}{a}Z^2(t). \tag{6.8}$$

Let us define $Q(t) = e^{\lambda_1 t} Z(t)$, so that

$$\frac{dQ(t)}{dt} \geq e^{-\lambda_1 t} \frac{1}{a} Q^2(t). \tag{6.9}$$

Integrating in time gives

$$-\frac{1}{Q(t)} + \frac{1}{Q(0)} \geq \frac{1}{a\lambda_1}(1 - e^{-\lambda_1 t}), \tag{6.10}$$

or

$$\frac{1}{Q(t)} \leq \frac{1}{Q(0)} - \frac{1}{a\lambda_1}(1 - e^{-\lambda_1 t}). \tag{6.11}$$

Thus, if $Q(0)$ is sufficiently large, so that

$$\frac{1}{Q(0)} < \frac{1}{a\lambda_1}, \tag{6.12}$$

then there exists a time T^* so that $Q(T^*) < 0$ which is a contradiction. Hence, no global in time solution may exist if the initial mass is sufficiently large, more precisely, if

$$\int_\Omega u_0(x)\phi(x)dx \geq \lambda_1 \int_\Omega \phi(x)dx. \tag{6.13}$$

Note that this condition would not change if we multiply ϕ by a constant!

In order to see what happens when the initial condition is small, note that $\phi_\mu = \mu\phi(x)$ is a super-solution: it satisfies

$$-\Delta\phi_\mu = \lambda_1\phi_\mu \geq \phi_\mu^2, \tag{6.14}$$

provided that

$$\mu\phi(x) \leq \lambda_1 \quad \text{for all } x \in \Omega.$$

This is true, if we set

$$\mu = \min_{x\in\Omega} \frac{\lambda_1}{\phi(x)}.$$

The minimum is achieved since $\phi(x) = 0$ on $\partial\Omega$. let us consider the difference

$$w(t,x) = u(t,x) - \phi_\mu(x),$$

with some fixed $b > 0$. It satisfies

$$w_t - \Delta w = u_t - \Delta u + \Delta\phi_\mu \leq u^2 - \phi_\mu^2 = (u - \phi_\mu)(u + \phi_\mu) = w(u + \phi_\mu). \tag{6.15}$$

We conclude that if $w(0,x) \leq 0$ for all $x \in \Omega$, then $w(t,x) \leq 0$ for all $x \in \Omega$ and all $t > 0$. Thus, if the initial condition is small, in the sense that

$$u_0(x) \leq \mu\phi(x), \tag{6.16}$$

then $u(t,x)$ remains finite for all $t > 0$ – diffusion wins over growth.

Blow-up in the semilinear heat equation: using the energy method

Let us now obtain a different size condition for the blow-up, without using the maximum principle, or positivity of the solution. We start with

$$\begin{aligned} &u_t = \Delta u + |u|^{p-1}u, \quad t \geq 0, \quad x \in \Omega, \\ &u = 0 \text{ on } \partial\Omega, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.17}$$

with some $p > 1$. This system has an energy:

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx. \tag{6.18}$$

Note that

$$\begin{aligned} \frac{dE}{dt} &= \int_\Omega (\nabla u \cdot \nabla u_t) dx - \int_\Omega |u|^{p-1} u u_t dx = -\int_\Omega u_t[\Delta u + |u|^{p-1}u] dx \\ &= -\int_\Omega (\Delta u + |u|^{p-1}u)^2 dx \leq 0, \end{aligned} \tag{6.19}$$

thus

$$E(t) \leq E(0) := \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1} dx. \tag{6.20}$$

Let us now see what happens with the L^2-norm of the solution. Multiplying the equation by u and integrating gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |u(t,x)|^2 dx = \int u[\Delta u + |u|^{p-1}u] dx = -\int |\nabla u(t,x)|^2 dx + \int |u|^{p+1} dx \\ &= -2E(t) - \frac{2}{p+1}\int_\Omega |u|^{p+1} dx + \int |u|^{p+1} dx \geq -2E_0 + \alpha \int |u|^{p+1} dx, \end{aligned} \tag{6.21}$$

with

$$\alpha = 1 - \frac{2}{p+1} > 0.$$

Thus, if $E_0 \leq 0$, then we have

$$\frac{1}{2}\frac{d}{dt}\int |u(t,x)|^2 dx \geq \alpha \int |u|^{p+1} dx. \tag{6.22}$$

Hölder's inequality implies, for $p > 1$ that

$$\int_\Omega |u|^2 dx \leq \Big(\int_\Omega |u|^{p+1} dx\Big)^{2/(p+1)} \Big(\int_\Omega dx\Big)^{1-2/(p+1)},$$

hence

$$\int_\Omega |u|^{p+1} dx \geq |\Omega|^{(1-p)/2} \Big(\int_\Omega |u|^2 dx\Big)^{(p+1)/2}.$$

Thus, setting

$$Z(t) = \int |u(t,x)|^2 d,$$

we get from (6.22) when $E_0 \le 0$:

$$\frac{dZ}{dt} \ge 2\alpha|\Omega|^{(1-p)/2} Z^{(p+1)/2}(t). \tag{6.23}$$

Thus, any solution with $E_0 \le 0$ blows up in a finite time. Note that this is also a size condition: if we fix any $g(x) \not\equiv 0$, and consider the initial condition $u_0(x) = rg(x)$, then

$$E_0(r) = \frac{1}{2}\int |\nabla u_0|^2 dx - \frac{1}{p+1}\int |u_0|^{p+1} dx = \frac{r^2}{2}\int |\nabla g|^2 dx - \frac{r^{p+1}}{p+1}\int |g|^{p+1} dx < 0,$$

for all

$$r > r_0 := \Big(\frac{p+1}{2}\int |\nabla g|^2 dx\Big)^{1/(p-1)} \Big(\int |g|^{p+1} dx\Big)^{-1/(p-1)}.$$

Diffusion can prevent extinction

Let us now consider a "negative nonlinear heat equation":

$$u_t = \Delta u - u^p, \quad x \in \mathbb{R}^d, \tag{6.24}$$

with $p > 1$ and $u(0,x) = u_0(x) \ge 0$. This a very simple model for reproduction, and the total mass

$$M(t) = \int_{\mathbb{R}^d} u(t,x) dx, \tag{6.25}$$

is the total mass that has not reproduced. Thus, small $M(t)$ corresponds to effective reproduction. Obviously, in the absence of diffusion we have

$$M(t) \to 0 \text{ as } t \to +\infty. \tag{6.26}$$

With diffusion we still have

$$\frac{dM}{dt} = D(t) : -\int_{\mathbb{R}^d} u^p(t,x) dx < 0, \tag{6.27}$$

so the total mass decreases. We would like to understand if diffusion may prevent extinction: will (6.26) still be true in the presence of diffusion?

First, we need an estimate for the dissipation $D(t)$ in (6.27). Note that solutions of the heat equation

$$v_t - \Delta v = 0, \quad x \in \mathbb{R}^d, \tag{6.28}$$

satisfy two well-known estimates:

$$\|v(t_2)\|_{L^1} \le \|v(t_1)\|_{L^1},$$

and

$$\|v(t_2)\|_{L^\infty} \le \frac{C}{(t_2 - t_1)^{d/2}} \|v(t_1)\|_{L^1},$$

for all $t_2 > t_1 \geq 0$. It follows that

$$\int_{\mathbb{R}^d} |v(t_2,x)|^p dx \leq \|v(t_2)\|_{L^1} \|v(t_2)\|_{L^\infty}^{p-1} \leq \frac{C}{(t_2-t_1)^{d(p-1)/2}} \|v(t_1)\|_{L^1}^p. \tag{6.29}$$

If $v(t,x)$ is the solution of (6.28) for $t > \tau$, with $v(\tau,x) = u(\tau,x)$, then we have $u(t,x) \leq v(t,x)$, whence intgrating (6.27) in time gives, for all $t_2 > \tau + a$, and $a \geq 0$:

$$\begin{aligned} M(t_2) &= M(\tau+a) - \int_{\tau+a}^{t_2} \int_{\mathbb{R}^d} u(t,x)^p dx dt \geq M(\tau+a) - \int_{\tau+a}^{t_2} \int_{\mathbb{R}^d} v(t,x)^p dx dt \\ &\geq M(\tau+a) - CM(\tau)^p \int_{\tau+a}^{t_2} \frac{dt}{(t-\tau)^{1+\delta}}, \end{aligned} \tag{6.30}$$

with

$$\delta = \frac{d(p-1)}{2} - 1. \tag{6.31}$$

In order to have $\delta > 0$ we need to assume that

$$p > 1 + \frac{2}{d}. \tag{6.32}$$

Then we have

$$M(t_2) \geq M(\tau+a) - CM(\tau)^p \int_{\tau+a}^{\infty} \frac{dt}{(t-\tau)^{1+\delta}} = M(\tau+a) - \frac{CM(\tau)^p}{a^\delta}, \tag{6.33}$$

for all $t_2 > \tau + a$. On the other hand, we also have, as $M(s)$ is decreasing in time:

$$\begin{aligned} M(\tau+a) &= M(\tau) - \int_\tau^{\tau+a} u^p(t,x) dx \geq M(\tau) - \|u_0\|_{L^\infty}^{p-1} \int_\tau^{\tau+a} \int_{\mathbb{R}^d} u(t,x) dx \\ &\geq M(\tau) - \|u_0\|_{L^\infty}^{p-1} \int_\tau^{\tau+a} M(s) ds = M(\tau) - a\|u_0\|_{L^\infty}^{p-1} M(\tau). \end{aligned} \tag{6.34}$$

Let us choose a so that

$$a\|u_0\|_{L^\infty}^{p-1} = \frac{1}{2}.$$

Then we have from (6.34):

$$M(\tau+a) \geq \frac{M(\tau)}{2}.$$

Using this in (6.33) gives us, for all $t_2 > \tau + a$:

$$M(t_2) \geq \frac{M(\tau)}{2} - \frac{CM(\tau)^p}{a^\delta}. \tag{6.35}$$

Then, if we have

$$M(t) \to 0 \text{ as } t \to +\infty,$$

we have a time τ so that

$$\frac{CM(\tau)^{p-1}}{a^\delta} < \frac{1}{3},$$

and (6.35) would imply

$$M(t) \geq \frac{M(\tau)}{6}, \text{ for all } t \geq \tau + a,$$

which is obviously incompatible with (6.36). We conclude that

$$\lim_{t \to +\infty} M(t) > 0, \tag{6.36}$$

and diffusion does, in fact, prevent the extinction.