Appetizers in Nonlinear PDEs

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Chapter 1

Maximum principle and the symmetry of solutions of elliptic equations

1.1 Act I. The maximum principle enters

We will have several main characters in this chapter: the maximum principle and the sliding and moving plane methods. The maximum principle and sliding will be introduced separately, and then blended together to study the symmetry properties of the solutions of elliptic equations. In this introductory section, we recall what the maximum principle is. This material is very standard and can be found in almost any undergraduate or graduate PDE text, such as the books by Evans [60], Han and Lin [83], and Pinchover and Rubinstein [123].

We will consider equations of the form

$$\begin{aligned} &\Delta u + F(x, u) = 0 \text{ in } \Omega, \\ &u = g \text{ on } \partial\Omega. \end{aligned} \tag{1.1.1}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary. There are many applications where such problems appear. We will mention just two – one is in the realm of probability theory, where $u(x)$ is an equilibrium particle density for some stochastic process, and the other is in classical physics. In the physics context, one may think of $u(x)$ as the equilibrium temperature distribution inside the domain Ω. The temperature flux is proportional to the gradient of the temperature – this is the Fourier law, which leads to the term Δu in the overall heat balance (1.1.1). The term $F(x, u)$ corresponds to the heat sources or sinks inside Ω, while $g(x)$ is the (prescribed) temperature on the boundary $\partial\Omega$. The maximum principle reflects a basic observation known to any child – first, if $F(x, u) = 0$ (there are neither heat sources nor sinks), or if $F(x, u) \leq 0$ (there are no heat sources but there may be heat sinks), the temeprature inside Ω may not exceed that on the boundary – without a heat source inside a room, you can not heat the interior of a room to a warmer temperature than its maximum on the boundary. The second observation is that if one considers two prescribed boundary conditions and heat sources such that

$$g_1(x) \leq g_2(x) \text{ and } F_1(x, u) \leq F_2(x, u),$$

then the corresponding solutions will satisfy $u_1(x) \leq u_2(x)$ – stronger heating leads to warmer rooms. It is surprising how such mundane considerations may lead to beautiful mathematics.

The maximum principle in complex analysis

Most mathematicians first encounter the maximum principle in a complex analysis course. Recall that the real and imaginary parts of an analytic function $f(z)$ have the following property.

Proposition 1.1.1 *Let $f(z) = u(z) + iv(z)$ be an analytic function in a smooth bounded domain $\Omega \subset \mathbb{C}$, continuous up to the boundary Ω. Then $u(z) = Ref(z)$, $v(z) = Imf(z)$ and $w(z) = |f(z)|$ all attain their respective maxima over Ω on its boundary. In addition, if one of these functions attains its maximum inside Ω, it has to be equal identically to a constant in Ω.*

This proposition is usually proved via the mean-value property of analytic functions (which itself is a consequence of the Cauchy integral formula): for any disk $B(z_0, r)$ contained in Ω we have

$$f(z_0) = \int_0^{2\pi} f(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \;\; u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \;\; v(z_0) = \int_0^{2\pi} v(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \quad (1.1.2)$$

and, as a consequence,

$$w(z) \leq \int_0^{2\pi} w(z_0 + re^{i\theta})\frac{d\theta}{2\pi}. \quad (1.1.3)$$

It is immediate to see that (1.1.3) implies that if one of the functions u, v and w attains a local maximum at a point z_0 inside Ω, it has to be equal to a constant in a disk around z_0. Thus, the set where it attains its maximum is both open and closed, hence it is all of Ω and this function equals identically to a constant.

The above argument while incredibly beautiful and simple, relies very heavily on the rigidity of analytic functions that is reflected in the mean-value property. The same rigidity is reflected in the fact that the real and imaginary parts of an analytic function satisfy the Laplace equation

$$\Delta u = 0, \quad \Delta v = 0,$$

while $w^2 = u^2 + v^2$ is subharmonic: it satisfies

$$\Delta(w^2) \geq 0.$$

We will see next that the mean-value principle is associated to the Laplace equation and not analyticity in itself, and thus applies to harmonic (and, in a modified way, to subharmonic) functions in higher dimensions as well. This will imply the maximum principle for solutions of the Laplace equation in an arbitrary dimension. One may ask whether a version of the mean-value property also holds for the solutions of general elliptic equations rather than just for the Laplace equation – the answer is "yes if understood properly": the mean value property survives as the general elliptic regularity theory, an equally beautiful sister of the complex analysis which is occasionally misunderstood as "technical".

Interlude: a probabilistic connection digression

Apart from the aforementioned connection to physics and the Fourier law, a good way to understand how the Laplace equation comes about, as well as many of its properties, including the maximum principle, is via its connection to the Brownian motion. It is easy to understand in terms of the discrete equations, which requires only very elementary probability theory. Consider a system of many particles on the n-dimensional integer lattice $\mathbb{Z}^n$. They all perform a symmetric random walk: at each integer time $t = k$ each particle jumps (independently from the others) from its current site $x \in \mathbb{Z}^n$ to one of its $2n$ neighbors, $x \pm e_k$ (e_k is the unit vector in the direction of the x_k-axis), with equal probability $1/(2n)$. At each step we may also insert new particles, the average number of inserted (or eliminated) particles per unit time at each site is $F(x)$. Let now $u_m(x)$ be the average number of particles at the site x at time m. The balance equation for $u_{m+1}(x)$ is

$$u_{m+1}(x) = \frac{1}{2n}\sum_{k=1}^{n}[u_m(x+e_k) + u_m(x-e_k)] + F(x). \tag{1.1.4}$$

Exercise 1.1.2 Derive (1.1.4) by considering how particles may appear at the position x at the time $m+1$ – they either jump from a neighbor, or are inserted.

If the system is in an equilibrium, so that $u_{m+1}(x) = u_m(x)$ for all x, then $u(x)$ (dropping the subscript m) satisfies the discrete equation

$$\frac{1}{2n}\sum_{k=1}^{n}[u(x+e_k) + u(x-e_k) - 2u(x)] + F(x) = 0.$$

If we now take a small mesh size h, rather than have particles jump be of size one, the above equation becomes

$$\frac{1}{2n}\sum_{k=1}^{n}[u(x+he_k) + u(x-he_k) - 2u(x)] + F(x) = 0.$$

A Taylor expansion in h leads to

$$\frac{h^2}{2n}\sum_{k=1}^{n}\frac{\partial^2 u(x)}{\partial x_k^2} + F(x) = \text{lower order terms}.$$

Taking the source of the form $F(x) = h^2/(2n)G(x)$ – the small factor h^2 prevents us from inserting or removing too many particles, we arrive, in the limit $h \downarrow 0$, at

$$\Delta u + G(x) = 0. \tag{1.1.5}$$

In this model, we interpret $u(x)$ as the local particle density, and $G(x)$ as the rate at which the particles are inserted (if $G(x) > 0$), or removed (if $G(x) < 0$). When equation (1.1.5) is posed in a bounded domain Ω, we need to supplement it with a boundary condition, such as

$$u(x) = g(x) \text{ on } \partial\Omega.$$

This boundary condition means the particle density on the boundary is prescribed – the particles are injected or removed if there are "too many" or "too little" particles at the boundary, to keep $u(x)$ at the given prescribed value $g(x)$.

The mean value property for sub-harmonic and super-harmonic functions

We now return to the world of analysis. A function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$ is harmonic if it satisfies the Laplace equation

$$\Delta u = 0 \text{ in } \Omega. \tag{1.1.6}$$

This is equation (1.1.1) with $F \equiv 0$, thus a harmonic function describes a heat distribution in Ω with neither heat sources nor sinks in Ω. We say that u is sub-harmonic if it satisfies

$$-\Delta u \leq 0 \text{ in } \Omega, \tag{1.1.7}$$

and it is super-harmonic if it satisfies

$$-\Delta u \geq 0 \text{ in } \Omega, \tag{1.1.8}$$

In other words, a sub-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \leq 0$ – it describes a heat distribution in Ω with only heat sinks present, and no heat sources, while a super-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \geq 0$ – it describes an equilibrium heat distribution in Ω with only heat sources present, and no sinks.

Exercise 1.1.3 Give an interpretation of the sub-harmonic and super-harmonic functions in terms of particle probability densities.

Note that any sub-harmonic function in one dimension is convex:

$$-u'' \leq 0,$$

and then, of course, for any $x \in \mathbb{R}$ and any $l > 0$ we have

$$u(x) \leq \frac{1}{2}\left(u(x+l) + u(x-l)\right), \text{ and } u(x) \leq \frac{1}{2l}\int_{x-l}^{x+l} u(y)dy.$$

The following generalization to sub-harmonic functions in higher dimensions shows that locally $u(x)$ is bounded from above by its spatial average. A super-harmonic function will be locally above its spatial average. A word on notation: for a set S we denote by $|S|$ its volume, and, as before, ∂S denotes its boundary.

Theorem 1.1.4 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $B(x,r)$ be a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ contained in Ω. Assume that the function $u(x)$ is sub-harmonic, that is, it satisfies*

$$-\Delta u \leq 0, \tag{1.1.9}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \le \frac{1}{|B(x,r)|}\int_{B(x,r)} udy, \quad u(x) \le \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} udS. \tag{1.1.10}$$

Next, suppose that the function $u(x)$ is super-harmonic:

$$-\Delta u \ge 0, \tag{1.1.11}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \ge \frac{1}{|B(x,r)|}\int_{B(x,r)} udy, \quad u(x) \ge \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} udS. \tag{1.1.12}$$

Moreover, if the function u is harmonic: $\Delta u = 0$, then we have equality in both inequalities in (1.1.10).

One reason to expect the mean-value property is from physics – if Ω is a ball with no heat sources, it is natural to expect that the equilibrium temperature in the center of the ball may not exceed the average temperature over any sphere concentric with the ball. The opposite is true if there are no heat sinks (this is true for a super-harmonic function). Another explanation can be seen from the discrete version of inequality (1.1.9):

$$u(x) \le \frac{1}{2n}\sum_{j=1}^{n}(u(x+he_j)+u(x-he_j)).$$

Here, h is the mesh size, and e_j is the unit vector in the direction of the coordinate axis for x_j. This discrete equation says exactly that the value $u(x)$ is smaller than the average of the values of u at the neighbors of the point x on the lattice with mesh size h, which is similar to the statement of Theorem 1.1.4 (though there is no meaning to "nearest" neighbor in the continuous case).

Proof. We will only consider a sub-harmonic function, the super-harmonic functions are treated identically. Let us fix the point $x \in \Omega$ and define

$$\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(z)dS(z). \tag{1.1.13}$$

It is easy to see that, since $u(x)$ is continuous, we have

$$\lim_{r\downarrow 0}\phi(r) = u(x). \tag{1.1.14}$$

Therefore, we would be done if we knew that $\phi'(r) \ge 0$ for all $r > 0$ small enough so that that the ball $B(x,r)$ is contained in Ω. To this end, passing to the polar coordinates $z = x + ry$, with $y \in \partial B(0,1)$, we may rewrite (1.1.13) as

$$\phi(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+ry)dS(y).$$

Then, differentiating in r gives

$$\phi'(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} y \cdot \nabla u(x+ry) dS(y).$$

Going back to the z-variables leads to

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{1}{r}(z-x) \cdot \nabla u(z) dS(z) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(z).$$

Here, we used the fact that the outward normal to $B(x,r)$ at a point $z \in \partial B(x,r)$ is

$$\nu = (z-x)/r.$$

Using Green's formula

$$\int_U \Delta g dy = \int_U \nabla \cdot (\nabla g) dy = \int_{\partial U} (\nu \cdot \nabla g) dS = \int_{\partial U} \frac{\partial g}{\partial \nu} dS,$$

gives now

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y) dy \geq 0.$$

It follows that $\phi(r)$ is a non-decreasing function of r, and then (1.1.14) implies that

$$u(x) \leq \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS, \tag{1.1.15}$$

which is the second identity in (1.1.10).

In order to prove the first equality in (1.1.10) we use the polar coordinates once again:

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} u dy = \frac{1}{|B(x,r)|} \int_0^r \left(\int_{\partial B(x,s)} u dS \right) ds \geq \frac{1}{|B(x,r)|} \int_0^r u(x) n\alpha(n) s^{n-1} ds$$
$$= u(x)\alpha(n) r^n \frac{1}{\alpha(n) r^n} = u(x).$$

We used above two facts: first, the already proved identity (1.1.15) about averages on spherical shells, and, second, that the area of an $(n-1)$-dimensional unit sphere is $n\alpha(n)$, where $\alpha(n)$ is the volume of the n-dimensional unit ball. Now, the proof of (1.1.10) is complete. The proof of the mean-value property for super-harmonic functions works identically. □

The maximum principle for the Laplacian

The first consequence of the mean value property is the maximum principle that says that a sub-harmonic function attains its maximum over any domain on the boundary and not inside the domain. From the physical point of view this is, again, obvious – a sub-harmonic function is nothing but the heat distribution in a room without heat sources, hence it is very natural that it attains its maximum on the boundary (the walls of the room). In one dimension this claim is also familiar: a sub-harmonic function of a one-dimensional variable is convex, and, of course, a smooth convex function does not have any local maxima.

Theorem 1.1.5 (**The maximum principle**) *Let $u(x)$ be a sub-harmonic function in a connected domain Ω and assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then*

$$\max_{x \in \bar{\Omega}} u(x) = \max_{y \in \partial\Omega} u(y). \tag{1.1.16}$$

Moreover, if $u(x)$ achieves its maximum at a point x_0 in the interior of Ω, then $u(x)$ is identically equal to a constant in Ω. Similarly, if $v \in C^2(\Omega) \cap C(\bar{\Omega})$ is a super-harmonic function in Ω, then

$$\min_{x \in \bar{\Omega}} v(x) = \min_{y \in \partial\Omega} v(y), \tag{1.1.17}$$

and if $v(x)$ achieves its minimum at a point x_0 in the interior of Ω, then $v(x)$ is identically equal to a constant in Ω.

Proof. Again, we only treat the case of a sub-harmonic function. Suppose that $u(x)$ attains its maximum at an interior point $x_0 \in \Omega$, and set

$$M = u(x_0).$$

Then, for any $r > 0$ sufficiently small (so that the ball $B(x_0, r)$ is contained in Ω), we have

$$M = u(x) \leq \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u dy \leq M,$$

with the equality above holding only if $u(y) = M$ for all y in the ball $B(x_0, r)$. Therefore, the set S of points where $u(x) = M$ is open. Since $u(x)$ is continuous, this set is also closed. Since S us both open and closed in Ω, and Ω is connected, it follows that $S = \Omega$, hence $u(x) = M$ at all points $x \in \Omega$. □

We should note the particularly simple proof above only applies to the Laplacian itself but the maximum principle applies to much more general elliptic operators than the Laplacian. In particular, already in this chapter, we will deal with slightly more general operators than the Laplacian, of the form

$$Lu = \Delta u(x) + c(x)u. \tag{1.1.18}$$

In order to anticipate that this issue is not totally trivial, consider the following exercise.

Exercise 1.1.6 Consider the boundary value problem

$$-u'' - au = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

with a given non-negative function $f(x)$, and a constant $a \geq 0$. Show that if $a < \pi^2$, then the function $u(x)$ is positive on the interval $(0, 1)$.

The reader may observe that $a = \pi^2$ is the leading eigenvalue of the operator $Lu = -u''$ on the interval $0 < x < 1$ with the boundary conditions $u(0) = u(1) = 0$. This transition will be generalized to much more general operators later on.

1.2 Act II. The moving plane method

1.2.1 The isoperimeteric inequality and sliding

We now bring in our second set of characters, the moving plane and sliding methods. As an introduction, we show how the sliding method can work alone, without the maximum principle. Maybe the simplest situation when the sliding idea proves useful is in an elegant proof of the isoperimetric inequality given by X. Cabré in [30] (see also [31]). The isoperimetric inequality says that among all domains of a given volume the ball has the smallest surface area.

Theorem 1.2.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Then,*

$$\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}, \tag{1.2.1}$$

where B_1 is the open unit ball in $\mathbb{R}^n$, $|\Omega|$ denotes the measure of Ω and $|\partial\Omega|$ is the perimeter of Ω (the $(n-1)$-dimensional measure of the boundary of Ω). In addition, equality in (1.2.1) holds if and only if Ω is a ball.

A technical aside: the area formula

The proof will use the area formula, a generalization of the usual change of variables formula in the multi-variable calculus. The latter says that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth one-to-one map (a change of variables), then

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} g(f^{-1}(y))dy. \tag{1.2.2}$$

Here, Jf is the Jacobian of the map f:

$$Jf(x) = \left|\det\left(\frac{\partial f_i}{\partial x_j}\right)\right|.$$

For general maps we have

Theorem 1.2.2 *(Area formula) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each function $g \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x\in f^{-1}\{y\}} g(x)\right] dy. \tag{1.2.3}$$

Note that if f is Lipschitz then it is differentiable almost everywhere by the Rademacher theorem [61], thus the Jacobian is defined almost everywhere as well. We will not prove the area formula here – see [61] for the proof. We will use the following corollary.

Corollary 1.2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each measurable set $A \subset \mathbb{R}^n$ we have*

$$|f(A)| \leq \int_A Jf(x)dx. \tag{1.2.4}$$

Proof. For a given set S we define its characteristic function as

$$\chi_S(x) = \begin{cases} 1, & \text{for } x \in S, \\ 0, & \text{for } x \notin S, \end{cases}$$

We use the area formula with $g(x) = \chi_A(x)$:

$$\begin{aligned} \int_A Jf(x)dx &= \int_{\mathbb{R}^n} \chi_A(x) Jf(x) dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A : \; f(x) = y] \, dy \geq \int_{\mathbb{R}^n} \chi_{f(A)}(y) dy = |f(A)|, \end{aligned}$$

and we are done. □

A more general form of this corollary is the following.

Corollary 1.2.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each nonnegative function $p \in L^1(\mathbb{R}^n)$ and each measurable set A, we have*

$$\int_{f(A)} p(y) dy \leq \int_A p(f(x)) Jf(x) dx. \tag{1.2.5}$$

Proof. The proof is as in the previous corollary. This time, we apply the area formula to the function $g(x) = p(f(x))\chi_A(x)$:

$$\begin{aligned} &\int_A p(f(x)) Jf(x) dx = \int_{\mathbb{R}^n} \chi_A(x) p(f(x)) Jf(x) dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x) p(f(x)) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A : \; f(x) = y] \, p(y) dy \geq \int_{f(A)} p(y) dy, \end{aligned}$$

and we are done. □

The proof of the isoperimetric inequality

We now proceed with Cabré's proof of the isoperimetric inequality in Theorem 1.2.1.

Step 1: sliding. Let $v(x)$ be the solution of the Neumann problem

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{1.2.6}$$

Here, ν is the outward normal at the boundary. Integrating the first equation above and using the boundary condition, we obtain

$$k|\Omega| = \int_\Omega \Delta v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.$$

Hence, solution exists only if

$$k = \frac{|\partial\Omega|}{|\Omega|}. \tag{1.2.7}$$

It is a classical result (see [79], for example) that with this particular value of k there exist infinitely many solutions that differ by addition of an arbitrary constant. We let v be any of them. As Ω is a smooth domain, v is also smooth.

Let Γ_v be the lower contact set of v, that is, the set of all $x \in \Omega$ such that the tangent hyperplane to the graph of v at x lies below that graph in all of $\bar{\Omega}$. More formally, we define

$$\Gamma_v = \{x \in \Omega : \ v(y) \geq v(x) + \nabla v(x) \cdot (y - x) \text{ for all } y \in \bar{\Omega}.\} \tag{1.2.8}$$

The crucial observation is that

$$B_1 \subset \nabla v(\Gamma_v). \tag{1.2.9}$$

Here, B_1 is the open unit ball centered at the origin.

Exercise 1.2.5 Explain why (1.2.9) is trivial in one dimension.

The geometric reason for this is as follows: take any $p \in B_1$ and consider the graphs of the functions

$$r_c(y) = p \cdot y + c.$$

We will now slide this plane upward – we will start with a "very negative" c, and start increasing it, moving the plane up. Note that there exists $M > 0$ so that if $c < -M$, then

$$r_c(y) < v(y) - 100 \text{ for all } y \in \bar{\Omega},$$

that is, the plane is below the graph in all of Ω. On the other hand, possibly after increasing M further, we may ensure that if $c > M$, then

$$r_c(y) > v(y) + 100 \text{ for all } y \in \bar{\Omega},$$

in other words, the plane is above the graph in all of Ω. Let then

$$\alpha = \sup\{c \in \mathbb{R} : \ r_c(y) < v(y) \text{ for all } y \in \bar{\Omega}\} \tag{1.2.10}$$

be the largest c so that the plane lies below the graph of v in all of Ω. It is easy to see that the plane $r_\alpha(y) = p \cdot y + \alpha$ has to touch the graph of v: there exists a point $y_0 \in \bar{\Omega}$ such that $r_\alpha(y_0) = v(y_0)$ and

$$r_\alpha(y) \leq v(y) \text{ for all } y \in \bar{\Omega}. \tag{1.2.11}$$

Furthermore, the point y_0 can not lie on the boundary $\partial\Omega$ since $|p| < 1$. Indeed, for all $y \in \partial\Omega$ we have

$$\left|\frac{\partial r_c}{\partial \nu}\right| = |p \cdot \nu| \leq |p| < 1 \text{ and } \frac{\partial v}{\partial \nu} = 1.$$

This means that if $r_c(y) = v(y)$ for some c, and y is on the boundary $\partial\Omega$, then there is a neighborhood $U \in \Omega$ of y such that $r_c(y) > v(y)$ for all $y \in U$. Comparing to (1.2.11), we see that $c \neq \alpha$, hence it is impossible that $y_0 \in \partial\Omega$. Thus, y_0 is an interior point of Ω, and, moreover, the graph of $r_\alpha(y)$ is the tangent plane to v at y_0. In particular, we have $\nabla v(y_0) = p$, and (1.2.11) implies that y_0 is in the contact set of v: $y_0 \in \Gamma_v$. We have now shown the inclusion (1.2.9): $B_1 \subset \nabla v(\Gamma_v)$. Note that the only information about the function $v(x)$ we have used so far is the Neumann boundary condition

$$\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega,$$

but not the Poisson equation for v in Ω.

Step 2: using the area formula. A trivial consequence of (1.2.9) is that

$$|B_1| \le |\nabla v(\Gamma_v)|. \tag{1.2.12}$$

Now, we will apply Corollary 1.2.3 to the map $\nabla v : \Gamma_v \to \nabla v(\Gamma_v)$. The Jacobian of this map is $|\det[D^2 v]|$.

Exercise 1.2.6 Show that if Γ_v is the contact set of a smooth function $v(x)$, then $\det[D^2 v]$ is non-negative for $x \in \Gamma_v$, and, moreover, all eigenvalues of $D^2 v$ are nonnegative on Γ_v.

As $\det[D^2 v]$ is non-negative for $x \in \Gamma_v$, we conclude from Corollary 1.2.3 and (1.2.12) that

$$|B_1| \le |\nabla v(\Gamma_v)| \le \int_{\Gamma_v} \det[D^2 v(x)]dx. \tag{1.2.13}$$

It remains to notice that by the classical arithmetic mean-geometric mean inequality applied to the (nonnegative) eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix $D^2 v(x)$, $x \in \Gamma_v$ we have

$$\det[D^2 v(x)] = \lambda_1 \lambda_2 \dots \lambda_n \le \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} \right)^n. \tag{1.2.14}$$

However, by a well-known formula from linear algebra,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{Tr}[D^2 v],$$

and, moreover, $\mathrm{Tr}[D^2 v]$ is simply the Laplacian Δv. This gives

$$\det[D^2 v(x)] \le \left(\frac{\mathrm{Tr}[D^2 v]}{n} \right)^n = \left(\frac{\Delta v}{n} \right)^n \text{ for } x \in \Gamma_v. \tag{1.2.15}$$

Recall that v is the solution of (1.2.6):

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{1.2.16}$$

with

$$k = \frac{|\partial\Omega|}{|\Omega|}.$$

Going back to (1.2.13), we deduce that

$$|B_1| \le \int_{\Gamma_v} \det[D^2 v(x)]dx \le \int_{\Gamma_v} \left(\frac{\Delta v}{n} \right)^n dx \le \left(\frac{k}{n} \right)^n |\Gamma_v| = \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Gamma_v| \le \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Omega|.$$

In addition, for the unit ball we have $|\partial B_1| = n|B_1|$, hence the above implies

$$\frac{|\partial B_1|^n}{|B_1|^{n-1}} \leq \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}, \tag{1.2.17}$$

which is nothing but the isoperimetric inequality (1.2.1).

In order to see that the inequality in (1.2.17) is strict unless Ω is a ball, we observe that it follows from the above argument that for the equality to hold in (1.2.17) we must have equality in (1.2.14), and, in addition, Γ_v has to coincide with Ω. This means that for each $x \in \Omega$ all eigenvalues of the matrix $D^2v(x)$ are equal to each other. That is, $D^2v(x)$ is a multiple of the identity matrix for each $x \in \Omega$.

Exercise 1.2.7 Show that if $v(x)$ is a smooth function such that

$$\frac{\partial^2 v(x)}{\partial x_i^2} = \frac{\partial^2 v(x)}{\partial x_j^2},$$

for all $1 \leq i, j \leq n$ and $x \in \Omega$, and

$$\frac{\partial^2 v(x)}{\partial x_i \partial x_j} = 0,$$

for all $i \neq j$ and $x \in \Omega$, then there exists $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$, so that

$$v(x) = b\left[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2\right] + c, \tag{1.2.18}$$

for all $x \in \Omega$.

Our function $v(x)$ does satisfy the assumptions of Exercise 1.2.7, hence it must be of the form (1.2.18). Finally, the boundary condition $\partial v/\partial \nu = 1$ on $\partial\Omega$ implies that Ω is a ball centered at the point $a \in \mathbb{R}^n$. $\square$

1.3 Act III. Their first meeting

The maximum principle returns, and we study it in a slightly greater depth. At the end of this act the maximum principle and the moving plane method are introduced to each other.

The Hopf lemma and the strong maximum principle

We now generalize the maximum principle to slightly more general operators than the Laplacian, to allow for a zero-order term. Let us begin with the following exercises.

Exercise 1.3.1 Show that if the function $u(x)$ satisfies an ODE of the form

$$u'' + c(x)u = 0, \quad a < x < b, \tag{1.3.1}$$

and $u(x_0) = 0$ for some $x_0 \in (a, b)$, and the function $c(x)$ is continuous on $[a, b]$, then u can not attain its maximum (or minimum) over the interval (a, b) at the point x_0 unless $u \equiv 0$.

This exercise is relatively easy – one has to think about the initial value problem for (1.3.1) with the data $u(x_0) = u'(x_0) = 0$. Now, look at the next exercise, which is slightly harder.

Exercise 1.3.2 Show that, once again, in one dimension, if $u(x)$, $x \in \mathbb{R}$ satisfies a differential inequality of the form

$$u'' + c(x)u \geq 0, \quad a < x < b,$$

the function $c(x)$ is continuous on $[a, b]$, and $u(x_0) = 0$ for some $x_0 \in (a, b)$ then u can not attain its maximum over the interval (a, b) at the point x_0 unless $u \equiv 0$.

The proof of the strong maximum principle relies on the Hopf lemma which guarantees that the point on the boundary where the maximum is attained is not a critical point of u.

Theorem 1.3.3 *(The Hopf Lemma) Let $B = B(y, r)$ be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$, and assume that $c(x) \leq 0$ in B. Suppose that a function $u \in C^2(B) \cap C(B \cup x_0)$ is a sub-solution, that is, it satisfies*

$$\Delta u + c(x)u \geq 0 \text{ in } B,$$

and that $u(x) < u(x_0)$ for any $x \in B$ and $u(x_0) \geq 0$. Then, we have $\dfrac{\partial u}{\partial \nu}(x_0) > 0$.

Proof. We may assume without loss of generality that B is centered at the origin: $y = 0$. We may also assume that $u \in C(\bar{B})$ and that $u(x) < u(x_0)$ for all $x \in \bar{B} \backslash \{x_0\}$ – otherwise, we would simply consider a smaller ball $B_1 \subset B$ that is tangent to B at x_0.

The idea is to modify u to turn it into a strict sub-solution of the form

$$w(x) = u(x) + \varepsilon h(x).$$

We also need w to inherit the other properties of u: it should attain its maximum over $\bar{B}$ at x_0, and we need to have $w(x) < w(x_0)$ for all $x \in B$. In addition, we would like to have

$$\frac{\partial h}{\partial \nu} < 0 \text{ on } \partial B,$$

so that the inequality

$$\frac{\partial w}{\partial \nu}(x_0) \geq 0$$

would imply

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

An appropriate choice is

$$h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2},$$

in a smaller domain

$$\Sigma = B \cap B(x_0, r/2).$$

Observe that $h > 0$ in B, $h = 0$ on ∂B (thus, h attains its minimum on ∂B – unlike u which attains its maximum there), and, in addition:

$$\begin{aligned}\Delta h + c(x)h &= e^{-\alpha|x|^2}\left[4\alpha^2|x|^2 - 2\alpha n + c(x)\right] - c(x)e^{-\alpha r^2}\\ &\geq e^{-\alpha|x|^2}\left[4\alpha^2|x|^2 - 2\alpha n + c(x)\right] \geq e^{-\alpha|x|^2}\left[4\alpha^2\frac{|r|^2}{4} - 2\alpha n + c(x)\right] > 0,\end{aligned}$$

for all $x \in \Sigma$ for a sufficiently large $\alpha > 0$. Hence, we have a strict inequality

$$\Delta w + c(x)w > 0, \text{ in } \Sigma, \tag{1.3.2}$$

for all $\varepsilon > 0$. Note that $w(x_0) = u(x_0) \geq 0$, thus the maximum of w over Σ is non-negative. Suppose that w attains this maximum at an interior point x_1, and $w(x_1) \geq 0$. As $\Delta w(x_1) \leq 0$ and $c(x_1) \leq 0$, it follows that

$$\Delta w(x_1) + c(x_1)w(x_1) \leq 0,$$

which is a contradiction to (1.3.2). Thus, w may not attain a non-negative maximum inside Σ but only on the boundary. We now show that if $\varepsilon > 0$ is sufficiently small, then w attains this maximum only at x_0. Indeed, as $u(x) < u(x_0)$ in B, we may find δ, so that

$$u(x) < u(x_0) - \delta \text{ for } x \in \partial\Sigma \cap B.$$

Take ε so that

$$\varepsilon h(x) < \delta \text{ on } \partial\Sigma \cap B,$$

then

$$w(x) < u(x_0) = w(x_0) \text{ for all } x \in \partial\Sigma \cap B.$$

On the other hand, for $x \in \partial\Sigma \cap \partial B$ we have $h(x) = 0$ and

$$w(x) = u(x) < u(x_0) = w(x_0).$$

We conclude that $w(x)$ attains its non-negative maximum in $\bar{\Sigma}$ at x_0 if ε is sufficiently small. This implies

$$\frac{\partial w}{\partial \nu}(x_0) \geq 0,$$

and, as a consequence

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial h}{\partial \nu}(x_0) = \varepsilon \alpha r e^{-\alpha r^2} > 0.$$

This finishes the proof. □

The next theorem is an immediate consequence of the Hopf lemma.

Theorem 1.3.4 *(The strong maximum principle) Assume that $c(x) \leq 0$ in Ω, and the function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\Delta u + c(x)u \geq 0,$$

and attains its maximum over $\bar{\Omega}$ at a point x_0. In this case, if $u(x_0) \geq 0$, then $x_0 \in \partial\Omega$ unless u is a constant. If the domain Ω has the internal sphere property, and $u \not\equiv const$, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Proof. Let $M = \sup_{\bar{\Omega}} u(x)$ and define the set $\Sigma = \{x \in \Omega : \ u(x) = M\}$, where the maximum is attained. We need to show that either Σ is empty or $\Sigma = \Omega$. Assume that Σ is non-empty but $\Sigma \neq \Omega$, and choose a point $p \in \Omega \backslash \Sigma$ such that

$$d_0 = d(p, \Sigma) < d(p, \partial\Omega).$$

Consider the ball $B_0 = B(p, d_0)$ and let $x_0 \in \partial B_0 \cap \partial\Sigma$. Then we have

$$\Delta u + c(x)u \geq 0 \text{ in } B_0,$$

and

$$u(x) < u(x_0) = M,\ M \geq 0 \text{ for all } x \in B_0.$$

The Hopf Lemma implies that

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the normal to B_0 at x_0. However, x_0 is an internal maximum of u in Ω and hence $\nabla u(x_0) = 0$. This is a contradiction. □

Now, we may state the strong comparison principle – note that we do not make any assumptions on the sign of the function $c(x)$ here.

Theorem 1.3.5 *(The strong comparison principle) Assume that $c(x)$ is a bounded function, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\Delta u + c(x)u \geq 0. \tag{1.3.3}$$

If $u \leq 0$ in Ω then either $u \equiv 0$ in Ω or $u < 0$ in Ω. Similarly, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\Delta u + c(x)u \leq 0 \text{ in } \Omega, \tag{1.3.4}$$

with $u \geq 0$ in Ω, with a bounded function $c(x)$. Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

Proof. If $c(x) \leq 0$, this follows directly from the strong maximum principle. In the general case, as $u \leq 0$ in Ω, the inequality (1.3.3) implies that, for any $M > 0$ we have

$$\Delta u + c(x)u - Mu \geq -Mu \geq 0.$$

However, if $M > \|c\|_{L^\infty(\Omega)}$ then the zero order coefficient satisfies

$$c_1(x) = c(x) - M \leq 0,$$

hence we may conclude, again from the strong maximum principle that either $u < 0$ in Ω or $u \equiv 0$ in Ω. The proof in the case (1.3.4) holds is identical. □

Separating sub- and super-solutions

A very common use of the strong maximum principle is to re-interpret it as the "untouchability" of a sub-solution and a super-solution of a linear or nonlinear problem – the basic principle underlying what we will see below. Assume that the functions $u(x)$ and $v(x)$ satisfy

$$-\Delta u \leq f(x, u), \quad -\Delta v \geq f(x, v) \text{ in } \Omega. \tag{1.3.5}$$

We say that $u(x)$ is a sub-solution, and $v(x)$ is a super-solution. Assume that, in addition, we know that

$$u(x) \leq v(x) \text{ for all } x \in \Omega, \tag{1.3.6}$$

that is, the sub-solution sits below the super-solution. In this case, we are going to rule out the possibility that they touch inside Ω (they can touch on the boundary, however): there can not be an $x_0 \in \Omega$ so that $u(x_0) = v(x_0)$. Indeed, if the function $f(x, s)$ is differentiable (or Lipschitz) in s, the quotient

$$c(x) = \frac{f(x, u(x)) - f(x, v(x))}{u(x) - v(x)}$$

is a bounded function, and the difference $w(x) = u(x) - v(x)$ satisfies

$$\Delta w + c(x)w \geq 0 \text{ in } \Omega. \tag{1.3.7}$$

As $w(x) \leq 0$ in all of Ω, the strong maximum principle implies that either $w(x) \equiv 0$, so that u and v coincide, or $w(x) < 0$ in Ω, that is, we have a strict inequality: $u(x) < v(x)$ for all $x \in \Omega$. In other words, a sub-solution and a super-solution can not touch at a point – this very simple principle will be extremely important in what follows.

Let us illustrate an application of the strong maximum principle, with a cameo appearance of the sliding method in a disguise as a bonus. Consider the boundary value problem

$$-u'' = e^u, \quad 0 < x < L, \tag{1.3.8}$$

with the boundary condition

$$u(0) = u(L) = 0. \tag{1.3.9}$$

If we think of $u(x)$ as a temperature distribution, then the boundary condition means that the boundary is "cold". On the other hand, the positive term e^u is a "heating term", which competes with the cooling by the boundary. A nonnegative solution $u(x)$ corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is sufficiently large, then no such equilibrium is possible – the physical reason is that the boundary is too far from the middle of the interval, so the heating term wins. This absence of an equilibrium is interpreted as an explosion, and this model was introduced exactly in that context in late 30's-early 40's. It is convenient to work with the function $w = u + \varepsilon$, which satisfies

$$-w'' = e^{-\varepsilon}e^w, \quad 0 < x < L, \tag{1.3.10}$$

with the boundary condition

$$w(0) = w(L) = \varepsilon. \tag{1.3.11}$$

Consider a family of functions

$$v_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right), \quad \lambda \geq 0, \quad 0 < x < L.$$

These functions satisfy (for any $\lambda \geq 0$)

$$v_\lambda'' + \frac{\pi^2}{L^2}v_\lambda = 0, \quad v_\lambda(0) = v_\lambda(L) = 0. \tag{1.3.12}$$

Therefore, if L is so large that

$$\frac{\pi^2}{L^2}s \leq e^{-\varepsilon}e^s, \quad \text{for all } s \geq 0,$$

we have

$$w'' + \frac{\pi^2}{L^2} w \leq 0, \tag{1.3.13}$$

that is, w is a super-solution for (1.3.12). In addition, when $\lambda > 0$ is sufficiently small, we have

$$v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L. \tag{1.3.14}$$

Let us now start increasing λ until the graphs of v_λ and w touch at some point:

$$\lambda_0 = \sup\{\lambda : \ v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L.\} \tag{1.3.15}$$

The difference

$$p(x) = v_{\lambda_0}(x) - w(x)$$

satisfies

$$p'' + \frac{\pi^2}{L^2} p \geq 0,$$

and $p(x) \leq 0$ for all $0 < x < L$. In addition, there exists x_0 such that $p(x_0) = 0$, and, as

$$v_\lambda(0) = v_\lambda(L) = 0 < \varepsilon = w(0) = w(L),$$

it is impossible that $x_0 = 0$ or $x_0 = L$. We conclude that $p(x) \equiv 0$, which is a contradiction. Hence, no solution of (1.3.8)-(1.3.9) may exist when L is sufficiently large.

In order to complete the picture, the reader may look at the following exercise.

Exercise 1.3.6 Show that there exists $L_1 > 0$ so that a nonnegative solution of (1.3.8)-(1.3.9) exists for all $0 < L < L_1$, and does not exist for all $L > L_1$.

The maximum principle for narrow domains

Before we allow the moving plane method to return, we describe the maximum principle for narrow domains, which is an indispensable tool in this method. Its proof will utilize the "ballooning method" we have seen in the analysis of the explosion problem. As we have discussed, the usual maximum principle in the form "$\Delta u + c(x)u \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ implies either $u \equiv 0$ or $u < 0$ in Ω" can be interpreted physically as follows. If u is the temperature distribution then the boundary condition $u \leq 0$ on $\partial\Omega$ means that "the boundary is cold". At the same time, the term $c(x)u$ can be viewed as a heat source if $c(x) \geq 0$ or as a heat sink if $c(x) \leq 0$. The conditions $u \leq 0$ on $\partial\Omega$ and $c(x) \leq 0$ together mean that both the boundary is cold and there are no heat sources – therefore, the temperature is cold everywhere, and we get $u \leq 0$. On the other hand, if the domain is such that each point inside Ω is "close to the boundary" then the effect of the cold boundary can dominate over a heat source, and then, even if $c(x) \geq 0$ at some (or all) points $x \in \Omega$, the maximum principle still holds.

Mathematically, the first step in that direction is the maximum principle for narrow domains. We use the notation $c^+(x) = \max[0, c(x)]$.

Theorem 1.3.7 *(The maximum principle for narrow domains) There exists $d_0 > 0$ that depends on the L^∞-norm $\|c^+\|_\infty$ so that if there exists a unit vector e such that $|(y-x)\cdot e| < d_0$*

for all $(x,y) \in \Omega$ then the maximum principle holds for the operator $\Delta + c(x)$. That is, if a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\Delta u(x) + c(x)u(x) \geq 0 \text{ in } \Omega, \tag{1.3.16}$$

and $u \leq 0$ on $\partial\Omega$ then either $u \equiv 0$ or $u < 0$ in Ω.

The main observation here is that in a narrow domain we need not assume $c \leq 0$ – but "the largest possible narrowness", depends, of course, on the size of the positive part $c^+(x)$ that competes against it.

Proof. Note that, according to the strong maximum principle, it is sufficient to show that $u(x) \leq 0$ in Ω. For the sake of contradiction, suppose that

$$\sup_{x \in \Omega} u(x) > 0. \tag{1.3.17}$$

Without loss of generality we may assume that e is the unit vector in the direction x_1, and that

$$\bar{\Omega} \subset \{0 < x_1 < d\}.$$

Suppose that d is so small that

$$c(x) \leq \pi^2/d^2, \quad \text{for all } x \in \Omega, \tag{1.3.18}$$

and consider the function

$$w(x) = \sin\big(\frac{\pi x_1}{d}\big).$$

It satisfies

$$\Delta w + \frac{\pi^2}{d^2} w = 0, \tag{1.3.19}$$

and $w(x) > 0$ in $\bar{\Omega}$, in particular

$$\inf_{\bar{\Omega}} w(x) > 0. \tag{1.3.20}$$

A consequence of the above is

$$\Delta w + c(x)w \leq 0, \tag{1.3.21}$$

so that $w(x)$ is a super-solution to (1.3.16), while $u(x)$ is a sub-solution. Given $\lambda \geq 0$, let us set $w_\lambda(x) = \lambda w(x)$. As a consequence of (1.3.20), there exists $\Lambda > 0$ so large that

$$\Lambda w(x) > u(x) \text{ for all } x \in \Omega.$$

We are going to push w_λ down until it touches $u(x)$: set

$$\lambda_0 = \inf\{\lambda : \; w_\lambda(x) > u(x) \text{ for all } x \in \Omega.\}$$

Note, that, because of (1.3.17), we know that $\lambda_0 > 0$. The difference

$$v(x) = u(x) - w_{\lambda_0}(x)$$

satisfies

$$\Delta v + c(x)v \geq 0.$$

The difference between $u(x)$, which satisfies the same inequality, and $v(x)$ is that we know already that $v(x) \leq 0$ – hence, we may conclude from the strong maximum principle again that either $v(x) \equiv 0$, or $v(x) < 0$ in Ω. As $w_\lambda(x) > 0$ on $\partial\Omega$, the former contradicts the boundary condition on $u(x)$. It follows that $v(x) < 0$ in Ω. As $v(x) < 0$ also on the boundary $\partial\Omega$, there exists $\varepsilon_0 > 0$ so that

$$v(x) < -\varepsilon_0 \text{ for all } x \in \bar{\Omega},$$

that is,

$$u(x) + \varepsilon_0 < w_{\lambda_0}(x) \text{ for all } x \in \bar{\Omega}.$$

But then we may choose $\lambda' < \lambda_0$ so that we still have

$$w_{\lambda'}(x) > u(x) \text{ for all } x \in \Omega.$$

This contradicts the minimality of λ_0. Thus, it is impossible that $u(x) > 0$ for some $x \in \Omega$, and we are done. □

The maximum principle for small domains

The maximum principle for narrow domains can be extended, dropping the requirement that the domain is narrow and replacing it by the condition that the domain has a small volume. We begin with the following lemma, a simple version of the Alexandrov-Bakelman-Pucci maximum principle, which measures how far from the maximum principle a force can push the solution.

Lemma 1.3.8 *(The baby ABP Maximum Principle) Assume that $c(x) \leq 0$ for all $x \in \Omega$, and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u + c(x)u \geq f \text{ in } \Omega, \tag{1.3.22}$$

and $u \leq 0$ on $\partial\Omega$. Then

$$\sup_\Omega u \leq C\text{diam}(\Omega) \left\| f^- \right\|_{L^n(\Omega)}, \tag{1.3.23}$$

with the constant C that depends only on the dimension n (but not on the function $c(x) \leq 0$).

Proof. The idea is very similar to what we have seen in the proof of the isoperimetric inequality. If $M := \sup_\Omega u \leq 0$, then there is nothing to prove, hence we assume that $M > 0$. As $u(x) \leq 0$ on $\partial\Omega$, the maximum is achieved at an interior point $x_0 \in \Omega$, so that $M = u(x_0)$. The function $v = -u^+$, satisfies $v \leq 0$ in Ω, $v \equiv 0$ on $\partial\Omega$ and

$$-M = \inf_\Omega v = v(x_0) < 0.$$

Let Γ be the lower contact set of the function v, defined as in (1.2.8): the collection of all points $x \in \Omega$ such that the graph of v lies above the tangent plane at x. As $v \leq 0$ in Ω, we must have $v < 0$ on Γ. Hence v is smooth on Γ, and

$$\Delta v = -\Delta u \leq -f(x) + c(x)u \leq -f(x), \text{ for } x \in \Gamma, \tag{1.3.24}$$

as $c(x) \le 0$ and $u(x) \ge 0$ on Γ. The analog of the inclusion (1.2.9) that we will now prove is

$$B(0; M/d) \subset \nabla v(\Gamma), \tag{1.3.25}$$

with $d = \mathrm{diam}(\Omega)$ and $B(0, M/d)$ the open ball centered at the origin of radius M/d. One way to see that is by sliding: let $p \in B(0; M/d)$ and consider the hyperplane that is the graph of

$$z_k(x) = p \cdot x - k.$$

Clearly, $z_k(x) < v(x)$ for k sufficiently large. As we decrease k, sliding the plane up, let $\bar{k}$ be the first value when the graphs of $v(x)$ and $z_{\bar{k}}(x)$ touch at a point x_1. Then we have

$$v(x) \ge z_{\bar{k}}(x) \text{ for all } x \in \Omega.$$

If x_1 is on the boundary $\partial\Omega$ then $v(x_1) = z_{\bar{k}}(x_1) = 0$, and we have

$$p \cdot (x_0 - x_1) = z_{\bar{k}}(x_0) - z_{\bar{k}}(x_1) \le v(x_0) - 0 = -M,$$

whence $|p| \ge M/d$, which is a contradiction. Therefore, x_1 is an interior point, which means that $x_1 \in \Gamma$ (by the definition of the lower contact set), and $p = \nabla v(x_1)$. This proves the inclusion (1.3.25).

Mimicking the proof of the isoperimetric inequality we use the area formula (c_n is the volume of the unit bal in $\mathbb{R}^n$):

$$c_n \left(\frac{M}{d}\right)^n = |B(0; M/d)| \le |\nabla v(\Gamma)| \le \int_\Gamma |\det(D^2 v(x))| dx. \tag{1.3.26}$$

Now, as in the aforementioned proof, for every point x in the contact set Γ, the matrix $D^2 v(x)$ is non-negative definite, hence (note that (1.3.24) implies that $f(x) \le 0$ on Γ)

$$|\det[D^2 v(x)]| \le \left(\frac{\Delta v}{n}\right)^n \le \frac{(-f(x))^n}{n^n}. \tag{1.3.27}$$

Integrating (1.3.27) and using (1.3.26), we get

$$M^n \le \frac{(\mathrm{diam}(\Omega))^n}{c_n n^n} \int_\Gamma |f^-(x)|^n dx, \tag{1.3.28}$$

which is (1.3.23). $\Box$

An important consequence of Lemma 1.3.8 is a maximum principle for a domain with a small volume [6].

Theorem 1.3.9 *(The maximum principle for domains of a small volume) Let a function* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfy*

$$\Delta u(x) + c(x) u(x) \ge 0 \text{ in } \Omega,$$

and assume that $u \le 0$ *on* $\partial\Omega$*. Then there exists a positive constant* δ *which depends on the spatial dimension* n*, the diameter of* Ω*, and* $\|c^+\|_{L^\infty}$*, so that if* $|\Omega| \le \delta$ *then* $u \le 0$ *in* Ω*.*

Proof. If $c \le 0$ then $u \le 0$ by the standard maximum principle. In general, assume that $u^+ \not\equiv 0$, and write $c = c^+ - c^-$. We have

$$\Delta u - c^- u \ge -c^+ u.$$

Lemma 1.3.8 implies that (with a constant C that depends only on the dimension n)

$$\sup_\Omega u \le C \mathrm{diam}(\Omega) \|c^+ u^+\|_{L^n(\Omega)} \le C \mathrm{diam}(\Omega) \|c^+\|_\infty |\Omega|^{1/n} \sup_\Omega u \le \frac{1}{2} \sup_\Omega u,$$

when the volume of Ω is sufficiently small:

$$|\Omega| \le \frac{1}{(2C\mathrm{diam}(\Omega)\|c^+\|_\infty)^n}. \tag{1.3.29}$$

We deduce that $\sup_\Omega u \le 0$ contradicting the assumption $u^+ \not\equiv 0$, Hence, we have $u \le 0$ in Ω under the condition (1.3.29). □

1.4 Act IV. Dancing together

We will now use a combination of the maximum principle (mostly for small domains) and the moving plane method to prove some results on the symmetry of the solutions to elliptic problems. We show just the tip of the iceberg – a curious reader will find many other results in the literature, the most famous being, perhaps, the De Giorgi conjecture, a beautiful connection between geometry and applied mathematics.

1.4.1 The Gidas-Ni-Nirenberg theorem

The following result on the radial symmetry of non-negative solutions is due to Gidas, Ni and Nirenberg. It is a basic example of a general phenomenon that positive solutions of elliptic equations tend to be monotonic in one form or other. We present the proof of the Gidas-Ni-Nirenberg theorem from [23]. The proof uses the moving plane method combined with the maximum principles for narrow domains, and domains of small volume.

Theorem 1.4.1 *Let $B_1 \in \mathbb{R}^n$ be the unit ball, and $u \in C(\bar{B}_1) \cap C^2(B_1)$ be a positive solution of the Dirichlet boundary value problem*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } B_1, \\ &u = 0 \ \text{on } \partial B_1, \end{aligned} \tag{1.4.1}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is radially symmetric in B_1 and

$$\frac{\partial u}{\partial r}(x) < 0 \text{ for } x \ne 0.$$

To address an immediate question the reader may have, we give the following simple exercise.

Exercise 1.4.2 Show that the conclusion that a function u satisfying (1.4.1) is radially symmetric is false in general without the assumption that the function u is positive. Hint: you may have to learn a little more about the Bessel functions and spherical harmonics.

The proof of Theorem 1.4.1 is based on the following lemma, which applies to general domains with a planar symmetry, not just balls.

Lemma 1.4.3 *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1 = 0\}$. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } \Omega, \\ &u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.4.2}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is symmetric with respect to x_1 and

$$\frac{\partial u}{\partial x_1}(x) < 0 \text{ for any } x \in \Omega \text{ with } x_1 > 0.$$

Proof of Theorem 1.4.1. Theorem 1.4.1 follows immediately from Lemma 1.4.3. Indeed, Lemma 1.4.3 implies that $u(x)$ is decreasing in any given radial direction, since the unit ball is symmetric with respect to any plane passing through the origin. It also follows from the same lemma that $u(x)$ is invariant under a reflection with respect to any hyperplane passing through the origin – this trivially implies that u is radially symmetric. □

Proof of Lemma 1.4.3

We use the coordinate system $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We will prove that

$$u(x_1, y) < u(x_1^*, y) \text{ for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1. \tag{1.4.3}$$

This, obviously, implies monotonicity in x_1 for $x_1 > 0$. Next, letting $x_1^* \to -x_1$, we get the inequality

$$u(x_1, y) \le u(-x_1, y) \text{ for any } x_1 > 0.$$

Changing the direction, we get the reflection symmetry: $u(x_1, y) = u(-x_1, y)$.

We now prove (1.4.3). Given any $\lambda \in (0, a)$, with $a = \sup_\Omega x_1$, we take the "moving plane"

$$T_\lambda = \{x_1 = \lambda\},$$

and consider the part of Ω that is "to the right" of T_λ:

$$\Sigma_\lambda = \{x \in \Omega : \ x_1 > \lambda\}.$$

Finally, given a point x, we let x_λ be the reflection of $x = (x_1, x_2, \ldots, x_n)$ with respect to T_λ:

$$x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n).$$

Consider the difference

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

The mean value theorem implies that w_λ satisfies

$$\Delta w_\lambda = f(u(x_\lambda)) - f(u(x)) = \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} w_\lambda = -c(x,\lambda) w_\lambda$$

in Σ_λ. This is a recurring trick: the difference of two solutions of a semi-linear equation satisfies a "linear" equation with an unknown function c. However, we know a priori that the function c is bounded:

$$|c(x)| \le \mathrm{Lip}(f), \text{ for all } x \in \Omega. \quad (1.4.4)$$

The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$, where $w_\lambda = -u(x_\lambda) < 0$ and of a part of the plane T_λ, where $x = x_\lambda$, thus $w_\lambda = 0$. Summarizing, we have

$$\begin{aligned} &\Delta w_\lambda + c(x,\lambda) w_\lambda = 0 \text{ in } \Sigma_\lambda \\ &w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 \text{ on } \partial\Sigma_\lambda, \end{aligned} \quad (1.4.5)$$

with a bounded function $c(x,\lambda)$. As the function $c(x,\lambda)$ does not necessarily have a definite sign, we may not directly apply the comparison principle and immediately conclude from (1.4.5) that

$$w_\lambda < 0 \text{ inside } \Sigma_\lambda \text{ for all } \lambda \in (0,a). \quad (1.4.6)$$

Nevertheless, using the moving plane method, we will be able to show that (1.4.6) holds. This implies in particular that w_λ assumes its maximum (equal to zero) over $\bar\Sigma_\lambda$ along T_λ. The Hopf lemma implies then

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1=\lambda} = 2\left.\frac{\partial u}{\partial x_1}\right|_{x_1=\lambda} < 0.$$

Given that λ is arbitrary, we conclude that

$$\frac{\partial u}{\partial x_1} < 0, \text{ for any } x \in \Omega \text{ such that } x_1 > 0.$$

Therefore, it remains only to show that $w_\lambda < 0$ inside Σ_λ to establish monotonicity of u in x_1 for $x_1 > 0$. Another consequence of (1.4.6) is that

$$u(x_1, x') < u(2\lambda - x_1, x') \text{ for all } \lambda \text{ such that } x \in \Sigma_\lambda,$$

that is, for all $\lambda \in (0, x_1)$, which is the same as (1.4.3).

In order to show that $w_\lambda < 0$ one would like to apply the maximum principle to the boundary value problem (1.4.5). However, as we have mentioned, a priori the function $c(x,\lambda)$ does not have a sign, so the usual maximum principle may not be used. On the other hand, there exists δ_c such that the maximum principle for narrow domains holds for the operator

$$Lu = \Delta u + c(x)u,$$

and domains of the width not larger than δ_c in the x_1-direction. Note that δ_c depends only on $\|c\|_{L^\infty}$ that is controlled in our case by (1.4.4). Moreover, when λ is sufficiently close to a:

$$a - \delta_c < \lambda < a,$$

the domain Σ_λ does have the width in the x_1-direction which is smaller than δ_c. Thus, for such λ the maximum principle for narrow domains implies that $w_\lambda < 0$ inside Σ_λ. This is because $w_\lambda \le 0$ on $\partial\Sigma_\lambda$, and $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$.

Let us now decrease λ (move the plane T_λ to the left, hence the name "the moving plane" method), and let (λ_0, a) be the largest interval of values so that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (\lambda_0, a)$. If $\lambda_0 = 0$, that is, if we may move the plane T_λ all the way to $\lambda = 0$, while keeping (1.4.6) true, then we are done – (1.4.6) follows. Assume, for the sake of a contradiction, that $\lambda_0 > 0$. Then, by continuity, we still know that

$$w_{\lambda_0} \le 0 \text{ in } \Sigma_{\lambda_0}.$$

Moreover, w_{λ_0} is not identically equal to zero on $\partial\Sigma_{\lambda_0}$. The strong comparison principle implies that

$$w_{\lambda_0} < 0 \text{ in } \Sigma_{\lambda_0}. \tag{1.4.7}$$

We will show that then

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \tag{1.4.8}$$

for sufficiently small $\varepsilon < \varepsilon_0$. This will contradict our choice of λ_0 (unless $\lambda_0 = 0$).

Here is the key step and the reason why the maximum principle for domains of small volume is useful for us here: choose a compact set K in Σ_{λ_0}, with a smooth boundary, which is "nearly all" of Σ_{λ_0}, in the sense that

$$|\Sigma_{\lambda_0} \backslash K| < \delta/2$$

with $\delta > 0$ to be determined. Inequality (1.4.7) implies that there exists $\eta > 0$ so that

$$w_{\lambda_0} \le -\eta < 0 \text{ for any } x \in K.$$

By continuity, there exits $\varepsilon_0 > 0$ so that

$$w_{\lambda_0-\varepsilon} < -\frac{\eta}{2} < 0 \text{ for any } x \in K, \tag{1.4.9}$$

for $\varepsilon \in (0, \varepsilon_0)$ sufficiently small. Let us now see what happens in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. As far as the boundary is concerned, we have

$$w_{\lambda_0-\varepsilon} \le 0$$

on $\partial\Sigma_{\lambda_0-\varepsilon}$ – this is true for $\partial\Sigma_\lambda$ for all $\lambda \in (0, a)$, and, in addition,

$$w_{\lambda_0-\varepsilon} < 0 \text{ on } \partial K,$$

because of (1.4.9). We conclude that

$$w_{\lambda_0-\varepsilon} \le 0 \text{ on } \partial(\Sigma_{\lambda_0-\varepsilon} \backslash K),$$

and $w_{\lambda_0-\varepsilon}$ does not vanish identically on $\partial(\Sigma_{\lambda_0-\varepsilon} \backslash K)$. Choose now δ (once again, solely determined by $\|c\|_{L^\infty(\Omega)}$), so small that we may apply the maximum principle for domains of small volume in domains of volume less than δ. When ε is sufficiently small, we have $|\Sigma_{\lambda_0-\varepsilon} \backslash K| < \delta$. Applying this maximum principle to the function $w_{\lambda_0-\varepsilon}$ in the domain $\Sigma_{\lambda_0-\varepsilon} \backslash K$, we obtain

$$w_{\lambda_0-\varepsilon} \le 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \backslash K.$$

The strong maximum principle implies that

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon}\backslash K.$$

Putting two and two together we see that (1.4.8) holds. This, however, contradicts the choice of λ_0. The proof of the Gidas-Ni-Nirenberg theorem is complete. □

1.4.2 The sliding method: moving sub-solutions around

The sliding method differs from the moving plane method in that one compares translations of a function rather than its reflections with respect to a plane. One elementary but beautiful application of the sliding method allows to extend lower bounds obtained on a solution of a semi-linear elliptic equation in one part of a domain to a different part by moving a sub-solution around the domain and observing that it may never touch a solution. This is a very simple and powerful tool in many problems.

Lemma 1.4.4 *Let u be a positive function in an open connected set D satisfying*

$$\Delta u + f(u) \leq 0 \text{ in } D$$

with a Lipschitz function f. Let B be a ball with its closure $\bar{B} \subset D$, and suppose z is a function in $\bar{B}$ satisfying

$$\begin{aligned}
&z \leq u \text{ in } B\\
&\Delta z + f(z) \geq 0, \text{ wherever } z > 0 \text{ in } B\\
&z \leq 0 \text{ on } \partial B.
\end{aligned}$$

Then for any continuous one-parameter family of Euclidean motions (rotations and translations) $A(t)$, $0 \leq t \leq T$, so that $A(0) = \mathrm{Id}$ and $A(t)\bar{B} \subset D$ for all t, we have

$$z^t(x) := z(A(t)^{-1}x) < u(x) \text{ in } B^t := A(t)B. \tag{1.4.10}$$

Proof. The rotational invariance of the Laplace operator implies that the function z^t satisfies

$$\begin{aligned}
&\Delta z^t + f(z^t) \geq 0, \text{ wherever } z^t > 0 \text{ in } B^t\\
&z^t \leq 0 \text{ on } \partial B^t.
\end{aligned}$$

Thus the difference $w^t = z^t - u$ satisfies

$$\Delta w^t + c^t(x)w^t \geq 0 \text{ wherever } z^t > 0 \text{ in } B^t, \tag{1.4.11}$$

with c^t bounded in B^t, where, as before,

$$c^t(x) = \begin{cases} \dfrac{f(z^t(x)) - f(u(x))}{z^t(x) - u(x)}, \text{ if } z^t(x) \neq u(x)\\ 0, \text{ otherwise.}\end{cases}$$

In addition, $w^t < 0$ on ∂B^t.

We now argue by contradiction. Suppose that there is a first t so that the graph of z^t touches the graph of u at a point x_0 – such t exists by continuity. Then, for that t, we still have $w^t \le 0$ in B^t, but also $w^t(x_0) = 0$. As $u > 0$ in D, and $z^t \le 0$ on ∂B^t, the point x_0 has to be inside B^t, which means that z^t satisfies

$$\Delta z^t + f(z^t) \ge 0$$

in the whole component G of the set of points in B^t where $z^t > 0$ that contains x_0. Thus, w^t satisfies (1.4.11) in G, and, in addition, $w^t \le 0$ and $w^t(x_0) = 0$. The comparison principle implies that $w^t \equiv 0$ in G. In particular, we have $w^t(\widetilde{x}) = 0$ for all $\widetilde{x} \in \partial G$. But then

$$z^t(\widetilde{x}) = u(\widetilde{x}) > 0 \text{ on } \partial G,$$

which contradicts the fact that $z^t = 0$ on ∂G. Hence the graph of z^t may not touch that of u and (1.4.10) follows. □

Lemma 1.4.4 is often used to "slide around" a sub-solution that is positive somewhere to show that solution itself is uniformly positive. We will use it repeatedly when we talk about the reaction-diffusion equations later on. Here is an exercise (to which we will return later) on how it can be applied.

Exercise 1.4.5 Let $u(x) > 0$ be a positive bounded solution of the equation

$$u_{xx} + u - u^2 = 0$$

on the real line $x \in \mathbb{R}$. Show that if L is sufficiently large and $\lambda > 0$ is sufficiently small, then the function

$$z_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right)$$

satisfies

$$\frac{\partial^2 z_\lambda}{\partial x^2} + z_\lambda - z_\lambda^2 \ge 0, \quad 0 < x < L,$$

and $u(x) \ge z_\lambda(x)$ for all $0 \le x \le L$. Use this to conclude that

$$\inf_{x \in \mathbb{R}} u(x) > 0.$$

Try to strengthen this result to prove that $u(x) \equiv 1$.

1.4.3 Monotonicity for the Allen-Cahn equation in $\mathbb{R}^n$

Our next example, taken from the paper [17] by Berestycki, Hamel and Monneau, shows one analog of the Gidas-Ni-Nirenberg theorem in the whole space $\mathbb{R}^n$. Recall that for the latter result we have considered a semi-linear elliptic equation in a ball with the Dirichlet boundary conditions, which are compatible with radially symmetric solutions, and have shown that the only possible non-negative solutions are, indeed, radially symmetric. In the whole space we will impose boundary conditions that allow solutions to depend on just one variable, say, x_n, and will show that any solution satisfying these boundary conditions depends only on x_n.

We consider solutions of

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n \tag{1.4.12}$$

which satisfy $|u| \leq 1$ together with the asymptotic conditions

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty \text{ uniformly in } x' = (x_1, \ldots, x_{n-1}). \tag{1.4.13}$$

We assume that f is a smooth (actually, just assuming that f is Lipschitz would be sufficient) function on $[-1, 1]$, and there exists $\delta > 0$ so that

$$f \text{ is non-increasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1], \text{ and } f(\pm 1) = 0. \tag{1.4.14}$$

The standard example to keep in mind is $f(u) = u - u^3$. In that case, (1.4.12) is known as the Allen-Cahn equation. Such problems appear in many applications, ranging from biology and combustion to the differential geometry, as a very basic model of a diffusive connection between two stable states. The main feature of the nonlinearity is that the corresponding time-dependent ODE

$$\frac{du}{dt} = f(u) \tag{1.4.15}$$

has two stable solutions $u \equiv -1$ and $u \equiv 1$. Solutions of the partial differential equation (1.4.12), on the other hand, describe the diffusive transitions between regions in space where u is close to the equilibrium $u \equiv -1$ and those where u is close to $u \equiv 1$.

In one dimension, this is simply the ODE

$$u_0'' + f(u_0) = 0, \quad x \in \mathbb{R}, \tag{1.4.16}$$

with the boundary conditions

$$u_0(\pm\infty) = \pm 1. \tag{1.4.17}$$

This equation may be solved explicitly: multiplying (1.4.16) by u_0' and integrating from $-\infty$ to x, using the boundary conditions, leads to

$$\frac{1}{2}(u_0')^2 + F(u_0) = 0, \quad u_0(\pm\infty) = \pm 1. \tag{1.4.18}$$

Here, we have defined

$$F(s) = \int_{-1}^{s} f(u)du. \tag{1.4.19}$$

Letting $x \to +\infty$ in (1.4.18) we see that a necessary condition for a solution of (1.4.18) to exist is that $F(1) = 0$, or

$$\int_{-1}^{1} f(u)du = 0. \tag{1.4.20}$$

Exercise 1.4.6 Show that the solutions of (1.4.16)-(1.4.17) are unique, up to a translation in the x-variable – note that if $u_0(x)$ is a solution to (1.4.16)-(1.4.17), then so is $\widetilde{u}(x) = u_0(x+\xi)$, for any $\xi \in \mathbb{R}$.

Exercise 1.4.7 Show that if $f(u) = u - u^3$ then $u_0(x)$ has an explicit expression

$$u_0(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big), \tag{1.4.21}$$

as well as all its translates $u_0(x+\xi)$, with a fixed $\xi \in \mathbb{R}$.

Our goal is to show that the asymptotic conditions (1.4.13) imply that the positive solutions of (1.4.12) are actually one-dimensional.

Theorem 1.4.8 *Let u be any solution of (1.4.12)-(1.4.13) such that $|u| \le 1$. Then it has the form $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$u_0'' + f(u_0) = 0 \text{ in } \mathbb{R},\ u_0(\pm\infty) = \pm 1. \tag{1.4.22}$$

Moreover, u is increasing with respect to x_n. Finally, such solution is unique up to a translation.

Without the uniformity assumption in (1.4.13), that is, imposing simply

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty, \tag{1.4.23}$$

this problem is known as "the weak form" of the De Giorgi conjecture, and was resolved by Savin [129] who showed that all solutions are one-dimensional in $n \le 8$, and del Pino, Kowalczyk and Wei [51] who showed that non-planar solutions exist $n \ge 9$. Their work is well beyond the scope of this chapter.

Note that (1.4.23), without the uniformity condition for the limits at infinity as in (1.4.13), does not imply that u depends only on the variable x_n. For example, any function of the form $u(x) = u_0(e \cdot x)$, where $e \in \mathbb{S}^{n-1}$ is a fixed vector with $|e| = 1$ and $e_n > 0$, and u_0 is any solution of (1.4.22), satisfies both (1.4.12) and (1.4.23). It will not, however, satisfy the uniformity assumption in (1.4.13). The additional assumption of uniform convergence at infinity made here makes this question much easier than the weak form of the De Giorgi conjecture. Nevertheless, the proof of Theorem 1.4.8 is both non-trivial and instructive. The full De Giorgi conjecture is that any solution of (1.4.14) in dimension $n \le 8$ with $f(u) = u - u^3$ (without imposing any boundary conditions on u at all) such that $-1 \le u \le 1$ is one-dimensional. It is still open in this generality, to the best of our knowledge. The motivation for the conjecture comes from the study of the minimal surfaces in differential geometry but we will not discuss this connection here.

A maximum principle in an unbounded domain

For the proof, we will need a version of the maximum principle for unbounded domains, interesting in itself.

Lemma 1.4.9 *Let D be an open connected set in $\mathbb{R}^n$, possibly unbounded. Assume that $\bar{D}$ is disjoint from the closure of an infinite open (solid) cone Σ. Suppose that a function $z \in C(\bar{D})$ is bounded from above and satisfies*

$$\begin{aligned} &\Delta z + c(x) z \ge 0 \ \text{in } D \\ &z \le 0 \ \text{on } \partial D. \end{aligned} \tag{1.4.24}$$

with some continuous function $c(x) \le 0$, then $z \le 0$.

Proof. If the function $z(x)$ would, in addition, vanish at infinity:

$$\limsup_{|x|\to+\infty} z(x) = 0, \tag{1.4.25}$$

then the proof would be easy. Indeed, if (1.4.25) holds then we can find a sequence $R_n \to +\infty$ so that

$$\sup_{\bar{D}\cap\{|x|=R_n\}} z(x) \le \frac{1}{n}. \tag{1.4.26}$$

The usual maximum principle applied in the bounded domain $D_n = D \cap B(0; R_n)$ implies then that $z(x) \le 1/n$ in D_n since this inequality holds on ∂D_n. Letting $n \to \infty$ gives

$$z(x) \le 0 \text{ in } D.$$

Our next task is to reduce the case of a bounded function z to (1.4.25). To do this, we will construct a harmonic function $g(x) > 0$ in D such that

$$|g(x)| \to +\infty \text{ as } |x| \to +\infty. \tag{1.4.27}$$

Since g is harmonic, the ratio $\sigma = z/g$ will satisfy a differential inequality in D:

$$\Delta\sigma + \frac{2}{g}\nabla g \cdot \nabla\sigma + c\sigma \ge 0. \tag{1.4.28}$$

This is similar to (1.4.24) but now σ does satisfy the asymptotic condition

$$\limsup_{x\in D,|x|\to\infty} \sigma(x) \le 0,$$

uniformly in $x \in D$. Moreover, $\sigma \le 0$ on ∂D. Hence one may apply the above argument to the function $\sigma(x)$, and conclude that $\sigma(x) \le 0$, which, in turn, implies that $z(x) \le 0$ in D.

Exercise 1.4.10 Note that we have brazenly applied the maximum principle above to the operator in the left side of (1.4.28), while we have previously only proved it for operators of the form $\Delta + c(x)$, with $c(x) \le 0$. To remedy this, consider a function ϕ which satisfies an inequality of the form

$$\Delta\phi + b(x) \cdot \nabla\phi + c(x)\phi \ge 0 \tag{1.4.29}$$

in a bounded domain D with $c(x) \le 0$. Show that ϕ can not attain a positive maximum inside D. Hint: mimic the proof of the strong maximum principle.

In order to construct such harmonic function $g(x)$ in D, the idea is to decrease the cone Σ to a cone $\widetilde{\Sigma}$ and to consider the principal eigenfunction $\psi > 0$ of the spherical Laplace-Beltrami operator in the region $G = \mathbb{S}^{n-1}\backslash\widetilde{\Sigma}$ with $\psi = 0$ on ∂G:

$$\begin{aligned} &\Delta_S\psi + \mu\psi = 0, \quad \psi > 0 \text{ in } G,\\ &\psi = 0 \text{ on } \partial G. \end{aligned}$$

Here, Δ_S is simply the restriction of the standard Laplacian operator to functions of the angular variable only (independent of the radial variable). Existence of such an eigenvalue that corresponds to a positive eigenfunction follows from the general spectral theory of elliptic operators. We do not expect the reader to be familiar with this theory, but for the moment, in order to keep the flow of the presentation, we simply ask to take for granted that such principal eigenvalue with a positive eigenfunction exists and is unique, or consult [60].

Exercise 1.4.11 Show that $\mu > 0$.

Going to the polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{n-1}$, we now define the function

$$g(x) = r^\alpha \psi(\xi), \quad x \in D,$$

with

$$\alpha(n + \alpha - 2) = \mu.$$

This choice of α makes the function g be harmonic:

$$\Delta g = \frac{\partial^2 g}{\partial r^2} + \frac{n-1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\Delta_S g = [\alpha(\alpha - 1) + \alpha(n-1) - \mu]r^{\alpha-2}\Psi = 0.$$

Moreover, as $\mu > 0$, we have $\alpha > 0$, and it is easy to see that there exists $c_0 > 0$ such that $\psi(x) \geq c_0$ for all $x \in D$. Thus (1.4.27) also holds, and the proof is complete. □

We will need the following corollary that we will use for half-spaces.

Corollary 1.4.12 *Let f be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1 and u_2 satisfy*

$$\Delta u_i + f(u_i) = 0 \text{ in } \Omega$$

and are such that $|u_i| \leq 1$. Assume furthermore that $u_2 \geq u_1$ on $\partial\Omega$ and that either $u_2 \geq 1-\delta$ or $u_1 \leq -1+\delta$ in Ω. If $\Omega \subset \mathbb{R}^n$ is an open connected set so that $\mathbb{R}^n \backslash \bar{\Omega}$ contains an open infinite cone then $u_2 \geq u_1$ in Ω.

Proof. Assume, for instance, that $u_2 \geq 1 - \delta$, and set $w = u_1 - u_2$. Then

$$\Delta w + c(x)w = 0 \text{ in } \Omega$$

with

$$c(x) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

Note that $c(x) \leq 0$ if $w(x) \geq 0$. Indeed, if $w(x) \geq 0$, then

$$u_1(x) \geq u_2(x) \geq 1 - \delta.$$

As, in addition, we know that $u_1 \leq 1$, and f is non-increasing on $[1-\delta, 1]$, it follows that $f(u_1(x)) \leq f(u_2(x))$, and thus $c(x) \leq 0$. Hence, if the set $G = \{w > 0\}$ is not empty, we may apply the maximum principle of Lemma 1.4.9 to the function w in G (note that $w \leq 0$ on ∂G), and conclude that $w \leq 0$ in G giving a contradiction. □

Proof of Theorem 1.4.8

We are going to prove that

$$u \text{ is increasing in any direction } \nu = (\nu_1, \ldots, \nu_n) \text{ with } \nu_n > 0. \tag{1.4.30}$$

This will mean that

$$\frac{1}{\nu_n}\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} \alpha_j \frac{\partial u}{\partial x_j} > 0$$

for any choice of $\alpha_j = \nu_j/\nu_n$. It follows that all $\partial u/\partial x_j = 0$, $j = 1, \ldots, n-1$, so that u depends only on x_n, and, moreover, $\partial u/\partial x_n > 0$. Hence, (1.4.30) implies the conclusion of Theorem 1.4.8 on the monotonicity of the solution.

We now prove (1.4.30). Monotonicity in the direction ν can be restated as

$$u^t(x) \geq u(x), \text{ for all } t \geq 0 \text{ and all } x \in D, \tag{1.4.31}$$

where $u^t(x) = u(x+t\nu)$ are the shifts of the function u in the direction ν. We start the sliding method with a very large t. The uniformity assumption in the boundary condition (1.4.13) implies that there exists a real $a > 0$ so that

$$u(x', x_n) \geq 1 - \delta \text{ for all } x_n \geq a,$$

and

$$u(x', x_n) \leq -1 + \delta \text{ for all } x_n \leq -a.$$

Take $t \geq 2a/\nu_n$, then the functions u and u^t are such that

$$\begin{aligned} u^t(x', x_n) &\geq 1 - \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and all } x_n \geq -a \\ u(x', x_n) &\leq -1 + \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and all } x_n \leq -a, \end{aligned} \tag{1.4.32}$$

and, in particular,

$$u^t(x', -a) \geq u(x', -a) \text{ for all } x' \in \mathbb{R}^{n-1}. \tag{1.4.33}$$

Hence, we may apply Corollary 1.4.12 separately in the half-spaces $\Omega_1 = \{(x', x_n) : \ x_n \leq -a\}$ and $\Omega_2 = \{(x', x_n) : \ x_n \geq -a\}$. In both cases, we conclude that $u^t \geq u$ and thus

$$u^t \geq u \text{ in all of } \mathbb{R}^n \text{ for } t \geq 2a/\nu_n.$$

Following the philosophy of the sliding method, we start to decrease t, and let

$$\tau = \inf\{t > 0, \ u^t(x) \geq u(x) \text{ for all } x \in \mathbb{R}^n\}.$$

By continuity, we still have $u^\tau \geq u$ in $\mathbb{R}^n$. Note that (1.4.31) is equivalent to $\tau = 0$, and we show this by contradiction. If $\tau > 0$, there are two possibilities.

Case 1. Suppose that

$$\inf_{D_a}(u^\tau - u) > 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{1.4.34}$$

The function u is globally Lipschitz continuous – the reader may either accept that this follows from the standard elliptic estimates [60], or do the following exercise.

Exercise 1.4.13 Let $u(x)$ be a uniformly bounded solution ($|u(x)| \le M$ for all $x \in \mathbb{R}^n$) of an equation of the form

$$-\Delta u = F(u)$$

in $\mathbb{R}^n$, with a differentiable function $F(u)$. Show that there exists a constant $C > 0$ which depends on the function F so that $|\nabla u(x)| \le CM$ for all $x \in \mathbb{R}^n$. Hint: fix $y \in \mathbb{R}^n$, and let $\chi(x)$ be a smooth cut-off function supported in the ball B centered at y of radius $r = 1$. Write an equation for the function $v(x) = \chi(x)u(x)$ of the form

$$-\Delta v = g,$$

with the function g that depends on u, F and χ, use the Green's function of the Laplacian to bound $\nabla v(y)$, and deduce a uniform bound on $\nabla u(y)$. Make sure you see why you need to pass from u to v.

The Lipschitz continuity of u together with assumption (1.4.34) implies that there exists $\eta_0 > 0$ so that for all $\tau - \eta_0 < t < \tau$ we still have

$$u^t(x', x_n) > u(x', x_n) \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and for all } -a \le x_n \le a. \tag{1.4.35}$$

As $u(x', x_n) \ge 1 - \delta$ for all $x_n \ge a$, we know that

$$u^t(x', x_n) \ge 1 - \delta \text{ for all } x_n \ge a \text{ and } t > 0. \tag{1.4.36}$$

We may then apply Corollary 1.4.12 in the half-spaces $\{x_n > a\}$ and $\{x_n < -a\}$ to conclude that

$$u^{\tau-\eta}(x) \ge u(x)$$

everywhere in $\mathbb{R}^n$ for all $\eta \in [0, \eta_0]$. This contradicts the choice of τ. Thus, the case (1.4.34) is impossible.

Case 2. Suppose that

$$\inf_{D_a}(u^\tau - u) = 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{1.4.37}$$

This would be a contradiction to the maximum principle if we could conclude from (1.4.37) that the graphs of u^τ and u touch at an internal point. This, however, is not clear, as there may exist a sequence of points ξ_k with $|\xi_k| \to +\infty$, such that $u^\tau(\xi_k) - u(\xi_k) \to 0$, without the graphs ever touching. In order to deal with this issue, we will use the usual trick of moving "the interesting part" of the domain to the origin and passing to the limit. We know from (1.4.37) that there exists a sequence $\xi_k \in D_a$ so that

$$u^\tau(\xi_k) - u(\xi_k) \to 0 \text{ as } k \to \infty. \tag{1.4.38}$$

Let us re-center: set

$$u_k(x) = u(x + \xi_k).$$

Differentiating the equation for u, we may bootstrap the claim of Exercise 1.4.13 to conclude that u is uniformly bounded in $C^3(\mathbb{R}^n)$, thus so is the sequence $u_k(x)$. The Ascoli-Arzela theorem implies that $u_k(x)$ converge along a subsequence to a function $u_\infty(x)$, uniformly on compact sets, together with the first two derivatives. The limit satisfies

$$\Delta u_\infty + f(u_\infty) = 0, \tag{1.4.39}$$

and, in addition, we have, because of (1.4.38):

$$u_\infty^\tau(0) = u_\infty(0),$$

and also

$$u_\infty^\tau(x) \geq u_\infty(x), \quad \text{for all } x \in \mathbb{R}^n,$$

because $u_k^\tau \geq u_k$ for all k. As both u_∞ and u_∞^τ satisfy (1.4.39), the strong maximum principle implies that $u_\infty^\tau = u_\infty$, that is,

$$u_\infty(x + \tau\nu) = u_\infty(x) \text{ for all } x \in \mathbb{R}^n.$$

In other words, the function u_∞ is periodic in the ν-direction. However, as all ξ_k lie in D_a, their n-th components are uniformly bounded $|(\xi_k)_n| \leq a$. Therefore, when we pass to the limit we do not lose the boundary conditions in x_n: the function u_∞ must satisfy the boundary conditions (1.4.13). This is a contradiction to the above periodicity. Hence, this case is also impossible, and thus $\tau = 0$. This proves monotonicity of $u(x)$ in x_n and the fact that u depends only on x_n: $u(x) = u(x_n)$.

In order to prove the uniqueness of such solution, assuming there are two such solutions u and v, one repeats the sliding argument above but applied to the difference

$$w^\tau(x_n) = u(x_n + \tau) - v(x_n).$$

Exercise 1.4.14 Use this sliding argument to show that there exists $\tau \in \mathbb{R}^n$ such that

$$u(x_n + \tau) = v(x_n) \text{ for all } x_n \in \mathbb{R},$$

showing uniqueness of such solution, up to a shift.

This completes the proof. □

Chapter 2

Diffusion equations

2.1 Introduction to the chapter

Parabolic equations of the form

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} = f(x, u, \nabla u), \tag{2.1.1}$$

are ubiquitous in mathematics and various applications in physics, biology, economics and other fields. While there are many textbooks on the subject, ranging from the most elementary to extremely advanced, most of them concentrate on the (highly non-trivial) questions of the existence and regularity of the solutions. We have chosen instead to focus on some striking qualitative properties of the solutions that, nevertheless, can be proved with almost no background in analysis, using only the very basic regularity results. The unifying link in this chapter will be the parabolic maximum principle and the Harnack inequality. Together with the parabolic regularity, they will be responsible for the seemingly very special behavior that we will observe in the solutions of these equations.

The chapter starts with a probabilistic introduction. While we do not try to motivate the basic diffusion equations by models in the applied sciences here, an interested reader would have no difficulty finding the connections between such equations and models in physics, biology, chemistry and ecology in many basic textbooks. On the other hand, the parabolic equations have a deep connection with probability. Indeed, some of the most famous results in the parabolic regularity theory were proved by probabilistic tools. It is, therefore, quite natural to introduce the chapter by explaining how the basic linear models arise, in a very simple manner, from limits of a random walk. We reassure the reader that the motivation from the physical or life sciences will not be absent from this book, as some of the later chapters will precisely be motivated by problems in fluid mechanics or biology.

The probabilistic section is followed by a brief interlude on the maximum principle. There is nothing original in the exposition, and we do not even present the proofs, as they can be found in many textbooks on PDE. We simply recall the statements that we will need.

We then proceed to the section on the existence and regularity theory for the nonlinear heat equations: the reaction-diffusion equations and viscous Hamilton-Jacobi equations. They arise in many models in physical and biological sciences, and our "true" interest is in the qualitative behavior of their solutions, as these reflect the corresponding natural phenomena. However, an unfortunate feature of the nonlinear partial differential equations is that, before talking knowledgeably about their solutions or their behavior, one first has to prove that they exist. This will, as a matter of fact, be a central problem in the last two chapters of this book, where we look at the fluid mechanics models, for which the existence of the solutions is quite non-trivial. As the reaction-diffusion equations that we have in mind here and in Chapter **??** both belong to a very well studied class and are much simpler, it would not be inconceivable to brush their existence theory under the rug, invoking some respectable treatises. This would not be completely right, for several reasons. The first is that we do not want to give the impression that the theory is inaccessible: it is quite simple and can be explained very easily. The second reason is that we wish to explain both the power and the limitation of the parabolic regularity theory, so that the difficulty of the existence issues for the fluid mechanics models in the latter chapters would be clearer to the reader. The third reason is more practical: even for the qualitative properties that we aim for, we still need to estimate derivatives. So, it is better to say how this is done.

The next section contains a rather informal guide to the regularity theory for the parabolic equations with inhomogeneous coefficients. We state the results we will need later, and outline the details of some of the main ideas needed for the proofs without presenting them in full – they can be found in the classical texts we mention below. We hope that by this point the reader will be able to study the proofs in these more advanced textbooks without losing sight of the main ideas. This section also contains the Harnack inequality. What is slightly different here is the statement of a (non-optimal) version of the Harnack inequality that will be of an immediate use to us in the first main application of this chapter, the convergence to the steady solutions in the one-dimensional Allen-Cahn equations on the line. The reason we have chosen this example is that it really depends on nothing else than the maximum principle and the Harnack inequality, illustrating how far reaching this property is. It is also a perfect example of how a technical information, such as bounds on the derivatives, has a qualitative implication – the long time behavior of the solutions.

The next section concerns the principal eigenvalue of the second order elliptic operators, a well-treated subject in its own right. We state the Krein-Rurman theorem and, in order to show the reader that we are not using any machinery heavier than the results we want to prove, we provide a proof in the context of the second order elliptic and parabolic operators. It shares many features with the convergence proof of the preceding section, and we hope the reader will realize the ubiquitous character of the ideas presented. We then treat another case of the large time behavior of the solutions of the viscous Hamilton-Jacobi equations, a class of nonlinear diffusion equations. Here, the convergence will not be to a steady state but to a one-parameter family of special solutions. The main challenge will be in the existence proof of this family of solutions, that relies on the Krein-Rutman theorem. Once it is at hand, the convergence will follow, with the same ideas as those of the preceding section. Finally, we end this chapter with a brief discussion of the viscosity solutions of the first order Hamilton-Jacobi equations, to give a glimpse of the difficulties encountered when the diffusion coefficient vanishes.

This chapter is rather long so we ask the reader to be prepared to persevere through the more technical places, with the hope that in the end the reader will find the effort rewarding.

A note on notation. We will follow throughout the book the summation convention:

the repeated indices are always summed over, unless specified otherwise. In particular, we will usually write equations such as (2.1.1) as

$$\frac{\partial u}{\partial t} - a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x)\frac{\partial u}{\partial x_j} = f(x, u, \nabla u), \tag{2.1.2}$$

or

$$\frac{\partial u}{\partial t} - a_{ij}(x)\partial_{x_i}\partial_{x_j} u + b_j(x)\partial_{x_j} u = f(x, u, \nabla u). \tag{2.1.3}$$

We hope the reader will get accustomed to this convention sufficiently fast so that it causes no confusion or inconvenience.

2.2 A probabilistic introduction to the evolution equations

Let us explain informally how the linear equations of the form (2.1.2), with $g \equiv 0$ arise from random walks, in a very simple way, in the spirit of what we have done for the elliptic equations in the previous chapter. One should emphasize that many of the qualitative properties of the solutions of the parabolic and integral equations, such as the maximum principle and regularity, on a very informal level, are an "obvious" consequence of the microscopic random walk model. For simplicity, we will mostly consider the one-dimensional case, the reader can, and should, generalize this approach to higher dimensions – this is quite straightforward.

Discrete equations and random walks

The starting point in our derivation of the evolution equations is a discrete time Markov jump process $X_{n\tau}$, with a time step $\tau > 0$, defined on a lattice with mesh size h:

$$h\mathbb{Z} = \{0, \pm h, \pm 2h, \dots\}.$$

The particle position evolves as follows: if the particle is located at a position $x \in h\mathbb{Z}$ at the time $t = n\tau$ then at the time $t = (n+1)\tau$ it jumps to a random position $y \in h\mathbb{Z}$, with the transition probability

$$P(X_{(n+1)\tau} = y | \; X_{n\tau} = x) = k(x - y), \quad x, y \in h\mathbb{Z}. \tag{2.2.1}$$

Here, $k(x)$ is a prescribed non-negative kernel such that

$$\sum_{y \in h\mathbb{Z}} k(y) = 1. \tag{2.2.2}$$

The classical symmetric random walk with a spatial step h and a time step τ corresponds to the choice $k(\pm h) = 1/2$, and $k(y) = 0$ otherwise – the particle may only jump to the nearest neighbor on the left and on the right, with equal probabilities.

In order to connect this process to an evolution equation, let us take a function $f : h\mathbb{Z} \to \mathbb{R}$, defined on our lattice, and introduce

$$u(t, x) = \mathbb{E}(f(X_t(x))). \tag{2.2.3}$$

Here, $X_t(x)$, $t \in \tau\mathbb{N}$, is the above Markov process starting at a position $X_0(x) = x \in h\mathbb{Z}$ at the time $t = 0$. If $f \geq 0$ then one may think of $u(t, x)$ as the expected value of a "prize" to be collected at the time t at a (random) location of $X_t(x)$ given that the process starts at the point x at the time $t = 0$. An important special case is when f is the characteristic function of a set A. Then, $u(t, x)$ is the probability that the jump process $X_t(x)$ that starts at the position $X_0 = x$ is inside the set A at the time t.

As the process $X_t(x)$ is Markov, the function $u(t, x)$ satisfies the following relation

$$u(t+\tau, x) = \mathbb{E}(f(X_{t+\tau}(x))) = \sum_{y \in h\mathbb{Z}} P(X_\tau = y | X_0 = x)\mathbb{E}(f(X_t(y))) = \sum_{y \in h\mathbb{Z}} k(x-y)u(t, y). \tag{2.2.4}$$

This is because after the initial step when the particle jumps at the time τ from the starting position x to a random position y, the process "starts anew", and runs for time t between the times τ and $t + \tau$. Equation (2.2.4) can be re-written, using (2.2.2) as

$$u(t+\tau, x) - u(t, x) = \sum_{y \in h\mathbb{Z}} k(x-y)[u(t, y) - u(t, x)]. \tag{2.2.5}$$

The key point of this section is that the discrete equation (2.2.5) leads to various interesting continuum limits as $h \downarrow 0$ and $\tau \downarrow 0$, depending on the choice of the transition kernel $k(y)$, and on the relative size of the spatial mesh size h and the time step τ. In other words, depending on the microscopic model – the particular properties of the random walk, we will end up with different macroscopic continuous models.

The heat equation and random walks

Before showing how a general parabolic equation with non-constant coefficients can be obtained via a limiting procedure from a random walk on a lattice, let us show how this can be done for the heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \tag{2.2.6}$$

with a constant diffusivity constant $a > 0$. We will assume that the transition probability kernel has the form

$$k(x) = \phi\left(\frac{x}{h}\right), \quad x \in h\mathbb{Z}, \tag{2.2.7}$$

with a non-negative function $\phi(m) \geq 0$ defined on $\mathbb{Z}$, such that

$$\sum_m \phi(m) = 1. \tag{2.2.8}$$

This form of $k(x)$ allows us to re-write (2.2.5) as

$$u(t+\tau, x) - u(t, x) = \sum_{y \in h\mathbb{Z}} \phi\left(\frac{x-y}{h}\right)[u(t, y) - u(t, x)], \tag{2.2.9}$$

or, equivalently,

$$u(t+\tau, x) - u(t, x) = \sum_{m \in \mathbb{Z}} \phi(m)[u(t, x - mh) - u(t, x)]. \tag{2.2.10}$$

In order to arrive to the heat equation in the limit, we will make the assumption that jumps are symmetric on average:

$$\sum_{m\in\mathbb{Z}} m\phi(m) = 0. \tag{2.2.11}$$

Then, expanding the right side of (2.2.10) in h and the left side in τ, we obtain

$$\tau\frac{\partial u(t,x)}{\partial t} = \frac{ah^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \text{lower order terms}, \tag{2.2.12}$$

with

$$a = \sum_m |m|^2\phi(m). \tag{2.2.13}$$

To balance the left and the right sides of (2.2.12), we need to take the time step $\tau = h^2$ – note that the scaling $\tau = O(h^2)$ is essentially forced on us if we want to balance the two sides of this equation. Then, in the limit $\tau = h^2 \downarrow 0$, we obtain the heat equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{a}{2}\frac{\partial^2 u(t,x)}{\partial x^2}. \tag{2.2.14}$$

The diffusion coefficient a given by (2.2.13) is the second moment of the jump size – in other words, it measures the "overall jumpiness" of the particles. This is a very simple example of how the microscopic information, the kernel $\phi(m)$, translates into a macroscopic quantity – the overall diffusion coefficient a in the macroscopic equation (2.2.14).

Exercise 2.2.1 Show that if (2.2.11) is violated and

$$b = \sum_{m\in\mathbb{Z}} m\phi(m) \neq 0, \tag{2.2.15}$$

then one needs to take $\tau = h$, and the (formal limit) is the advection equation

$$\frac{\partial u(t,x)}{\partial t} + b\frac{\partial u(t,x)}{\partial x} = 0, \tag{2.2.16}$$

without any diffusion.

Exercise 2.2.2 A reader familiar with the basic probability theory should relate the limit in (2.2.16) to the law of large numbers and explain the relation $\tau = h$ in these terms. How can (2.2.14) and the relation $\tau = h^2$ between the temporal and spatial steps be explained in terms of the central limit theorem?

Parabolic equations with variable coefficients and drifts and random walks

In order to connect a linear parabolic equation with inhomogeneous coefficients, such as (2.1.2) with the right side $g \equiv 0$:

$$\frac{\partial u}{\partial t} - a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} = 0, \tag{2.2.17}$$

to a continuum limit of random walks, we consider a slight modification of the microscopic dynamics that led to the heat equation in the macroscopic limit. We go back to (2.2.4):

$$u(t+\tau,x) = \mathbb{E}(f(X_{t+\tau}(x))) = \sum_{y\in h\mathbb{Z}} P(X_\tau = y|X_0 = x)\mathbb{E}(f(X_t(y))) = \sum_{y\in h\mathbb{Z}} k(x,y)u(t,y). \tag{2.2.18}$$

Here, $k(x,y)$ is the probability to jump to the position y from a position x. Note that we no longer assume that the law of the jump process is spatially homogeneous: the transition probabilities depend not only on the difference $x-y$ but both on x and y. However, we will assume that $k(x,y)$ is "locally homogeneous". This condition translates into taking it of the form

$$k(x,y;h) = \phi(x, \frac{x-y}{h}; h). \tag{2.2.19}$$

The "slow" spatial dependence of the transition probability density is encoded in the dependence of the function $\phi(x,z,h)$ on the "macroscopic" variable x, while its "fast" spatial variations are described by the dependence of $\phi(x,z,h)$ on the variable z.

Exercise 2.2.3 Make sure you can interpret this point. Think of "freezing" the variable x and only varying the z-variable.

We will soon see why we introduce the additional dependence of the transition density on the mesh size h – this will lead to a non-trivial first order term in the parabolic equation we will obtain in the limit. We assume that the function $\phi(x,m;h)$, with $x\in\mathbb{R}$, $m\in\mathbb{Z}$ and $h\in(0,1)$, satisfies

$$\sum_{m\in\mathbb{Z}} \phi(x,m;h) = 1 \text{ for all } x\in\mathbb{R} \text{ and } h\in(0,1), \tag{2.2.20}$$

which leads to the analog of the normalization (2.2.2):

$$\sum_{y\in h\mathbb{Z}} k(x,y) = 1 \text{ for all } x\in h\mathbb{Z}. \tag{2.2.21}$$

This allows us to re-write (2.2.18) in the familiar form

$$u(t+\tau,x) - u(t,x) = \sum_{y\in h\mathbb{Z}} \phi(x, \frac{x-y}{h}; h)[u(t,y) - u(t,x)], \tag{2.2.22}$$

or, equivalently,

$$u(t+\tau,x) - u(t,x) = \sum_{m\in\mathbb{Z}} \phi(x,m;h)[u(t,x-mh) - u(t,x)], \tag{2.2.23}$$

We will make the assumption that the average asymmetry of the jumps is of the size h. In other words, we suppose that

$$\sum_{m\in\mathbb{Z}} m\phi(x,m;h) = b(x)h + O(h^2), \tag{2.2.24}$$

that is,

$$\sum_{m\in\mathbb{Z}} m\phi(x,m;0) = 0 \text{ for all } x\in\mathbb{R} \ ,$$

and

$$b(x)=\sum_{m\in\mathbb{Z}} m\frac{\partial\phi(x,m;h=0)}{\partial h} \tag{2.2.25}$$

is a given smooth function. The last assumption we will make is that the time step is $\tau = h^2$, as before. Expanding the left and the right side of (2.2.23) in h now leads to the parabolic equation

$$\frac{\partial u}{\partial t} = -b(x)\frac{\partial u(t,x)}{\partial x} + a(x)\frac{\partial^2 u(t,x)}{\partial x^2}, \tag{2.2.26}$$

with

$$a(x)=\frac{1}{2}\sum_{m\in\mathbb{Z}}|m|^2\phi(x,m;h=0). \tag{2.2.27}$$

This is a parabolic equation of the form (2.1.2) in one dimension. We automatically satisfy the condition $a(x) > 0$ (known as the ellipticity condition) unless $\phi(x,m;h=0)=0$ for all $m \in \mathbb{Z}\setminus\{0\}$. That is, $a(x) = 0$ only at the positions where the particles are completely stuck and can not jump at all. Note that the asymmetry in (2.2.24), that is, the mismatch in the typical jump sizes to the left and right, leads to the first order term in the limit equation (2.2.26) – because of that the first-order coefficient $b(x)$ is known as the drift, while the second-order coefficient $a(x)$ (known as the diffusivity) measures "the overall jumpiness" of the particles, as seen from (2.2.27).

Exercise 2.2.4 Relate the above considerations to the method of characteristics for the first order linear equation

$$\frac{\partial u}{\partial t} + b(x)\frac{\partial u}{\partial x} = 0.$$

How does it arise from similar considerations?

Exercise 2.2.5 It is straightforward to generalize this construction to higher dimensions leading to general parabolic equations of the form (2.1.2). Verify that the diffusion matrices $a_{ij}(x)$ in (2.1.2) that arise in this fashion, will always be nonnegative, in the sense that for any $\xi \in \mathbb{R}^n$ and all x, we have (once again, as the repeated indices are summed over):

$$a_{ij}(x)\xi_i\xi_j \geq 0. \tag{2.2.28}$$

This is very close to the lower bound in the ellipticity condition on the matrix $a_{ij}(x)$ which says that there exists a constant $c > 0$ so that for any $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have

$$c|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq c^{-1}|\xi|^2. \tag{2.2.29}$$

We see that the ellipticity condition appears very naturally in the probabilistic setting.

Summarizing, we see that parabolic equations of the form (2.1.2) arise as limits of random walks that make jumps of the size $O(h)$, with a time step $\tau = O(h^2)$. Thus, the overall number of jumps by a time $t = O(1)$ is very large, and each individual jump is very small. The drift vector $b_j(x)$ appears from the local non-zero mean of the jump direction and size, and the diffusivity matrix $a_{ij}(x)$ measures the typical jump size. In addition, the diffusivity matrix is nonegative-definite: condition (2.2.28) is satisfied.

Parabolic equations and branching random walks

Let us now explain how random walks can lead to parabolic equations with a zero-order term:

$$\frac{\partial u}{\partial t} - a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = 0. \tag{2.2.30}$$

This will help us understand qualitatively the role of the coefficient $c(x)$. Once again, we will consider the one-dimensional case for simplicity, and will only give the details for the case

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + c(x)u = 0, \tag{2.2.31}$$

as the non-constant diffusion matrix $a_{ij}(x)$ and drift $b_j(x)$ can be treated exactly as in the case $c = 0$.

In order to incorporate the zero order term we need to allow the particles not only jump but also branch – this is the reason why the zero-order term will appear in (2.2.30). As before, our particles make jumps on the lattice $h\mathbb{Z}$, at the discrete times $t \in \tau\mathbb{N}$. We start at $t = 0$ with one particle at a position $x \in h\mathbb{Z}$. Let us assume that at the time $t = n\tau$ we have a collection of N_t particles $X_1(t,x), \dots, X_{N_t}(t,x)$ (the number N_t is random, as will immediately see). At the time t, each particle $X_m(t,x)$ behaves independently from the other particles. With the probability

$$p_0 = 1 - |c(X_m(t))|\tau,$$

it simply jumps to a new location $y \in h\mathbb{Z}$, chosen with the transition probability $k(X_m(t)-y)$, as in the process with no branching. If the particle at $X_m(t,x)$ does not jump – this happens with the probability $p_1 = 1 - p_0$, there are two possibilities. If $c(X_m(t)) < 0$, then it is replaced by two particles at the same location $X_m(t,x)$ that remain at this position until the time $t+\tau$. If $c(X_m(t)) > 0$ and the particle does not jump, then it is removed. This process is repeated independently for all particles $X_1(t,x), \dots, X_{N_t}(t,x)$, giving a new collection of particles at the locations $X_1(t+\tau,x), \dots, X_{N_{t+\tau}}(t+\tau,x)$ at the time $t+\tau$. If $c(x) > 0$ at some positions, then the process can terminate when there are no particles left. If $c(x) \le 0$ everywhere, then the process continues forever.

To connect this particle system to an evolution equation, given a function f, we define, for $t \in \tau N$, and $x \in h\mathbb{Z}$,

$$u(t,x) = \mathbb{E}[f(X_1(t,x)) + f(X_2(t,x)) + \cdots + f(X_{N_t}(t,x))].$$

The convention is that $f = 0$ inside the expectation if there are no particles left. This is similar to what we have done for particles with no branching. If f is the characteristic function of a set A, then $u(t,x)$ is the expected number of particles inside A at the time $t > 0$.

In order to get an evolution equation for $u(t,x)$, we look at the initial time when we have just one particle at the position x: if $c(x) \le 0$, then this particle either jumps or branches, leading to the balance

$$u(t+\tau,x) = (1 + c(x)\tau)\sum_{y \in h\mathbb{Z}} k(x-y)u(t,y) - 2c(x)\tau u(t,x), \quad \text{if } c(x) \le 0, \tag{2.2.32}$$

which is the analog of (2.2.4). If $c(x) > 0$ the particle either jumps or is removed, leading to

$$u(t+\tau,x) = (1-|c(x)|\tau)\sum_{y\in h\mathbb{Z}} k(x-y)u(t,y). \tag{2.2.33}$$

In both cases, we can re-write the balances similarly to (2.2.5):

$$u(t+\tau,x) - u(t,x) = (1-|c(x)|\tau)\sum_{y\in h\mathbb{Z}} k(x-y)(u(t,y)-u(t,x)) - c(x)\tau u(t,x). \tag{2.2.34}$$

We may now take the transition probability kernel of the familiar form

$$k(x) = \phi(\frac{x}{h}),$$

with a function $\phi(m)$ as in (2.2.7)-(2.2.8). Taking $\tau = h^2$ leads, as in (2.2.12), to the diffusion equation but now with a zero-order term:

$$\frac{\partial u}{\partial t} = \frac{a}{2}\frac{\partial^2 u}{\partial x^2} - c(x)u. \tag{2.2.35}$$

Thus, the zero-order coefficient $c(x)$ can be interpreted as the branching (or killing, depending on the sign of $c(x)$) rate of the random walk. The elliptic maximum principle for $c(x) \geq 0$ that we have seen in the previous chapter, simply means, on this informal level, that if the particles never branch, and can only be removed, their expected number can not grow in time.

Exercise 2.2.6 Add branching to the random walk we have discussed in Section 2.2 of this chapter, and obtain a more general parabolic equation, in higher dimensions:

$$\frac{\partial u}{\partial t} - a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = 0. \tag{2.2.36}$$

2.3 The maximum principle interlude: the basic statements

As the parabolic maximum principle underlies most of the parabolic existence and regularity theory, we first recall some basics on the maximum principle for parabolic equations. They are very similar in spirit to what we have described in the previous chapter for the Laplace and Poisson equations. This material can, once again, be found in many standard textbooks, such as [60], so we will not present most of the proofs but just recall the statements we will need.

We consider a (more general than the Laplacian) elliptic operator of the form

$$Lu(x) = -a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(t,x)\frac{\partial u}{\partial x_j}, \tag{2.3.1}$$

in a bounded domain $x \in \Omega \subset \mathbb{R}^n$ and for $0 \leq t \leq T$. Note that the zero-order coefficient is set to be zero for the moment. As we have mentioned, the ellipticity of L means that the matrix $a_{ij}(t,x)$ is uniformly positive-definite and bounded. That is, there exist two positive constants $\lambda > 0$ and $\Lambda > 0$ so that, for any $\xi \in \mathbb{R}^n$, and $0 \le t \le T$, and any $x \in \Omega$, we have

$$\lambda|\xi|^2 \le a_{ij}(t,x)\xi_i\xi_j \le \Lambda|\xi|^2. \tag{2.3.2}$$

We also assume that all coefficients $a_{ij}(t,x)$ and $b_j(t,x)$ are continuous and uniformly bounded. Given a time $T > 0$, define the parabolic cylinder $\Omega_T = [0,T) \times \Omega$ and its parabolic boundary as

$$\Gamma_T = \{x \in \Omega,\ 0 \le t \le T:\ \text{either } x \in \partial\Omega \text{ or } t = 0\}.$$

In other words, Γ_T is the part of the boundary of Ω_T without "the top" $\{(t,x):\ t = T, x \in \Omega\}$.

Theorem 2.3.1 *(The weak maximum principle) Let a function $u(t,x)$ satisfy*

$$\frac{\partial u}{\partial t} + Lu \le 0, \quad x \in \Omega, \quad 0 \le t \le T, \tag{2.3.3}$$

and assume that Ω is a smooth bounded domain. Then $u(t,x)$ attains its maximum over Ω_T on the parabolic boundary Γ_T, that is,

$$\sup_{\Omega_T} u(t,x) = \sup_{\Gamma_T} u(t,x). \tag{2.3.4}$$

As in the elliptic case, we also have the strong maximum principle.

Theorem 2.3.2 *(The strong maximum principle) Let a smooth function $u(t,x)$ satisfy*

$$\frac{\partial u}{\partial t} + Lu = 0, \quad x \in \Omega, \quad 0 \le t \le T, \tag{2.3.5}$$

in a smooth bounded domain Ω. Then if $u(t,x)$ attains its maximum over $\bar{\Omega}_T$ at an interior point $(t_0,x_0) \notin \Gamma_T$ then $u(t,x)$ is equal to a constant in Ω_T.

We will not prove these results here, the reader may consult [60] or other standard textbooks on PDEs for a proof. One standard generalization of the maximum principle is to include the lower order term with a sign, as in the elliptic case – compare to Theorem 1.3.4 in Chapter 1. Namely, it is quite straightforward to show that if $c(x) \ge 0$ then the maximum principle still holds for parabolic equations (2.3.5) with an operator L of the form

$$Lu(x) = -a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u. \tag{2.3.6}$$

The proof can, once again, be found in [60]. However, as we have seen in the elliptic case, in the maximum principles for narrow domains (Theorem 1.3.7 in Chapter 1) and domains of a small volume (Theorem 1.3.9 in the same chapter), the sign condition on the coefficient $c(t,x)$ is not necessary for the maximum principle to hold. Below, we will discuss a more general condition that quantifies the necessary assumptions on the operator L for the maximum principle to hold in a unified way.

A consequence of the maximum principle is the comparison principle, a result that holds also for operators with zero order coefficients and in unbounded domains. In general, the comparison principle in unbounded domains holds under a proper restriction on the growth of the solutions at infinity. Here, for simplicity we assume that the solutions are uniformly bounded.

Theorem 2.3.3 *Let the smooth uniformly bounded functions $u(t,x)$ and $v(t,x)$ satisfy*

$$\frac{\partial u}{\partial t} + Lu + c(t,x)u \geq 0, \quad 0 \leq t \leq T, \quad x \in \Omega \tag{2.3.7}$$

and

$$\frac{\partial v}{\partial t} + Lv + c(t,x)v \leq 0, \quad 0 \leq t \leq T, \quad x \in \Omega, \tag{2.3.8}$$

in a smooth (and possibly unbounded) domain Ω, with a bounded function $c(t,x)$. Assume that $u(0,x) \geq v(0,x)$ and

$$u(t,x) \geq v(t,x) \text{ for all } 0 \leq t \leq T \text{ and } x \in \partial\Omega.$$

Then, we have

$$u(t,x) \geq v(t,x) \text{ for all } 0 \leq t \leq T \text{ and all } x \in \Omega.$$

Moreover, if in addition, $u(0,x) > v(0,x)$ on an open subset of Ω then $u(t,x) > v(t,x)$ for all $0 < t < T$ and all $x \in \Omega$.

The assumption that both $u(t,x)$ and $v(t,x)$ are uniformly bounded is important if the domain Ω is unbounded – without this condition even the Cauchy problem for the standard heat equation in $\mathbb{R}^n$ may have more than one solution, while the comparison principle implies uniqueness trivially. An example of non-uniqueness is discussed in detail in [87] – such solutions grow very fast as $|x| \to +\infty$ for any $t > 0$, while the initial condition $u(0,x) \equiv 0$. The extra assumption that $u(t,x)$ is bounded allows to rule out this non-uniqueness issue. Note that the special case $\Omega = \mathbb{R}^n$ is included in Theorem 2.3.3, and in that case only the comparison at the initial time $t = 0$ is needed for the conclusion to hold for bounded solutions. Once again, a reader not interested in treating the proof as an exercise should consult [60], or another of his favorite basic PDE textbooks. We should stress that in the rest of this book we will only consider solutions, for which the uniqueness holds.

A standard corollary of the parabolic maximum principle is the following estimate.

Exercise 2.3.4 Let Ω be a (possibly unbounded) smooth domain, and $u(t,x)$ be the solution of the initial boundary value problem

$$\begin{aligned} &u_t + Lu + c(t,x)u = 0, \text{ in } \Omega, \\ &u(t,x) = 0 \text{ for } x \in \partial\Omega, \\ &u(0,x) = u_0(x). \end{aligned} \tag{2.3.9}$$

Assume (to ensure the uniqueness of the solution) that u is locally in time bounded: for all $T > 0$ there exists $C_T > 0$ such that $|u(t,x)| \leq C_T$ for all $t \in [0,T]$ and $x \in \Omega$. Assume that the function $c(t,x)$ is bounded, with $c(t,x) \geq -M$ for all $x \in \Omega$, then $u(t,x)$ satisfies

$$|u(t,x)| \leq \|u_0\|_{L^\infty} e^{Mt}, \quad \text{for all } t > 0 \text{ and } x \in \Omega. \tag{2.3.10}$$

The estimate (2.3.10) on the possible growth (or decay) of the solution of (2.3.9) is by no means optimal, and we will soon see how it can be improved.

We also have the parabolic Hopf Lemma, of which we will only need the following version.

Lemma 2.3.5 *(The parabolic Hopf Lemma) Let $u(t,x) \geq 0$ be a solution of*

$$u_t + Lu + c(t,x)u = 0, \quad 0 \leq t \leq T,$$

in a ball $B(z,R)$. Assume that there exists $t_0 > 0$ and $x_0 \in \partial B(z,R)$ such that $u(t_0,x_0) = 0$, then we have

$$\frac{\partial u(t_0,x_0)}{\partial \nu} < 0. \tag{2.3.11}$$

The proof is very similar to that of the elliptic Hopf Lemma, and can be found, for instance, in [82].

2.4 Regularity for the nonlinear heat equations

The regularity theory for the parabolic equations is an extremely rich and fascinating subject that is often misunderstood as "technical". To keep things relatively simple, we are not going to delve into it head first. Rather, we have in mind two particular parabolic models, for which we would like to understand the large time behavior: the semi-linear and quasi-linear equations of the simplest form. The truth is that these two examples contain some of the main features under which the more general global existence and regularity results hold: the Lipschitz behavior of the nonlinearity, and the smooth spatial dependence of the coefficients in the equation.

2.4.1 The forced linear heat equation

Before we talk about the theory of the semi-linear and quasi-linear diffusion equations, let us have a look at the forced linear heat equation

$$u_t = \Delta u + g(t,x), \tag{2.4.1}$$

posed in the whole space $x \in \mathbb{R}^n$, and with an initial condition

$$u(0,x) = u_0(x). \tag{2.4.2}$$

The immediate question for us is how regular the solution of (2.4.1)-(2.4.2) is, in terms of the regularity of the initial condition $u_0(x)$ and the forcing term $g(t,x)$. The function $u(t,x)$ is given explicitly by the Duhamel formula

$$u(t,x) = v(t,x) + \int_0^t w(t,x;s)ds. \tag{2.4.3}$$

Here, $v(t,x)$ is the solution of the homogeneous heat equation

$$v_t = \Delta v, \quad x \in \mathbb{R}^n, \ t > 0, \tag{2.4.4}$$

with the initial condition $v(0,x) = u_0(x)$, and $w(t,x;s)$ is the solution of the Cauchy problem

$$w_t(t,x;s) = \Delta w(t,x;s), \quad x \in \mathbb{R}^n, \ t > s, \tag{2.4.5}$$

that runs starting at the time s, and is supplemented by the initial condition at $t = s$:

$$w(t = s, x; s) = g(s, x). \tag{2.4.6}$$

Let us denote the solution of the Cauchy problem (2.4.4) as

$$v(t, x) = e^{t\Delta} u_0. \tag{2.4.7}$$

This defines the operator $e^{t\Delta}$. It maps the initial condition to the heat equation to its solution at the time t, and is given explicitly as

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-(x-y)^2/(4t)} f(y) dy. \tag{2.4.8}$$

With this notation, another way to write the Duhamel formula (2.4.1) is

$$u(t, x) = e^{t\Delta} u_0(x) + \int_0^t e^{(t-s)\Delta} g(s, x) ds, \tag{2.4.9}$$

or, more explicitly:

$$u(t, x) = \frac{1}{(4\pi t)^{n/2}} \int e^{-(x-y)^2/(4t)} u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} g(s, y) dy ds. \tag{2.4.10}$$

The first term in (2.4.10) is rather benign as far as regularity is concerned.

Exercise 2.4.1 Show that if the initial condition $u_0(y)$ is continuous and bounded then the function $v(t, x)$ given by the first integral in the right side of (2.4.10) is infinitely differentiable in t and x for all $t > 0$ and $x \in \mathbb{R}^n$.

The second term in (2.4.10),

$$J(t, x) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} g(s, y) dy ds \tag{2.4.11}$$

is potentially more problematic because of the term $(t-s)^{-n/2}$ that blows up as $s \uparrow t$. A simple change of variables shows that if $g(t, x)$ is uniformly bounded then so is $J(t, x)$:

$$|J(t, x)| \leq \|g\|_{L^\infty} \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} dy ds = \frac{t\|g\|_{L^\infty}}{\pi^{n/2}} \int e^{-z^2} dz = t\|g\|_{L^\infty}. \tag{2.4.12}$$

Exercise 2.4.2 Deduce this upper bound for $J(t, x)$ directly from the parabolic maximum principle, without explicit computations.

Let us see what we can say the regularity of $J(t, x)$.

Proposition 2.4.3 *Let $g(t, x)$ be a bounded uniformly continuous function over $[0, T] \times \mathbb{R}^n$, that satisfies*

$$|g(t, x) - g(t', x')| \leq C\big(|t - t'|^{\alpha/2} + |x - x'|^\alpha\big) \tag{2.4.13}$$

for some $\alpha \in (0, 1)$. Then $J(t, x)$ given by (2.4.11) is twice continuously differentiable in x, and once continuously differentiable in t over $(0, T) \times \mathbb{R}^n$, with the corresponding derivatives bounded over every interval of the form $[\varepsilon, T]$, $0 < \varepsilon < T$.

Note the difference in the Hölder exponents in t and x in the assumption (2.4.13) on the function $g(t,x)$. It will be clear from the proof how this disparity comes about. More generally, it is related to the different scaling of the heat equation and other parabolic problems in time and space.

Proof. The first inclination in an attempt to show the regularity of $J(t,x)$ would be to simply differentiate the integrand in (2.4.11). Let us fix some $1 \le i \le n$, and compute

$$\frac{\partial}{\partial x_i}\Big(\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}}\Big) = -\frac{x_i-y_i}{2(t-s)}\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}},$$

so that after differentiation the integrand can be bounded as

$$\Big|\frac{\partial}{\partial x_i}\Big(\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}}\Big)g(s,y)\Big| \le \frac{\|g\|_{L^\infty}}{\sqrt{t-s}}\frac{|z|e^{-|z|^2}}{(4\pi(t-s))^{n/2}}, \qquad z=\frac{x-y}{\sqrt{t-s}}. \tag{2.4.14}$$

As the volume element has the form

$$dz = \frac{dx}{(t-s)^{n/2}},$$

this shows that the x_i-derivative of the integrand is dominated by an integrable function in t and z.

Exercise 2.4.4 Use estimate (2.4.14) to conclude that $J(t,x)$ is C^1 in the x-variables, with bounded derivatives. Note that this bound does not use any information on the function $g(t,x)$ except that it is uniformly bounded. In other words, even if we only assume that $g(t,x)$ is a uniformly bounded measurable function, the solution of the heat equation will be continuously differentiable in x.

The above argument can not be repeated for the time derivative: if we differentiate the integrand in time, and make the same change of variable to z as in (2.4.14), that would bring a non-integrable $(t-s)^{-1}$ singularity instead of a $(t-s)^{-1/2}$ term, as before.

Exercise 2.4.5 Verify that differentiating the integrand twice in x leads to the same kind of singularity in $(t-s)$ as differentiating once in t.

Our strategy, instead, will be to take $\delta \in (0,t)$ small, and consider an approximation

$$J_\delta(t,x) = \int_0^{t-\delta} e^{(t-s)\Delta}g(s,.)(x)ds = \int_0^{t-\delta}\int_{\mathbb{R}^n}\frac{1}{(4\pi(t-s))^{n/2}}e^{-(x-y)^2/(4(t-s))}g(s,y)dyds. \tag{2.4.15}$$

Note that $J_\delta(s,x)$ is the solution of the Cauchy problem (in the variable s, with t fixed)

$$\frac{\partial J_\delta}{\partial s} = \Delta J_\delta + H(t-s-\delta)g(s,x), \tag{2.4.16}$$

with the initial condition $J_\delta(0,x)=0$. Here, we have introduced the cut-off $H(s)=1$ for $s<0$ and $H(s)=0$ for $s>0$.

The function $J_\delta(t,x)$ is smooth both in t and x for all $\delta > 0$ – this is easy to check simply by differentiating the integrand in (2.4.15) in t and x, since that does not produce any singularity because $t-s>\delta$. Moreover, $J_\delta(t,x)$ converges uniformly to $J(t,x)$ as $\delta\downarrow 0$ – this follows from the estimate

$$|J(t,x)-J_\delta(t,x)|\le \delta\|g\|_{L^\infty}, \tag{2.4.17}$$

that can be checked as in (2.4.12). As a consequence, the derivatives of $J_\delta(t,x)$ converge weakly, in the sense of distributions, to the corresponding weak derivatives of $J(t,x)$. Thus, to show that, say, the second derivatives (understood in the sense of distributions) $\partial_{x_ix_j}J(t,x)$ are actually continuous functions, it suffices to prove that the partial derivatives $\partial_{x_ix_j}J_\delta(t,x)$ converge uniformly to a continuous function, and that is what we will do.

We will look in detail at $\partial_{x_ix_j}J_\delta$, with $i\neq j$. As the integrand for J_δ has no singularity at $s=t$, we may simply differentiate under the integral sign

$$\frac{\partial^2 J_\delta(t,x)}{\partial x_i\partial x_j}=\int_0^{t-\delta}\int_{\mathbb{R}^n}\frac{(x_i-y_i)(x_j-y_j)}{4(t-s)^2(4\pi(t-s))^{n/2}}e^{-|x-y|^2/4(t-s)}g(s,y)dsdy.$$

The extra factor $(t-s)^2$ in the denominator can not be removed simply by the change of variable used in (2.4.14) – as the reader can immediately check, this would still leave a non-integrable extra factor of $(t-s)^{-1}$ that would cause an obvious problem in passing to the limit $\delta\downarrow 0$.

A very simple but absolutely crucial observation that will come to our rescue here is that, as $i\neq j$, we have

$$\int_{\mathbb{R}^n}(x_i-y_i)(x_j-y_j)e^{-|x-y|^2/4(t-s)}dy=0. \tag{2.4.18}$$

This allows us to write

$$\frac{\partial^2 J_\delta(t,x)}{\partial x_i\partial x_j}=\int_0^{t-\delta}\int_{\mathbb{R}^n}\frac{(x_i-y_i)(x_j-y_j)}{4(t-s)^2(4\pi(t-s))^{n/2}}e^{-|x-y|^2/4(t-s)}\big(g(s,y)-g(t,x)\big)dsdy.$$

Now, we can use the regularity of $g(s,y)$ to help us. In particular, the Hölder continuity assumption (2.4.13) gives

$$\begin{aligned}\Big|\frac{(x_i-y_i)(x_j-y_j)}{4(t-s)^2(4\pi(t-s))^{n/2}}e^{-|x-y|^2/4(t-s)}\big(g(s,y)-g(t,x)\big)\Big| &\le \frac{C|z|^2e^{-|z|^2}(|t-s|^{\alpha/2}+|x-y|^\alpha)}{(t-s)}\\ &\le \frac{C}{(t-s)^{1-\alpha/2}}\frac{k(z)}{(4\pi(t-s))^{n/2}},\end{aligned} \tag{2.4.19}$$

still with $z=(x-y)/\sqrt{t-s}$, as in (2.4.14), and

$$k(z)=|z|^2e^{-|z|^2/4}(1+|z|^\alpha).$$

As before, the factor of $(t-s)^{n/2}$ in the right side of (2.4.19) goes into the volume element

$$dz=\frac{dx}{(t-s)^{n/2}},$$

and we only have the factor $(t-s)^{1-\alpha/2}$ left in the denominator in (2.4.19), which is integrable in s, unlike the factor $(t-s)^{-1}$ one would get without using the cancellation in (2.4.18) and the Hölder regularity of $g(t,x)$. Thus, after accounting for the Jacobian factor, the integrand in the expression for $\partial_{x_ix_j}J_\delta$ is dominated by an integrable function in z, which entails the uniform convergence of $\partial_{x_ix_j}J_\delta$ as $\delta\downarrow 0$. In particular, the continuity of the limit follows as well.

Exercise 2.4.6 Complete the argument by looking at the remaining derivatives $\partial_t J_\delta(t,x)$ and $\partial_{x_i x_i} J(t,x)$. The key step in both cases is to find a cancellation such as in (2.4.18).

Let us summarize the results of this section as follows.

Theorem 2.4.7 *Let $u(t,x)$ be the solution of the Cauchy problem*

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + g(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \\ &u(0,x) = u_0(x). \end{aligned} \tag{2.4.20}$$

Assume that $u_0(x)$ is a uniformly bounded function on $\mathbb{R}^n$, and $g(t,x)$ is a bounded uniformly continuous function over $[0,T] \times \mathbb{R}^n$, that satisfies a Hölder regularity estimate

$$|g(t,x) - g(t',x')| \le C\big(|t-t'|^{\alpha/2} + |x-x'|^\alpha\big) \tag{2.4.21}$$

for some $\alpha \in (0,1)$. Then $u(t,x)$ is twice continuously differentiable in x, and once continuously differentiable in t over $(0,T) \times \mathbb{R}^n$, with the corresponding derivatives bounded over every interval of the form $[\varepsilon, T]$, $0 < \varepsilon < T$.

We will come back to Theorem 2.4.7 in Section 2.5, where we will consider the regularity of the solution of more general parabolic equations, after we have a look at the solutions of the nonlinear heat equations.

2.4.2 Existence and regularity for a semi-linear diffusion equation

We now turn to a semi-linear parabolic equation of the form

$$u_t = \Delta u + f(x,u). \tag{2.4.22}$$

Such equations are commonly known as the reaction-diffusion equations, and are very common in biological and physical sciences. We will consider (2.4.22) posed in $\mathbb{R}^n$, and equipped with a bounded, nonnegative, and uniformly continuous initial condition

$$u(0,x) = u_0(x). \tag{2.4.23}$$

Needless to say, the continuity and non-negativity assumptions on the initial conditions could be relaxed, but this is a sufficiently general set-up, to which many problems can reduced, and which allows us to explain the basic ideas. The function f is assumed to be smooth in all its variables and uniformly Lipschitz in u:

$$|f(x,u) - f(x,u')| \le C_f|u-u'|, \text{ for all } x \in \mathbb{R}^n \text{ and } u,u' \in \mathbb{R}. \tag{2.4.24}$$

In addition, we assume that

$$f(x,0) = 0, \qquad f(x,u) < 0 \quad \text{if } u \ge M, \text{ for some large } M \ge \|u_0\|_\infty. \tag{2.4.25}$$

One example to keep in mind is the Fisher-KPP equation

$$u_t = \Delta u + u(1-u), \tag{2.4.26}$$

with the predator-prey nonlinearity $f(u) = u(1-u)$. We refer the reader to Chapter **??** for the discussion of how this equation arises in the biological modeling and other applications, as well as to the explanation of its name.

Another important example is the time-dependent version of the Allen-Cahn equation we have encountered in Chapter 1:

$$u_t = \Delta u + u - u^3. \tag{2.4.27}$$

In that case, we usually assume that $|u| \le 1$ rather than positivity of u – the reader should check that the function $v = 1 + u$ satisfies an equation

$$v_t = \Delta v + f(v), \tag{2.4.28}$$

with a nonlinearity f that satisfies assumptions we will need below.

The assumption $f(x,0) = 0$ and the smoothness of $f(x,u)$ mean that if $u(t,x)$ is a smooth bounded solution of (2.4.22)-(2.4.23), then $u(t,x)$ satisfies

$$u_t = \Delta u + c(t,x)u, \tag{2.4.29}$$

with a smooth function

$$c(t,x) = \frac{f(x,u(t,x))}{u(t,x)}.$$

As $v(t,x) \equiv 0$ is a solution of (2.4.29), the comparison principle, Theorem 2.3.3, implies that if $u_0(x) \ge 0$ for all $x \in \mathbb{R}^n$, then $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^n$, unless $u_0(x) \equiv 0$. We will prove the following existence result.

Theorem 2.4.8 *Under the above assumptions on the nonlinearity f, given a bounded and continuous initial condition $u_0(x) \ge 0$, there exists a unique bounded smooth solution $u(t,x)$ to* (2.4.22)-(2.4.23)*, which, in addition, satisfies $0 \le u(t,x) \le M$. Moreover, for all $T > 0$ each derivative of u is bounded over $[T,+\infty) \times \mathbb{R}^n$.*

Uniqueness of the solutions is straightforward. If the Cauchy problem (2.4.22)-(2.4.23) has two smooth bounded solutions $u_1(t,x)$ and $u_2(t,x)$, then $w = u_1 - u_2$ satisfies

$$w_t = \Delta w + c(t,x)w, \tag{2.4.30}$$

with the initial condition $w(0,x) = 0$ and a bounded function

$$c(t,x) = \frac{f(x,u(t,x)) - f(x,v(t,x))}{u(t,x) - v(t,x)}.$$

The maximum principle then implies that $w(t,x) \le 0$ and $w(t,x) \ge 0$, thus $w(t,x) \equiv 0$, proving the uniqueness.

The typical approach to the existence proofs in nonlinear problems is to use a fixed point argument, and this is exactly what we will do. To this end, it is useful, and standard, to rephrase the parabolic initial value problem (2.4.22)-(2.4.23) as an integral equation. This is done as follows. Given a fixed $T > 0$ and the given initial condition $u_0(x)$, we define an operator $\mathcal{T}$ as a mapping of the space $C([0,T] \times \mathbb{R}^n)$ to itself via

$$\begin{aligned}[\mathcal{T}u](t,x) &= e^{t\Delta}u_0(x) + \int_0^t e^{(t-s)\Delta} f(\cdot, u(s,\cdot))(x)ds \qquad (2.4.31)\\ &= e^{t\Delta}u_0(x) + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} f(y,u(s,y))dyds,\end{aligned}$$

with the operator $e^{t\Delta}$ defined in (2.4.8). The Duhamel formula for the solution of the Cauchy problem (2.4.22)-(2.4.23) can be now succinctly restated as

$$u(t,x) = [\mathcal{T}u](t,x). \tag{2.4.32}$$

In other words, any solution of the initial value problem is a fixed point of the operator $\mathcal{T}$. On the other hand, to show that a fixed point of $\mathcal{T}$ is a solution of the initial value problem, we need to know that any such fixed point is differentiable.

A priori regularity

The key boot-strap observation is the following.

Lemma 2.4.9 *Let $u(t,x)$ be a fixed point of the operator $\mathcal{T}(t,x)$ on the time interval $[0,T]$. If $u(t,x)$ is a uniformly continuous function over $[0,T]\times\mathbb{R}^n$, then $u(t,x)$ is infinitely differentiable on any time interval of the form $[\varepsilon,T]$ with $\varepsilon>0$.*

Proof. The first term in the right side of (2.4.31) is infinitely differentiable for any $t>0$ and $x\in\mathbb{R}^n$ simply because it is a solution of the heat equation with a bounded and continuous initial condition u_0. Thus, we only need to deal with the Duhamel term

$$D[u](t,x) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} f(y,u(s,y))dyds. \tag{2.4.33}$$

Assume for a moment that we can show that $u(t,x)$ is Hölder continuous in t and x. Then, so is $f(x,u(t,x))$, and Proposition 2.4.3 tells us that $D[u](t,x)$ is differentiable in t and x. Then, we see from (2.4.31) or (2.4.32) that so is $u(t,x)$, and thus $f(x,u(t,x))$ is differentiable as well. But then we can show that $D[u](t,x)$ is twice differentiable, hence so are $u(t,x)$ and $f(x,u(t,x))$. We may iterate this argument, each time gaining derivatives in t and x, and conclude that, actually, $u(t,x)$ is infinitely differentiable in t and x. Therefore, it suffices to show that any uniformly continuous solution $u(t,x)$ of (2.4.32) is Hölder continuous. This is a consequence of the following strengthened version of Proposition 2.4.3. Recall that we denote

$$J(t,x) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} g(s,y)dyds. \tag{2.4.34}$$

Lemma 2.4.10 *Let $g(t,x)$ be a bounded uniformly continuous function. Then $J(t,x)$ given by (2.4.34) is Hölder continuous for all $t>0$ and $x\in\mathbb{R}^n$.*

Proof of Lemma 2.4.10. Let us freeze $t>0$ and prove that J is differentiable in x_i, with some $1\le i\le n$. The proof of Proposition 2.4.3 had plenty of room, as far as differentiability in x is concerned, so that part of the argument is identical: we write

$$\Big|\frac{\partial}{\partial x_i}\Big(\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}}\Big)\Big| = \Big| -\frac{x_i-y_i}{2(t-s)}\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}}\Big| \le \frac{2\|g\|_\infty}{\sqrt{t-s}}\frac{|z|e^{-|z|^2}}{(4\pi(t-s))^{n/2}}, \tag{2.4.35}$$

with

$$z = \frac{x-y}{\sqrt{t-s}}. \tag{2.4.36}$$

This shows that the x_i-derivative of the integrand is dominated by an integrable function in z, as

$$dz = \frac{dx}{(t-s)^{n/2}},$$

which implies that J is C^1 in its x variables, with bounded derivatives.

On the other hand, the proof of the differentiability in t of $J(t,x)$ in Proposition 2.4.3 did rely on the Hölder regularity of the function $g(t,x)$, something that we do not have now. Thus, we will have to compute a difference instead of a derivative. Assume, for convenience, that $t' \geq t$, and write

$$\begin{aligned} J(t,x) - J(t',x) = &\int_0^t \int_{\mathbb{R}^n} \Big(\frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}} - \frac{e^{-|x-y|^2/4(t'-s)}}{(4\pi(t'-s))^{n/2}} \Big) g(s,y)dyds \\ &- \int_t^{t'} \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4(t'-s)}}{(4\pi(t'-s))^{n/2}} g(s,y)dyds = I_1(t,t',x) + I_2(t,t',x). \end{aligned} \tag{2.4.37}$$

The second term above satisfies the simple estimate

$$|I_2(t,t',x)| \leq \|g\|_{L^\infty}|t'-t|, \tag{2.4.38}$$

obtained via the by now automatic change of variables as in (2.4.36). As for I_1, we write

$$\frac{e^{-|x-y|^2/4(t'-s)}}{(4\pi(t'-s))^{n/2}} - \frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{n/2}} = \int_t^{t'} \frac{h(z)}{(4\pi(\tau-s))^{n/2+1}} d\tau, \quad z = \frac{x-y}{\sqrt{\tau-s}},$$

with

$$h(z) = \Big(-\frac{n}{2} + \frac{|z|^2}{4} \Big) e^{-|z|^2}.$$

Thus, we have, changing the variables $y \to z$ in the integral over $\mathbb{R}^n$, and integrating z out:

$$\begin{aligned} |I_1(t,t',x)| &\leq C\|g\|_{L^\infty} \int_0^t \int_t^{t'} \frac{d\tau}{\tau-s} ds = C\|g\|_{L^\infty} \int_0^t \log\Big(\frac{t'-s}{t-s}\Big) ds \\ &= C\|g\|_{L^\infty}(t'\log t' - t\log t - (t'-t)\log(t'-t)). \end{aligned} \tag{2.4.39}$$

This proves that

$$|I_1(t,t',x)| \leq C\|g\|_{L^\infty}|t'-t|^\alpha, \tag{2.4.40}$$

for all $\alpha \in (0,1)$, finishing the proof of Lemma 2.4.10, and thus also that of Lemma 2.4.9. □

The Picard iteration argument

Thus, we know that if a uniformly continuous solution $u(t,x)$ of (2.4.32) exists, then $u(t,x)$ is smooth. Let us now prove the existence of such solution. We will first show this for a time interval $[0,T]$ sufficiently small but independent of the initial condition $u_0(x)$. We will use the standard Picard iteration: set $u^{(0)} = 0$ and define

$$u^{(n+1)}(t,x) = \mathcal{T}u^{(n)}(t,x). \tag{2.4.41}$$

The previous analysis shows that all $u^{(n)}(t,x)$ are smooth for $t>0$ and $x\in\mathbb{R}^n$, and, using the uniform Lipschitz property (2.4.24) of the function $f(x,u)$ in u, we obtain

$$
\begin{aligned}
|u^{(n+1)}(t,x)-u^{(n)}(t,x)| &\le \int_0^t\int_{\mathbb{R}^n}\frac{e^{-(x-y)^2/(4(t-s))}}{(4\pi(t-s))^{n/2}}|f(y,u^{(n)}(s,y))-f(y,u^{(n-1)}(s,y))|dyds\\
&\le C_f\int_0^t\int_{\mathbb{R}^n}\frac{e^{-(x-y)^2/(4(t-s))}}{(4\pi(t-s))^{n/2}}|u^{(n)}(s,y)-u^{(n-1)}(s,y)|dyds\\
&\le C_fT\sup_{0\le s\le T,y\in\mathbb{R}^n}|u^{(n)}(s,y)-u^{(n-1)}(s,y)|. \qquad (2.4.42)
\end{aligned}
$$

This shows that if $T<C_f^{-1}$, then the mapping $\mathcal{T}$ is a contraction and thus has a unique fixed point within the uniformly continuous functions $u(t,x)$ over $[0,T]\times\mathbb{R}^n$. The only assumption we used about the initial condition $u_0(x)$ is that it is continuous and satisfies

$$
0\le u_0(x)\le M.
$$

However, as we have mentioned, the maximum principle and the regularity of $u(t,x)$ imply that $u(T,x)$ satisfies the same properties if $u_0(x)$ does. The key point is that the time T does not depend on u_0. Therefore, we can repeat this argument on the time intervals $[T,2T]$, $[2T,3T]$, and so on, eventually constructing a global in time solution to the Cauchy problem. This finishes the proof of Theorem 2.4.8.

Exercise 2.4.11 Consider the same setting as in Theorem 2.4.8 but without the assumption that $M\ge\|u_0\|_{L^\infty}$ in (2.4.25). In other words, replace (2.4.25) by the assumption

$$
f(x,0)=0,\qquad f(x,u)<0\quad\text{if } u\ge M,\text{ for some large } M, \qquad (2.4.43)
$$

together with the assumption that $u_0(x)$ is a bounded non-negative function. Show that the conclusion of Theorem 2.4.8 still holds – the Cauchy problem (2.4.22)-(2.4.23) admits a unique smooth bounded solution for such initial conditions as well, and $u(t,x)$ satisfies

$$
0\le u(t,x)\le\max\big(M,\sup_{x\in\mathbb{R}^n}u_0(x)\big). \qquad (2.4.44)
$$

2.4.3 The regularity of the solutions of a quasi-linear heat equation

One may wonder if the treatment that we have given to the semi-linear heat equation (2.4.22) is too specialized as it relies on particular cancellations. To dispel this fear, we show how this approach can be extended to equations with a drift and other quasi-linear heat equations of the form

$$
u_t-\Delta u=f(x,\nabla u), \qquad (2.4.45)
$$

posed for $t>0$ and $x\in\mathbb{R}^n$. The nonlinearity is now stronger: it depends not on u itself but on its gradient ∇u. This time we do not make any sign assumptions on f, but we ask that the nonlinear term $f(x,p)$ satisfies the following two hypotheses: there exist $C_1>0$ so that

$$
|f(x,0)|\le C_1\text{ for all } x\in\mathbb{R}^n, \qquad (2.4.46)
$$

and $C_2 > 0$ so that

$$|f(x,p_1) - f(x,p_2)| \le C_2|p_1 - p_2|, \text{ for all } x, p_1, p_2 \in \mathbb{R}^n. \tag{2.4.47}$$

One consequence of (2.4.46) and (2.4.47) is a uniform bound

$$|f(x,p)| \le C_3(1 + |p|), \tag{2.4.48}$$

showing that $f(x,p)$ grows at most linearly in p. We also ask that $f(x,p)$ is smooth in x and p.

Two of the standard examples of equations of the form (2.4.45) are parabolic equations with constant diffusion and nonuniform drifts, such as

$$u_t = \Delta u + b_j(x)\frac{\partial u}{\partial x_j}, \tag{2.4.49}$$

with a prescribed drift $b(x)$, and the viscous regularizations of the Hamilton-Jacobi equations, such as

$$u_t = \Delta u + f(|\nabla u|). \tag{2.4.50}$$

We will encounter both of them in the sequel. Our goal is the following.

Theorem 2.4.12 *Equation* (2.4.45)*, equipped with a bounded uniformly continuous initial condition u_0, has a unique smooth solution over* $(0, +\infty) \times \mathbb{R}^n$*, which is bounded with all its derivatives over every set of the form* $(\varepsilon, T) \times \mathbb{R}^n$*.*

We will use the ideas previously displayed in the proof of Theorem 2.4.8. One of the main difficulties in looking at (2.4.45) is that it involves a nonlinear function of the gradient of the function u, which, a priori, may not be smooth at all. Thus, a natural idea is to regularize that term, and then pass to the limit. A relatively painless approach is to consider the following nonlocal approximation:

$$u_t^\varepsilon - \Delta u^\varepsilon = f(x, \nabla v^\varepsilon), \quad v^\varepsilon = e^{\varepsilon\Delta}u^\varepsilon. \tag{2.4.51}$$

When $\varepsilon > 0$ is small, one expects the solutions to (2.4.45) and (2.4.51) to be close as

$$e^{\varepsilon\Delta}\psi \to \psi, \text{ as } \varepsilon \to 0. \tag{2.4.52}$$

Exercise 2.4.13 In which function spaces does the convergence in (2.4.52) hold? For instance, does it hold in L^2 or L^∞? How about $C^1(\mathbb{R})$?

A damper on our expectations is that the convergence in (2.4.52) does not automatically translate into the convergence of the corresponding gradients, unless we already know that ψ is differentiable. In other words, if ψ is not differentiable, there is no reason to expect that

$$\nabla(e^{\varepsilon\Delta}\psi) \to \nabla\psi,$$

simply because the right side may not exist. And this is what we need to understand the convergence of the term $f(x, \nabla v^\varepsilon)$ in (2.4.51). Thus, something will have to be done about this.

Nevertheless, a huge advantage of the (2.4.51) over (2.4.45) is that the function v^ε that appears inside the nonlinearity is smooth if u^ε is merely continuous, as long as $\varepsilon > 0$. This can be used to show that the Cauchy problem for (2.4.51) has a unique smooth solution.

Exercise 2.4.14 Show that, for every $\varepsilon > 0$ and every bounded function $u(x)$ bounded, we have

$$\|\nabla(e^{\varepsilon\Delta}u)\|_{L^\infty} \leq \frac{C}{\sqrt{\varepsilon}}\|u\|_{L^\infty}. \tag{2.4.53}$$

Use this fact, and the strategy in the proof of Theorem 2.4.8, to prove that (2.4.51), equipped with the bounded uniformly continuous initial condition u_0, has a unique smooth solution u^ε over a set of the form $(0, T_\varepsilon] \times \mathbb{R}^n$, with a time $T_\varepsilon > 0$ that depends on $\varepsilon > 0$ but not on the initial condition u_0.

Having constructed solutions to (2.4.51) on a finite time interval $[0, T_\varepsilon]$, in order to obtain a global in time solution to the original equation (2.4.45), we need to do two things: (1) extend the existence of the solutions to the approximate equation (2.4.51) to all $t > 0$, and (2) pass to the limit $\varepsilon \to 0$ and show that the limit of u^ε exists (possibly along a sub-sequence) and satisfies "the true equation" (2.4.45). The latter step will require uniform bounds on ∇u^ε that do not depend on ε – something much better than what is required in Exercise 2.4.14.

Global in time existence of the approximate solution

To show that the solution to (2.4.51) exists for all $t > 0$, and not just on the interval $[0, T_\varepsilon]$, we use the Duhamel formula

$$u^\varepsilon(t,x) = e^{t\Delta}u_0(x) + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))} f(y, \nabla v^\varepsilon(s,y))dyds. \tag{2.4.54}$$

Assumption (2.4.48), together with the gradient bound (2.4.53), implies an estimate

$$|f(x, \nabla v^\varepsilon(t,x))| \leq C\Big(1 + \frac{\|u^\varepsilon(t,\cdot)\|_{L^\infty}}{\sqrt{\varepsilon}}\Big),$$

that can be used in (2.4.54) to yield

$$\|u^\varepsilon(t,\cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + Ct + \frac{C}{\sqrt{\varepsilon}} \int_0^t \|u^\varepsilon(s,\cdot)\|_{L^\infty} ds. \tag{2.4.55}$$

We set

$$Z_\varepsilon(t) = \int_0^t \|u^\varepsilon(s,\cdot)\|_{L^\infty} ds,$$

and write (2.4.55) as

$$\frac{dZ_\varepsilon}{dt} \leq \|u_0\|_{L^\infty} + Ct + \frac{C}{\sqrt{\varepsilon}} Z_\varepsilon. \tag{2.4.56}$$

Multiplying by $\exp(-Ct/\sqrt{\varepsilon})$ and integrating gives

$$Z_\varepsilon(t) \leq \frac{\sqrt{\varepsilon}}{C} e^{Ct/\sqrt{\varepsilon}} \big(\|u_0\|_{L^\infty} + Ct\big). \tag{2.4.57}$$

Using this in (2.4.55) gives a uniform estimate

$$\|u^\varepsilon(t,\cdot)\|_{L^\infty} \leq \big(\|u_0\|_{L^\infty} + Ct\big)\big(1 + e^{Ct/\sqrt{\varepsilon}}\big). \tag{2.4.58}$$

Therefore, the L^∞-norm of the solution can grow by at most a fixed factor over the time interval $[0, T_\varepsilon]$. This allows us to restart the Cauchy problem on the time interval $[T_\varepsilon, 2T_\varepsilon]$, and then on $[2T_\varepsilon, 3T_\varepsilon]$, and so on, showing that the regularized problem (2.4.51) admits a global in time solution.

Passing to the limit $\varepsilon \downarrow 0$

A much more serious challenge is to send $\varepsilon \downarrow 0$, and recover a solution of the original equation (2.4.45). To do this, we will obtain the Hölder estimates of u^ε and its derivatives up to the second order in space and the first order in time, that will be independent of ε. The Ascoli-Arzola theorem will then provide us with the compactness of the family u^ε, and allow us to pass to the limit and obtain a solution of (2.4.45).

Exercise 2.4.15 Assume that there exists $\alpha \in (0,1)$ such that, for all $\delta > 0$ and $T > \delta$, there is $C_\delta(T) > 0$, locally bounded for $T \in [\delta, +\infty)$, and independent of $\varepsilon \in (0,1)$, for which we have the following Hölder regularity estimates:

$$\Big|\frac{\partial}{\partial t}\Big(u^\varepsilon(t,x)-u^\varepsilon(t',x')\Big)\Big|+\Big|D_x^2\Big(u^\varepsilon(t,x)-u^\varepsilon(t',x')\Big)\Big| \le C_\delta(T)\Big(|t-t'|^{\alpha/2}+|x-x'|^\alpha\Big), \quad (2.4.59)$$

for all $t, t' \in [\delta, T]$ and $x, x' \in \mathbb{R}^n$. Write down a complete proof that then there exists a subsequence u^{ε_k} converges to a limit $u(t,x)$ as $k \to +\infty$, and $u(t,x)$ is a solution to (2.4.45). In particular, pay attention as to why we know that $\nabla v^\varepsilon \to \nabla u$. In which space does the convergence take place?

This exercise gives us the road map to the construction of a solution to (2.4.45): we "only" need to establish the Hölder estimates (2.4.59) for the solutions of the approximate equation (2.4.51). We will use the following lemma, that is a slight generalization of the Gronwall lemma, and which is very useful in estimating the derivatives for the solutions of the parabolic equations.

Lemma 2.4.16 *Let $\varphi(t)$ be a nonnegative bounded function that satisfies, for all $0 \le t \le T$:*

$$\varphi(t) \le \frac{a}{\sqrt{t}} + b\int_0^t \frac{\varphi(s)}{\sqrt{t-s}}ds. \quad (2.4.60)$$

Then, for all $T > 0$, there is $C(T) > 0$ that depends on T and b, such that

$$\varphi(t) \le \frac{C(T)a}{\sqrt{t}}. \quad (2.4.61)$$

Proof. First, note that we can write $\phi(t) = a\psi(t)$, leading to

$$\psi(t) \le \frac{1}{\sqrt{t}} + b\int_0^t \frac{\psi(s)}{\sqrt{t-s}}ds. \quad (2.4.62)$$

Then, iterating (2.4.60) we obtain

$$\psi(t) \le \sum_{k=0}^n I_n(t) + R_{n+1}(t), \quad (2.4.63)$$

for any $n \ge 0$, with

$$I_{n+1}(t) = b\int_0^t \frac{I_n(s)}{\sqrt{t-s}}ds, \quad I_0(t) = \frac{1}{\sqrt{t}}, \quad (2.4.64)$$

and

$$R_{n+1}(t) = b\int_0^t \frac{R_n(s)}{\sqrt{t-s}}, \quad R_0(t) = \varphi(t). \tag{2.4.65}$$

We claim that there exist a constant $c > 0$, and $p > 1$ so that

$$I_n(t) \le \frac{1}{\sqrt{t}} \frac{(ct)^{n/2}}{(n!)^{1/p}}. \tag{2.4.66}$$

Indeed, this bound holds for $n = 0$, and if it holds for $I_n(t)$, then we have

$$\begin{aligned} I_{n+1}(t) &= b\int_0^t \frac{I_n(s)}{\sqrt{t-s}} ds \le \frac{bc^{n/2}}{(n!)^{1/p}} \int_0^t \frac{s^{(n-1)/2} ds}{\sqrt{t-s}} = \frac{bc^{n/2} t^{(n+1)/2}}{\sqrt{t}(n!)^{1/p}} \int_0^1 \frac{\tau^{(n-1)/2} d\tau}{\sqrt{1-\tau}} \\ &= \frac{bc^{n/2} t^{(n+1)/2}}{(n!)^{1/p}\sqrt{t}} \Big(\int_0^1 \tau^{3(n-1)/2} d\tau\Big)^{1/3} \Big(\int_0^1 \frac{d\tau}{(1-\tau)^{3/4}}\Big)^{2/3} \\ &= \frac{bc^{n/2} t^{(n+1)/2}}{\sqrt{t}(n!)^{1/p}} \frac{4^{2/3}}{(3n/2-1)^{1/3}} \le \frac{bc^{n/2} t^{(n+1)/2}}{\sqrt{t}(n!)^{1/p}} \frac{4^{2/3} 2^{1/3}}{(n+1)^{1/3}}. \end{aligned} \tag{2.4.67}$$

Thus, the bound (2.4.66) holds with $p = 3$ and $c = b^2 \cdot 2^{10/3}$.

Exercise 2.4.17 Use the same argument to estimate $R_n(t)$ and show that if $\varphi(t)$ is uniformly bounded on $[0, T]$, then

$$\varphi(t) \le \frac{a}{\sqrt{t}} + a\sum_{n=1}^{\infty} I_n(t). \tag{2.4.68}$$

Now, the estimate (2.4.61) follows from (2.4.68) and (2.4.66). □

This lemma will now (somewhat effortlessly) bring us to (2.4.59). First, let us use the Duhamel formula (2.4.54) to get the Hölder bound on ∇u^ε. The maximum principle implies that

$$\|e^{t\Delta} u_0\|_{L^\infty} \le \|u_0\|_{L^\infty}, \tag{2.4.69}$$

and also that the gradient

$$\nabla v^\varepsilon = e^{\varepsilon\Delta} \nabla u^\varepsilon,$$

satisfies the bound

$$\|\nabla v^\varepsilon(t,\cdot)\|_{L^\infty} \le \|\nabla u^\varepsilon(t,\cdot)\|_{L^\infty}. \tag{2.4.70}$$

We use these estimates, together with (2.4.48) in the Duhamel formula (2.4.54), leading to

$$\|u^\varepsilon(t,\cdot)\|_{L^\infty} \le \|u_0\|_{L^\infty} + Ct + C\int_0^t \|\nabla u^\varepsilon(s,\cdot)\|_{L^\infty} ds. \tag{2.4.71}$$

The next step is to take the gradient of the Duhamel formula. The first term is estimated as in (2.4.53):

$$\|\nabla(e^{t\Delta} u_0)\|_{L^\infty} \le \frac{C}{\sqrt{t}} \|u_0\|_{L^\infty}. \tag{2.4.72}$$

To bound the gradient of the integral term in the Duhamel formula, we note that

$$\|\nabla e^{(t-s)\Delta} f(\cdot, \nabla v^\varepsilon)\|_{L^\infty} \le \frac{C}{\sqrt{t-s}} \|f(\cdot, \nabla v^\varepsilon(s,\cdot)\|_{L^\infty}. \tag{2.4.73}$$

The term in the right side is bounded, once again, using the linear growth bound (2.4.48) and (2.4.70). Altogether these estimates lead to

$$\|\nabla u^\varepsilon(t,\cdot)\|_{L^\infty} \leq \frac{C}{\sqrt{t}}\|u_0\|_{L^\infty} + C\sqrt{t} + C\int_0^t \frac{\|\nabla u^\varepsilon(s,.)\|_{L^\infty}}{\sqrt{t-s}}ds. \tag{2.4.74}$$

Writing

$$\frac{C}{\sqrt{t}}\|u_0\|_{L^\infty} + C\sqrt{t} \leq \frac{C\|u_0\|_{L^\infty} + CT}{\sqrt{t}}, \quad 0 \leq t \leq T,$$

we can put (2.4.74) into the form of (2.4.60). Lemma 2.4.16 implies then that there exists a constant $C(T) > 0$, independent of ε, such that

$$\|\nabla u^\varepsilon(t,\cdot)\|_{L^\infty} \leq \frac{C(T)}{\sqrt{t}}, \quad 0 < t \leq T. \tag{2.4.75}$$

Using this estimate in (2.4.71) gives a uniform bound on u^ε itself:

$$\|u^\varepsilon(t,\cdot)\|_{L^\infty} \leq C(T), \quad 0 < t \leq T. \tag{2.4.76}$$

In other words, the family $u^\varepsilon(t,\cdot)$ is uniformly bounded in the Sobolev space $W^{1,\infty}(\mathbb{R}^n)$ – the space of L^∞ functions with gradients (in the sense of distributions) that are also L^∞ functions:

$$\|u^\varepsilon(t,\cdot)\|_{W^{1,\infty}} \leq \frac{C(T)}{\sqrt{t}}, \quad 0 < t \leq T. \tag{2.4.77}$$

The uniform bound on the gradient in (2.4.77) seems a far cry from what we need in (2.4.59) – that estimate requires the second derivatives to be Hölder continuous, and so far we only have a uniform bound on the first derivative – we do not even know yet that the first derivatives are Hölder continuous. Surprisingly, the end of the proof is actually near. Take some $1 \leq i \leq n$, and set

$$z_i^\varepsilon = \frac{\partial u^\varepsilon}{\partial x_i}.$$

The equation for z_i^ε is (using the summation convention for repeated indices)

$$\partial_t z_i^\varepsilon - \Delta z_i^\varepsilon = \partial_{x_i} f(x, \nabla v^\varepsilon) + \partial_{p_j} f(x, \nabla v^\varepsilon)\partial_{x_j} q_i^\varepsilon, \quad q_i^\varepsilon = e^{\varepsilon\Delta} z_i^\varepsilon. \tag{2.4.78}$$

We look at (2.4.78) as an equation for z_i^ε, with a given ∇v^ε that satisfies the already proved uniform bound

$$\|\nabla v^\varepsilon(t,\cdot)\|_{L^\infty} \leq \frac{C(T)}{\sqrt{t}}, \quad 0 < t \leq T, \tag{2.4.79}$$

that follows immediately from (2.4.75). Thus, (2.4.78) is of the form

$$\partial_t z_i^\varepsilon - \Delta z_i^\varepsilon = G(x, \nabla q_i^\varepsilon), \quad q_i^\varepsilon = e^{\varepsilon\Delta} z_i^\varepsilon, \tag{2.4.80}$$

with

$$G(x,p) = \partial_{x_i} f(x, \nabla v^\varepsilon) + \partial_{p_j} f(x, \nabla v^\varepsilon) p_j. \tag{2.4.81}$$

The function $G(x,p)$ satisfies the assumptions on the nonlinearity $f(x,p)$ stated at the beginning of this section – it is simply a linear function in the variable p, and it x-dependence is only via the function ∇v^ε that is uniformly bounded on any time interval $[\delta, T]$ with $\delta > 0$. Hence, on any such time interval z_i^ε satisfies an equation of the type we have just analyzed for u^ε, and our previous analysis shows that

$$\|\nabla z_i^\varepsilon(t,\cdot)\|_{L^\infty} \le \frac{C(T,\delta)}{\sqrt{t-\delta}}, \quad \delta < t \le T. \tag{2.4.82}$$

In other words, we have the bound

$$\|D^2 u^\varepsilon(t,\cdot)\|_{L^\infty} \le \frac{C(T,\delta)}{\sqrt{t-\delta}}, \quad \delta < t \le T. \tag{2.4.83}$$

This is almost what we need in (2.4.59) – we also need to show that D^2u^ε are Hölder continuous. One way to see this is to note that with the information we have already obtained, we know that the right side in (2.4.80) is a uniformly bounded function, on any time interval $[\delta, T]$, with $\delta > 0$. Lemma 2.4.10 implies then immediately that $\nabla z_i(t,x)$ is Hölder continuous on the time interval $[2\delta, T]$, which is exactly what we seek for D^2u^ε in (2.4.59). The bound on the time derivative follows then immediately from the equation (2.4.51) for u^ε – the reader may pause for a second to see why the term $f(x, \nabla v^\varepsilon)$ is Hölder continuous.

Exercise 2.4.18 So far, we have proved that (2.4.45) has a solution $u(t,x)$ that is uniformly bounded in the Hölder space $C^{2,\alpha}(\mathbb{R}^n)$ on any time interval $[\delta, T]$ with $\delta > 0$. Differentiate the equation for u and iterate this argument, showing that the solution is actually infinitely differentiable.

All that is left in the proof of Theorem 2.4.12 is to prove the uniqueness of a smooth solution. We will invoke the maximum principle again. Recall that we are looking for smooth solutions, so the difference $w = u_1 - u_2$ between any two solutions u_1 and u_2 simply satisfies an equation with a drift:

$$w_t - \Delta w = b(t,x) \cdot \nabla w, \tag{2.4.84}$$

with a smooth drift $b(t,x)$ such that

$$f(x, \nabla u_1(t,x)) - f(x, \nabla u_2(t,x)) = b(t,x) \cdot [\nabla u_1(t,x) - \nabla u_2(t,x)].$$

As $w(0,x) \equiv 0$, the maximum principle implies that $w(t,x) \equiv 0$ and $u_1 \equiv u_2$. This completes the proof of Theorem 2.4.12. □

Exercise 2.4.19 Prove that, if u_0 is smooth, then smoothness holds up to $t = 0$. Prove that equation (2.4.45) holds up to $t = 0$, that is:

$$u_t(0,x) = \Delta u_0(x) + f(x, \nabla u_0(x)).$$

Exercise 2.4.20 Extend the result of Theorem 2.4.12 to equations of the form

$$u_t = \Delta u + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u, \tag{2.4.85}$$

with smooth coefficients $b_j(t,x)$ and $c(t,x)$.

2.5 A survival kit in the jungle of regularity

In our noble endeavor to carry out out as few computations as possible, we have not touched a very important subject: that of optimal regularity. In other words, given a linear, possibly inhomogeneous equation of the form

$$u_t - a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x) = f(t,x), \tag{2.5.1}$$

the coefficients a_{ij}, b_j c and the right side f having a certain given degree of smoothness, what is the maximal regularity that one may expect from u? The question is a little different from what we did for the nonlinear equations, where one would first prove a certain, possibly small, amount of regularity, in the hope that this would be sufficient for an iteration leading to a much better regularity than in one iteration step. The answer to the question of maximal regularity is, in a nutshell: if the coefficients have a little bit of continuity, such as the Hölder continuity, then the derivatives u_t and D^2u have the same regularity as f. This, however, is true up to some painful exceptions: continuity for f will not entail, in general, the continuity of u_t and D^2u.

The question of the maximal regularity for linear parabolic equations has a certain degree of maturity, an interested reader should consult [97] to admire the breadth, beauty and technical complexity of the available results. Our goal here is much more modest: we will explain why the Hölder continuity of f will entail the Hölder continuity of u_t and D^2u – the result we have already seen for the heat equation using the explicit computations with the Duhamel term. In the context of the parabolic equations, we say that a function $g(t,x)$ is α-Hölder continuous on $(0,T]\times\mathbb{R}^n$, with $0<\alpha<1$, if for every $\varepsilon>0$ there is $C_\varepsilon>0$ such that

$$|g(t,x)-g(t',x')| \le C_\varepsilon\big(|t-t'|^{\alpha/2}+|x-x'|^{\alpha}\big), \quad \text{for all } \varepsilon<t,t'<T \text{ and } x\in,x'\in\mathbb{R}^n. \tag{2.5.2}$$

This is what we have already seen for the heat equation, for example, in Theorem 2.4.7. If C_ε does not blow up as $\varepsilon\to 0$, then we say that g is α-Hölder continuous on $[0,T]\times\mathbb{R}^n$. The Hölder norm of g over $[\varepsilon,T]\times\mathbb{R}^n$ is

$$\sup_{\varepsilon\le t,t'\le T,\ x\in\mathbb{T}^N} \frac{|g(t,x)-g(t',x')|}{|x-x'|^{\alpha}+|t-t'|^{\alpha/2}}. \tag{2.5.3}$$

When $a_{ij}(t,x)=\delta_{ij}$ (the Kronecker symbol), the second order term in (2.5.1) is the Laplacian, and our work was already almost done in Theorem 2.4.7. We will try to convince the reader, without giving the full details of all the proofs, that this carries over to variable diffusion coefficients, and, importantly, to problems with boundary conditions. Our main message here is that all the ideas necessary for the various proofs have already been displayed, and that "only" technical complexity and dexterity are involved. Our discussion follows Chapter 4 of [97], which presents various results with much more details.

When the diffusion coefficients are not continuous, but merely bounded, the methods described in this chapter break down. Chapter **??**, based on the Nash inequality, explains to some extent how to deal with such problems by a very different approach.

The Hölder regularity for the forced heat equation

We begin still with the inhomogeneous heat equation, strengthening what we have done in Theorem 2.4.7: we show that if the forcing term is α-Hölder continuous in the sense of (2.5.2), then the Hölder continuity passes on to the corresponding derivatives of the solution.

Theorem 2.5.1 *Let g be α-Hölder on $(0,T]\times\mathbb{R}^n$, and let $v(t,x)$ solve*

$$v_t - \Delta v = g \text{ on } (0,T]\times\mathbb{R}^n, \quad v(0,x)=0. \tag{2.5.4}$$

Then $\partial_t v$ and $D^2 v$ are α-Hölder continuous over $(0,T]\times\mathbb{T}^n$. Further, their Hölder norms over any set of the form $(\varepsilon,T]\times\mathbb{R}^n$, with $\varepsilon>0$, are controlled by that of g over the same set.

Proof. Our analysis follows what we did in Section 2.4.1 except we have to look at the Hölder differences for the second derivatives. The function $v(t,x)$ is given by the Duhamel formula

$$v(t,x)=\int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi(t-s))^{n/2}}e^{-(x-y)^2/(4(t-s))}g(s,y)dyds. \tag{2.5.5}$$

We are going to examine only $\partial_{x_ix_j}v$, with $i\neq j$, leaving the other derivatives to the reader as a lengthy but straightforward exercise. Let us set

$$h_{ij}(z)=\frac{z_iz_j}{(4\pi)^{n/2}}e^{-|z|^2}, \quad D(s,t,x,y)=h_{ij}\Big(\frac{x-y}{2\sqrt{t-s}}\Big)\frac{g(s,y)-g(t,x)}{t-s},$$

so that, using the fact that h_{ij} has a zero integral, we may write

$$\frac{\partial^2 v(t,x)}{\partial x_i\partial x_j}=\int_0^t\int_{\mathbb{R}^n}D(s,t,x,y)\frac{dsdy}{(t-s)^{n/2}}. \tag{2.5.6}$$

We remind the reader that the justification of expression (2.5.6) can be found in the proof of Proposition 2.4.3. Recall, in particular, that the mean zero property of h_{ij} is absolutely crucial as it allows us to bring the difference $g(s,y)-g(t,x)$ under the integral sign – otherwise, the integral would be divergent.

Now, for $\varepsilon\le t\le t'\le T$ and x,x' in $\mathbb{R}^n$, we have

$$\begin{aligned}\frac{\partial^2 v(t',x')}{\partial x_i\partial x_j}-\frac{\partial^2 v(t,x)}{\partial x_i\partial x_j}&=\int_t^{t'}\int_{\mathbb{R}^n}D(s,t',x',y)dsdy\\&+\int_0^t\int_{\mathbb{R}^n}\Big(D(s,t',x',y)-D(s,t,x,y)\Big)dsdy=J_1+J_2.\end{aligned}$$

Exercise 2.5.2 Verify that no additional ideas other than what has already been developed in Section 2.4.1 are required to prove that the integral J_1 satisfies an inequality of the form (2.5.2), with the control by the Hölder norm of g.

As for the integral J_2, we need to look at it a little deeper. The change of variables

$$z=x+2\sqrt{t-s}y$$

transforms (2.5.6) into

$$\frac{\partial^2 v(t,x)}{\partial x_i \partial x_j} = \int_0^t \int_{\mathbb{R}^n} h_{ij}(z) \frac{(g(s, x + 2\sqrt{t-s}z) - g(t,x))}{t-s} \frac{dsdz}{\pi^{n/2}},$$

and J_2 becomes

$$\begin{aligned}
&J_2(t,t',x,x') \\
=&\int_0^t \int_{\mathbb{R}^n} h_{ij}(z) \Big[\frac{g(s,x'+2\sqrt{t'-s}z) - g(t',x')}{t'-s} - \frac{g(s,x+2\sqrt{t-s}z) - g(t,x)}{t-s}\Big] \frac{dsdz}{\pi^{n/2}} \\
=&\int_0^t \int_{\mathbb{R}^n} h_{ij}(z) \Big[\frac{g(s,x'+2\sqrt{t'-s}z) - g(t',x')}{t'-s} - \frac{g(s,x+2\sqrt{t'-s}z) - g(t',x)}{t'-s}\Big] \frac{dsdz}{\pi^{n/2}} \\
+&\int_0^t \int_{\mathbb{R}^n} h_{ij}(z) \Big[\frac{g(s,x+2\sqrt{t'-s}z) - g(t',x)}{t'-s} - \frac{g(s,x+2\sqrt{t-s}z) - g(t,x)}{t-s}\Big] \frac{dsdz}{\pi^{n/2}} \\
=& J_{21}(t,t',x,x') + J_{22}(t,t',x,x'). \qquad (2.5.7)
\end{aligned}$$

We estimate each term separately.

The estimate of $J_{22}(t,t',x,x)$. We split the time integration domain into the intervals

$$A = \{s: \ t - \frac{1}{2}(t'-t) \le s \le t\}, \quad B = \{0 \le s \le t - \frac{1}{2}(t'-t)\}.$$

The Hölder regularity of $g(t,x)$ implies that

$$|g(s, x + 2\sqrt{t'-s}z) - g(t',x)| \le C(t'-s)^{\alpha/2}|z|,$$

and

$$|g(s, x + 2\sqrt{t-s}z) - g(t,x)| \le C(t-s)^{\alpha/2}|z|,$$

Note that for $s \in A$ we have

$$t' - s \le \frac{3}{2}(t'-t), \quad t - s \le \frac{1}{2}(t'-t),$$

hence the contribution to J_{22} by the integral over the interval A can be bounded as

$$\begin{aligned}
J_{22}^A(t,t',x,x') &\le C \int_{t'-(t'-t)/2}^t \int_{\mathbb{R}^n} |h_{ij}(z) z| \Big[\frac{1}{(t'-s)^{1-\alpha/2}} + \frac{1}{(t-s)^{1-\alpha/2}}\Big] \frac{dsdz}{\pi^{n/2}} \\
&\le C(t'-t)^{\alpha/2}. \qquad (2.5.8)
\end{aligned}$$

To estimate the contribution to J_{22} by the integral over the interval B, note that for $s \in B$ both increments $t-s$ and $t'-s$ are strictly positive. Let us also recall that h_{ij} has zero integral. Thus, we may remove both $g(t,x)$ and $g(t',x')$ from the integral. In other words, we have

$$J_{22}^B(t,t',x,x') = \int_0^{t-1/2(t'-t)} \int_{\mathbb{R}^n} \Big(\frac{g(s,x+2\sqrt{t'-s}z)}{t'-s} - \frac{g(s,x+2\sqrt{t-s}z)}{t-s}\Big) h_{ij}(z) \frac{dsdz}{\pi^{n/2}}. \qquad (2.5.9)$$

Exercise 2.5.3 Show that the integrand in (2.5.9) can be bounded from above by

$$C|z|^3e^{-|z|^2}\Big(\frac{|(\sqrt{t'-s}-\sqrt{t-s})z|)^\alpha}{t'-s}+\frac{1}{t-s}-\frac{1}{t'-s}\Big), \tag{2.5.10}$$

with the constant C that depends on the Hölder constant and the L^∞-norm of the function g.

Integrating out the z-variable, and using (2.5.10) we obtain

$$\begin{aligned} J_{22}^B(t,t',x,x') &\le C\int_0^{t-1/2(t'-t)}\frac{(\sqrt{t'-s}-\sqrt{t-s})^\alpha}{t'-s}ds+C\int_0^{t-1/2(t'-t)}\Big(\frac{1}{t-s}-\frac{1}{t'-s}\Big)ds \\ &\le C(t'-t)^{\alpha/2}+C_\varepsilon(t'-t),\quad \varepsilon\le t\le t'. \end{aligned} \tag{2.5.11}$$

We conclude that

$$J_{22}(t,t',x,x')\le C_\varepsilon(t'-t)^{\alpha/2},\quad \varepsilon\le t\le t' \tag{2.5.12}$$

The estimate of $J_{21}(t,t',x,x')$. Now, we estimate

$$\begin{aligned} &J_{21}(t,t',x,x') \\ &=\int_0^t\int_{\mathbb{R}^n}h_{ij}(z)\Big[\frac{g(s,x'+2\sqrt{t'-s}z)-g(t',x')}{t'-s}-\frac{g(s,x+2\sqrt{t'-s}z)-g(t',x)}{t'-s}\Big]\frac{dsdz}{\pi^{n/2}} \\ &=J_{21}^A+J_{21}^B. \end{aligned} \tag{2.5.13}$$

The two terms above refer to the integration over the time interval $A=\{t-|x-x'|^2\le s\le t\}$ and its complement B. In the first domain, we just use the bounds

$$|g(s,x'+2\sqrt{t'-s}z)-g(t',x')|\le C(t'-s)^{\alpha/2}|z|\le C(|t'-t|^{\alpha/2}+|x-x'|^\alpha)|z|,$$

and

$$|g(s,x+2\sqrt{t'-s}z)-g(t',x)|\le C(t'-s)^{\alpha/2}|z|\le C(|t'-t|^{\alpha/2}+|x-x'|^\alpha)|z|.$$

After integrating out the z-variable, this leads to

$$|J_{21}^A(t,t',x,x')|\le C(|t'-t|^{\alpha/2}+|x-x'|^\alpha). \tag{2.5.14}$$

As h_{ij} has zero mass, and $t'-s$ is strictly positive when $s\in B$, we can drop the terms involving $g(t',x')$ and $g(t',x)$, leading to

$$\begin{aligned} J_{21}^B(t,t',x,x') &=\int_0^{t-|x-x'|^2}\int_{\mathbb{R}^n}h_{ij}(z)\frac{g(s,x'+2\sqrt{t'-s}z)-g(s,x+2\sqrt{t'-s}z)}{t'-s}\frac{dsdz}{\pi^{n/2}} \\ &=\int_0^{t-|x-x'|^2}\int_{\mathbb{R}^n}\Big(h_{ij}\Big(\frac{x'-y}{2\sqrt{t'-s}}\Big)-h_{ij}\Big(\frac{x-y}{2\sqrt{t'-s}}\Big)\Big)\frac{g(s,y)}{t'-s}\frac{dsdy}{(4\pi(t'-s))^{n/2}}. \end{aligned} \tag{2.5.15}$$

Once again, because h_{ij} has zero mass we have

$$J_{21}^B(t,t',x,x')=\int_0^{t-|x-x'|^2}\int_{\mathbb{R}^n}\Big(h_{ij}\Big(\frac{x'-y}{2\sqrt{t'-s}}\Big)-h_{ij}\Big(\frac{x-y}{2\sqrt{t'-s}}\Big)\Big)\frac{g(s,y)-g(t',x')}{t'-s}\frac{dsdy}{(4\pi(t'-s))^{n/2}}.$$

The integrand above can be re-written as

$$
\begin{aligned}
&\Big(h_{ij}\Big(\frac{x'-y}{2\sqrt{t'-s}}\Big)-h_{ij}\Big(\frac{x-y}{2\sqrt{t'-s}}\Big)\Big)\frac{g(s,y)-g(t',x')}{t'-s} \\
&=\frac{1}{2}\int_0^1\frac{g(s,y)-g(s,x_\sigma)+g(s,x_\sigma)-g(t',x')}{(t'-s)^{3/2}}(x'-x)\cdot\nabla h_{ij}\Big(\frac{x_\sigma-y}{2\sqrt{t'-s}}\Big)d\sigma,
\end{aligned}
\tag{2.5.16}
$$

with $x_\sigma=\sigma x+(1-\sigma)x'$. It follows that

$$
\begin{aligned}
|J_{21}^B(t,t',x,x')|\le C_g|x-x'|&\int_0^{t-|x-x'|^2}\int_0^1\int_{\mathbb{R}^n}\Big|\nabla h_{ij}\Big(\frac{x_\sigma-y}{2\sqrt{t'-s}}\Big)\Big| \\
&\times\frac{|y-x_\sigma|^\alpha+|x'-x_\sigma|^\alpha+|t'-s|^{\alpha/2}}{(t'-s)^{3/2}}\frac{dsdyd\sigma}{(t'-s)^{n/2}},
\end{aligned}
\tag{2.5.17}
$$

with the constant C_g that is a multiple of the Hölder constant of g. Estimating $|\nabla h(z)|$ by $|z|^3e^{-|z|^2}$ and $|x'-x_\sigma|$ by $|x-x'|$, and making the usual change of variable

$$
y\to z=\frac{x_\sigma-y}{2\sqrt{t'-s}},
$$

and integrating out the z-variable, we arrive at

$$
|J_{21}^B(t,t',x,x')|\le C_g|x-x'|\int_0^{t-|x-x'|^2}\Big(\frac{1}{(t-s)^{(3-\alpha)/2}}+\frac{|x-x'|^\alpha}{(t-s)^{3/2}}\Big)ds. \tag{2.5.18}
$$

Integrating out the s-variable, we obtain

$$
|J_{21}^B(t,t',x,x')|\le C|x-x'|\big(|x-x'|^{2(\alpha/2-1/2)}+|x-x'|^\alpha|x-x'|^{-1}\big)\le C|x-x'|^\alpha, \tag{2.5.19}
$$

thus J_{21} is also Hölder continuous, finishing the proof. □

A remark on the constant coefficients case

Let us now consider solutions of general constant coefficients equations of the form

$$
u_t-a_{ij}\partial_{x_ix_j}u+b_j\partial_{x_j}u+cu=f(t,x). \tag{2.5.20}
$$

We assume that a_{ij}, b_i and c are constants, and the matrix $A:=(a_{ij})$ is positive definite: there exists a constant $\lambda>0$ so that for any vector $\xi\in\mathbb{R}^n$ we have

$$
a_{ij}\xi_i\xi_j\ge\lambda|\xi|^2. \tag{2.5.21}
$$

Assume also that f is an α-Hölder function over $[0,T]\times\mathbb{R}^n$, and take the initial condition $v(0,x)\equiv 0$. The function $v(t,x)=u(t,x+Bt)\exp(ct)$, with $B=(b_1,\dots,b_n)$, solves

$$
v_t-a_{ij}\partial_{x_ix_j}v=f(t,x+Bt). \tag{2.5.22}
$$

The change of variable $w(t,x)=v(t,\sqrt{A}x)$ brings us back to the forced heat equation:

$$
w_t-\Delta w=f(t,\sqrt{A}(x+Bt)). \tag{2.5.23}
$$

We see that the conclusion of Theorem 2.5.1 also applies to other parabolic equations with constant coefficients, as long as the ellipticity condition (2.5.21) holds.

Exercise 2.5.4 Consider the solutions of the equation

$$u_t - u_{xx} + u_y = f(t, x, y), \tag{2.5.24}$$

in $\mathbb{R}^2$ and use this example to convince yourself that the ellipticity condition is necessary for the Hölder regularity as in Theorem 2.5.1 to hold.

The Cauchy problem for the inhomogeneous coefficients

Theorem 2.5.1 is the last one that we proved fully in this section. In the rest, we will only give a sketch of the proofs, and sometimes we will not state the results in a formal way. However, we have all the ideas to attack the first big piece of this section, the Cauchy problem for the parabolic equations with variable coefficients:

$$\begin{aligned} &u_t - a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u = 0, \quad t > 0,\ x \in \mathbb{R}^n, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{2.5.25}$$

We make the following assumptions on the coefficients: first, they are sufficiently regular – the functions $(a_{ij}(t,x))_{1\le i,j\le N}$, $(b_j(t,x))_{1\le j\le N}$ and $c(t,x)$, all α-Hölder continuous over $[0,T]\times\mathbb{R}^n$. Second, we make the ellipticity assumption, generalizing (2.5.21): there exist $\lambda > 0$ and $\Lambda > 0$ so that for any vector $\xi \in \mathbb{R}^n$ and any $x \in \mathbb{R}^n$ we have

$$\lambda|\xi|^2 \le a_{ij}(t,x)\xi_i\xi_j \le \Lambda|\xi|^2. \tag{2.5.26}$$

We assume that the initial condition $u_0(x)$ is a continuous function – this assumption can be very much weakened but we do not focus on it right now.

Theorem 2.5.5 *The Cauchy problem* (2.5.25) *has a unique solution $u(t,x)$, whose Hölder norm on the sets of the form $[\varepsilon,T]\times\mathbb{R}^n$ is controlled by the L^∞ norm of u_0.*

Exercise 2.5.6 Show that the uniqueness of the solution is an immediate consequence of the maximum principle.

Thus, the main issue is the construction of a solution with the desired regularity. The idea is to construct the fundamental solution of (2.5.25), that is, the solution $E(t,s,x,y)$ of (2.5.25) on the time interval $s \le t \le T$, instead of $0 \le t \le T$:

$$\partial_t E - a_{ij}(t,x)\frac{\partial E}{\partial x_i}x_j + b_j(t,x)\frac{\partial E}{\partial x_j} + c(t,x)E = 0, \quad t > s,\ x \in \mathbb{R}^n, \tag{2.5.27}$$

with the initial condition

$$E(t = s, s, x, y) = \delta(x - y), \tag{2.5.28}$$

the Dirac mass at $x = y$. The solution of (2.5.25) can then be written as

$$u(t,x) = \int_{\mathbb{R}^n} E(t,0,x,y)u_0(y)dy. \tag{2.5.29}$$

Thus, if can show that $E(t,s,x,y)$ is smooth enough (at least away from $t=s$), $u(t,x)$ will satisfy the desired estimates as well. Note that this is a very strong property: the initial condition in (2.5.28) at $t=s$ is a measure – and we need to show that for all $t>s$ the solution is actually a smooth function. On the other hand, this is exactly what happens for the heat equation

$$u_t = \Delta u,$$

where the fundamental solution is

$$E(t,s,x,y) = \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))}.$$

Exercise 2.5.7 Go back to the equation

$$u_t - u_{xx} + u_y = 0.$$

considered in Exercise 2.5.4. Show that its fundamental solution is not a smooth function in the y-variable. Thus, the ellipticity condition is important for this property.

The understanding of the regularity of the solutions of the Cauchy problem is also a key to the inhomogeneous problem because of the Duhamel principle.

Exercise 2.5.8 Let $f(t,x)$ be a Hölder-continuous function over $[0,T]\times\mathbb{R}^n$. Use the Duhamel principle to write down the solution of

$$\begin{aligned} & u_t - a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} u + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u = f(t,x), \quad t>0,\ x\in\mathbb{R}^n, \\ & u(0,x) = u_0(x), \quad x\in\mathbb{R}^n, \end{aligned} \tag{2.5.30}$$

in terms of $E(t,s,x,y)$.

Thus, everything boils down to constructing the fundamental solution $E(t,s,x,y)$, and a way to do it is via the parametrix method. Let us set $b_j = c = 0$ – this does not affect the essence of the arguments but simplifies the notation. The philosophy is that the possible singularities of $E(t,s,x,y)$ are localized at $t=s$ snd $x=y$ (as for the heat equation). Therefore, in order to capture the singularities of $E(t,s,x,y)$ we may try to simply freeze the coefficients in the equation at $t=s$ and $x=y$, and compare $E(t,s,x,y)$ to the fundamental solution $E_0(s,t,x,y)$ of the resulting equation:

$$\begin{aligned} & \partial_t E_0 - a_{ij}(s,y)\frac{\partial^2 E_0}{\partial x_i \partial x_j} = 0, \quad t>s,\ x\in\mathbb{R}^n, \\ & E_0(t=s,x) = \delta(x-y), \quad x\in\mathbb{R}^n. \end{aligned} \tag{2.5.31}$$

There is no reason to expect the two fundamental solutions to be close – they satisfy different equations. Rather, the expectation is that that E will be a smooth perturbation of E_0 – and, since E_0 solves an equation with constant coefficients (remember that s and y are fixed here), we may compute it exactly.

To this end, let us write the equation for $E(t,s,x,y)$ as

$$\begin{aligned}
&\partial_t E - a_{ij}(s,y)\frac{\partial^2 E}{\partial x_i \partial x_j} = F(t,x), \quad t > s, \; x \in \mathbb{R}^n, \\
&E(t = s, x) = \delta(x - y), \quad x \in \mathbb{R}^n,
\end{aligned} \tag{2.5.32}$$

with the right side

$$F(t,x,y) = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E}{\partial x_i \partial x_j}. \tag{2.5.33}$$

The difference

$$R_0 = E - E_0$$

satisfies

$$\partial_t R_0 - a_{ij}(s,y)\frac{\partial^2 R_0}{\partial x_i \partial x_j} = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E_0}{\partial x_i \partial x_j} + F_0(t,x), \quad t > s, \tag{2.5.34}$$

with the initial condition $R_0(t = s, x) = 0$, and

$$F_0(t,x) = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 R_0}{\partial x_i \partial x_j}. \tag{2.5.35}$$

Let us further decompose

$$R_0 = E_1 + R_1.$$

Here, E_1 is the solution of

$$\partial_t E_1 - a_{ij}(s,y)\frac{\partial^2 E_1}{\partial x_i \partial x_j} = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E_0}{\partial x_i \partial x_j}, \quad t > s, \tag{2.5.36}$$

with the initial condition $E_1(t = s, x) = 0$. The remainder R_1 solves

$$\partial_t R_1 - a_{ij}(s,y)\frac{\partial^2 R_1}{\partial x_i \partial x_j} = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E_1}{\partial x_i \partial x_j} + F_1(t,x), \quad t > s, \tag{2.5.37}$$

with $R_1(t = s, x) = 0$, and

$$F_1(t,x) = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 R_1}{\partial x_i \partial x_j}. \tag{2.5.38}$$

Equation (2.5.36) for E_1 is a forced parabolic equation with constant coefficients – as we have seen, its solutions behave exactly like those of the standard heat equation with a forcing, except for rotations and dilations. We may assume without loss of generality that $a_{ij}(s,y) = \delta_{ij}$, so that the reference fundamental solution is

$$E_0(t,s,x,y) = \frac{1}{(4\pi(t-s))^{n/2}} e^{-(x-y)^2/(4(t-s))}, \tag{2.5.39}$$

and (2.5.36) is simply a forced heat equation:

$$\partial_t E_1 - \Delta E_1 = [a_{ij}(t,x) - \delta_{ij}]\frac{\partial^2 E_0(t,s,x,y)}{\partial x_i \partial x_j}, \quad t > s, \; x \in \mathbb{R}^n. \tag{2.5.40}$$

The functions $a_{ij}(t,x)$ Hölder continuous, with $a_{ij}(s,y) = \delta_{ij}$. The regularity of E_1 can be approached by the tools of the previous sections – after all, (2.5.36) is just another forced heat equation! The next exercise may be useful for understanding what is going on. .

Exercise 2.5.9 Consider, instead of (2.5.36) the solution of

$$\partial_t z - \Delta z = \frac{\partial^2 E_0(t,s,x,y)}{\partial x_i \partial x_j}, \quad t > s, \ x \in \mathbb{R}^n, \tag{2.5.41}$$

with $z(t = s, x) = 0$. How does its regularity compare to that of E_0? Now, what can you say about the regularity of the solution to (2.5.40), how does the factor $[a_{ij}(t,x) - \delta_{ij}]$ help to make E_1 more regular than z? In which sense is E_1 more regular than E_0?

With this understanding in hand, one may consider the iterative process: write

$$R_1 = E_2 + R_2,$$

with E_2 the solution of

$$\partial_t E_2 - a_{ij}(s,y)\frac{\partial^2 E_2}{\partial x_i \partial x_j} = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E_1}{\partial x_i \partial x_j}, \quad t > s, \tag{2.5.42}$$

with $E_2(t = s, x) = 0$, and R_2 the solution of

$$\partial_t R_2 - a_{ij}(s,y)\frac{\partial^2 R_2}{\partial x_i \partial x_j} = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 E_2}{\partial x_i \partial x_j} + F_2(t,x), \quad t > s, \tag{2.5.43}$$

with $R_2(t = s, x) = 0$, and

$$F_2(t,x) = (a_{ij}(t,x) - a_{ij}(s,y))\frac{\partial^2 R_2}{\partial x_i \partial x_j}. \tag{2.5.44}$$

Continuing this process, we have a representation for $E(t,s,x,y)$ as

$$E = E_0 + E_1 + \cdots + E_n + R_n, \tag{2.5.45}$$

with each next term E_j more regular than $E_0, \ldots, E_{j-1}$. Regularity of all E_j can be inferred as in Exercise 2.5.9. One needs, of course, also to estimate the remainder R_n to obtain a "true theorem" but we leave this out of this chapter, to keep the presentation short. An interested reader should consult the aforementioned references for full details. We do, however, offer the reader another (not quite trivial) exercise.

Exercise 2.5.10 Prove that $E(s,t,s,y)$ has Gaussian estimates of the form:

$$m\frac{e^{-|x-y|^2/Dt}}{(t-s)^{n/2}} \le E(s,t,x,y) \le M\frac{e^{-|x-y|^2/dt}}{(t-s)^{n/2}}.$$

The constants m and M, unfortunately, depend very much on T; however the constants d and D do not.

Interior regularity

So far, we have considered parabolic problems in the whole space $\mathbb{R}^n$, without any boundaries. One of the miracles of the second order diffusion equations is that the regularity properties are *local.* That is, the regularity of the solutions in a given region only depends on how regular the coefficients are in a slightly larger region. Consider, again, the inhomogeneous parabolic equation

$$u_t - a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u = f(t,x), \quad t > 0, \tag{2.5.46}$$

and assume that the coefficients $a_{ij}(t,x)$, $b_j(t,x)$ and $c(t,x)$, and forcing $f(t,x)$, are α-Hölder in $S = [0,T] \times B_R(x_0)$. It turns out that the derivatives $D^2u(t,x)$ and $\partial_t u(t,x)$ are α-Hölder in a smaller set of the form $S = [\varepsilon, T] \times B_{(1-\varepsilon)R}(x_0)$, for any $\varepsilon > 0$. The most important point is that the Hölder norm of u in S is controlled only by ε, R, and the Hölder norms of the coefficients and the L^∞ bound of u, both inside the larger set S. Note that the Hölder estimates on u do not hold in the original set S, we need a small margin, going down to the smaller set S_ε.

Exercise 2.5.11 Prove this fact. One standard way to do it is to pick a nonnegative and smooth function $\gamma(x)$, equal to 1 in $B_{R/2}(x_0)$ and 0 outside of $B_R(x)$, and to write down an equation for $v(t,x) = \gamma(x)u(t,x)$. Note that this equation is now posed on $(0,T] \times \mathbb{R}^n$, and that the whole spacee theory can be applied. The computations should be, at times cumbersome. If in doubt, consult [60]. Looking ahead, we will use this strategy in the proof of Lemma 2.8.3 in Section 2.8.1 below, so the reader may find it helpful to read this proof now.

Specifying the Dirichlet boundary conditions allows to get rid of this small margin, and this is the last issue that we are going to discuss in this section. Let us consider equation (2.5.46), posed this time in $(0,T] \times \Omega$, where Ω is bounded smooth open subset of $\mathbb{R}^n$. As a side remark, it is not crucial that Ω be bounded. However, if Ω is unbounded, we should ask its boundary to oscillate not too much at infinity. Let us supplement (2.5.46) by an initial condition $u(0,x) = u_0(x)$ in Ω, with a continuous function u_0, and the Dirichlet boundary condition

$$u(t,x) = 0 \text{ for } 0 \le t \le T \text{ and } x \in \partial\Omega. \tag{2.5.47}$$

Theorem 2.5.12 *Assume $a_{ij}(t,x)$, $b_j(t,x)$, $c(t,x)$, and $f(t,x)$ are α-Hölder in $(0,T] \times \overline{\Omega}$ – note that, here, we do need the closure of Ω. The above initial-boundary value problem has a unique solution $u(t,x)$ such that $D^2u(t,x)$ and $\partial_t u(t,x)$ are α-Hölder in $[\varepsilon, T] \times \bar{\Omega}$, with their Hölder norms controlled by the L^∞ bound of u_0, and the Hölder norms of the coefficients and f.*

The way to prove this result parallels the way we followed to establish Theorem 2.5.5. First, we write down an explicit solution on a model situation. Then, we prove the regularity in the presence of a Hölder forcing in the model problem. Once this is done, we turn to general constant coefficients. Then, we do the parametric method on the model situation. Finally, we localize the problem and reduce it to the model situation.

Let us be more explicit. The model situation is the heat equation in a half space

$$\Omega_n = \mathbb{R}^n_+ := \{x = (x_1, \dots x_n) \in \mathbb{R}^n : \ x_n > 0\}.$$

Setting $x' = (x_1, \dots x_{n-1})$, we easily obtain the solution of the initial boundary value problem

$$\begin{aligned} &u_t - \Delta u = 0, \quad t > 0, \ x \in \Omega_n, \\ &u(t, x', 0) = 0, \\ &u(0, x) = u_0(x), \end{aligned} \tag{2.5.48}$$

as

$$u(t, x) = \int_{\mathbb{R}^n} E_0(t, x, y) u_0(y) dy, \tag{2.5.49}$$

with the fundamental solution

$$E_0(t, x, y) = \frac{e^{-(x'-y')^2/4t}}{(4\pi t)^{n/2}} \Big(e^{-(x_n - y_n)^2/4t} - e^{-(x_n+y_n)^2/4t} \Big). \tag{2.5.50}$$

Let us now generalize step by step: for an equation with a constant drift

$$u_t - \Delta u + b_j \partial_{x_j} u = 0, \quad t > 0, \ x \in \Omega_n, \tag{2.5.51}$$

the change of unknowns $u(t, x) = e^{x_n b_n/2} v(t, x)$ transforms the equation into

$$v_t - \Delta v + b_j \partial_{x'_j} v - \frac{b_n^2}{4} v = 0, \quad t > 0, \ x \in \Omega_n. \tag{2.5.52}$$

Thus, the fundamental solution, for (2.5.51) is

$$E(t, x, y) = e^{tb_n^2/4 - xb_n/2} E_0(t, x - tB', y), \quad B' = (b_1, \dots B_{n-1}, 0). \tag{2.5.53}$$

For an equation of the form

$$u_t - a_{ij} \partial_{x_i x_j} u = 0, \quad t > 0, \ x \in \Omega_n, \tag{2.5.54}$$

with a constant positive-definite diffusivity matrix a_{ij}, we use the fact that the function

$$u(t, x) = v(t, \sqrt{A^{-1}} x),$$

with $v(t, x)$ a solution of the heat equation

$$v_t = \Delta v,$$

solves (2.5.54). For an equation mixing the two sets of coefficients, one only has to compose the transformations. At that point, one can, with a nontrivial amount of computations, prove the desired regularity for the solutions of

$$u_t - a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_j \frac{\partial u}{\partial x_j} + cu = f(t, x) \tag{2.5.55}$$

with constant coefficients, and the Dirichlet boundary conditions on $\partial\Omega_n$. Then, one can use the parametrix method to obtain the result for general inhomogeneous coefficients. This is how one proves Theorem 2.5.12 for $\Omega_n = \mathbb{R}^n_+$.

How does one pass to a general Ω? Unfortunately, the work is not at all finished yet. One still has to prove a local version of the already proved theorem in Ω_n, in the spirit of the local regularity in $\mathbb{R}^n$, up to the fact that we must not avoid the boundary. Once this is done, consider a general Ω. Cover its boundary $\partial\Omega$ with balls such that, in each of them, $\partial\Omega$ is a graph in a suitable coordinate system. By using this new coordinate system, one retrieves an equation of the form (2.5.30), and one has to prove that the diffusion coefficients satisfy a coercivity inequality. At this point, maximal regularity for the Dirichlet problem is proved.

Of course, all kinds of local versions (that is, versions of Theorem 2.5.12 where the coefficients are α-Höder only in a part of $\overline{\Omega}$) are available. Also, most of the above material is valid for the Neumann boundary conditions

$$\partial_\nu u = 0 \text{ on } \partial\Omega,$$

or Robin boundary conditions

$$\partial_\nu u + \gamma(t,x)u = 0 \text{ on } \partial\Omega.$$

We encourage the reader who might still be interested in the subject to try to produce the full proofs, with an occasional help from the books we have mentioned.

The Harnack inequalities

We will only touch here on the Harnack inequalities, a very deep and involved topic of parabolic equations. In a nutshell, the Harnack inequalities allow to control the infimum of a positive solution of a parabolic equation by a universal fraction of its maximum, modulo a time shift. They provide one possible path to regularity, but we will ignore this aspect here. They are also mostly responsible for the behaviors that are very specific to the diffusion equations, as will be seen in the next section.

We are going to prove what is, in a sense, a "‘poor man's"’ version. It is not as scale invariant as one would wish, and uses the regularity theory instead of proving it. It is, however, suited to what we wish to do, and already gives a good account of what is going on. Consider our favorite equation

$$u_t - \Delta u + b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u = 0, \tag{2.5.56}$$

with smooth coefficients b_j and c, posed for $t \in (0,T)$, and $x \in B_{R+1}(0)$. We stress that the variable smooth diffusion coefficients could be put in the picture.

Theorem 2.5.13 *Let $u(t,x) \geq 0$ be a non-negative bounded solution of (2.5.56) for $0 \leq t \leq T$ and $x \in B_{R+1}(0)$, and assume that for all $t \in [0,T]$:*

$$\sup_{|x|\leq R+1} u(t,x) \leq k_2, \quad \sup_{|x|\leq R} u(t,x) \geq k_1. \tag{2.5.57}$$

There is a constant $h_R > 0$, that does not depend on T, but that depends on k_1 and k_2, such that, for all $t \in [1, T]$:

$$h_R \leq \inf_{|x| \leq R} u(t, x). \tag{2.5.58}$$

Proof. The proof is by contradiction. Assume that there exists a sequence u_n of the solutions of (2.5.56) satisfying (2.5.57), and $t_n \in [1, T]$, and $x_n \in B_R(0)$, such that

$$\lim_{n \to +\infty} u_n(t_n, x_n) = 0. \tag{2.5.59}$$

Up to a possible extraction of a subsequence, we may assume that

$$t_n \to t_\infty \in [1, T] \text{ and } x_n \to x_\infty \in B_R(0).$$

The Hölder estimates on u_n and its derivatives in Theorem 2.5.12 together with the Ascoli-Arzela theorem, imply that the sequence u_n is relatively compact in $C^2([t_\infty - 1/2] \times B_{R+1/2}(0))$. Hence, again, after a possible extraction of a subsequence, we may assume that u_n converges to $u_\infty \in C^2([t_\infty - 1/2, T] \times B_{R+1/2}(0))$, together with its first two derivatives in x and the first derivatives in t. Thus, the limit $u_\infty(t, x)$ satisfies (2.5.56) for $t_\infty - 1/2 \leq t \leq T$, and $x \in B_{R+1/2}(0))$, and is non-negative. It also satisfies the bounds in (2.5.57), hence it is not identically equal to zero. Moreover it satisfies $u_\infty(t_\infty, x_\infty) = 0$. This contradicts the strong maximum principle. □

2.6 The long time behavior for the Allen-Cahn equation

We will see in this section how the possibility of comparing two solutions of the same problem will imply their convergence in the long time limit, putting to work the two main characters we have seen so far in this chapter: the comparison principle and the Harnack inequality.

We consider the one-dimensional Allen-Cahn equation

$$u_t - u_{xx} = f(u), \tag{2.6.1}$$

with

$$f(u) = u - u^3. \tag{2.6.2}$$

Recall that we have already considered the steady solutions of this equation in Section 1.4.3 of Chapter 1, and, in particular, the role of its explicit time-independent solutions

$$\phi(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \tag{2.6.3}$$

and its translates $\phi_{x_0}(x) := \phi(x + x_0)$, $x_0 \in \mathbb{R}$.

Exercise 2.6.1 We have proved in Chapter 1 that, if $\psi(x)$ is a steady solution to (2.6.1) that satisfies

$$\lim_{x \to -\infty} \psi(x) = -1, \quad \lim_{x \to +\infty} \psi(x) = 1,$$

then ψ is a translate of ϕ. For an alternative proof, draw the phase portrait of the equation

$$-\psi'' = f(\psi) \tag{2.6.4}$$

in the (ψ,ψ') plane. For an orbit (ψ,ψ') connecting $(-1,0)$ to $(1,0)$, show that the solution tends to $(-1,0)$ exponentially fast. Multiply then (2.6.4) by ψ', integrate from $-\infty$ to x and conclude.

Recall that the Allen-Cahn equation is a simple model for a physical situation when two phases are stable, corresponding to $u=\pm 1$. The time dynamics of the initial value problem for (2.6.1) corresponds to a competition between these two states. The fact that

$$\int_{-1}^{1} f(u)du = 0 \tag{2.6.5}$$

means that the two states are "equally stable" – this is a necessary condition for (2.6.1) to have a time-independent solution $\phi(x)$ such that

$$\phi(x) \to \pm 1, \quad \text{as } x \to \pm\infty. \tag{2.6.6}$$

In other words, such connection between $+1$ and -1 exists only if (2.6.5) holds.

Since the two phases $u=\pm 1$ are equally stable, one expects that if the initial condition $u_0(x)$ for (2.6.1) satisfies

$$\lim_{x\to-\infty} u_0(x) = -1, \qquad \lim_{x\to+\infty} u_0(x) = 1, \tag{2.6.7}$$

then, as $t\to+\infty$, the solution $u(t,x)$ will converge to a steady equilibrium, that has to be a translate of ϕ. This is the subject of the next theorem, that shows, in addition, that the convergence rate is exponential.

Theorem 2.6.2 *There exists $\omega>0$ such that for any uniformly continuous and bounded initial condition u_0 for (2.6.1) that satisfies (2.6.7), we can find $x_0\in\mathbb{R}$ and $C_0>0$ such that*

$$|u(t,x)-\phi(x+x_0)| \le C_0 e^{-\omega t}, \quad \textit{for all } x\in\mathbb{R} \textit{ and } t>0. \tag{2.6.8}$$

Since there is a one parameter family of steady solutions, naturally, one may ask how the solution of the initial value problem chooses a particular translation of ϕ in the long time limit. In other words, one would like to know how the shift x_0 depends on the initial condition u_0. However, this dependence is quite implicit and there is no simple expression for x_0.

There are at least two ways to prove Theorem 2.6.2. The first one starts with the following:

Exercise 2.6.3 Verify that the energy functional

$$J(u) = \int_{\mathbb{R}} \left(\frac{1}{2}|u_x|^2 - F(u)\right)dx, \quad F(u) = \int_{-1}^{u} f(v)dv,$$

decreases in time for any solution $u(t,x)$ of (2.6.1).

With the aid of an estimate showing that the level sets of u do not escape to infinity, one then proves that the solution eventually comes very close to a translate $\phi_{x_0}(x)$, uniformly on $\mathbb{R}$, at some (possibly very large) time τ. Next, one uses a stability argument, based on the analysis of the first eigenvalue – something we will encounter soon in this chapter – of the operator

$$\mathcal{M}u = -u_{xx} - f'(\phi_{x_0})u.$$

This stability result shows that if $u(t,x)$ is close to $\phi_{x_0}(x)$ at $t = \tau$, then $u(t,x)$ will stay close to $\phi_{x_0}(x)$ for all $t > \tau$. An iteration of this argument completes the proof. This is the method devised in the beautiful paper of Fife and McLeod [71]. It has been generalized to gradient systems in a remarkable paper of Risler [126], which proves very precise spreading estimates of the leading edge of the solutions, only based on a one-dimensional set of energy functionals. Risler's ideas were put to work on the simpler example (2.6.1) in a paper by Gallay and Risler [78].

We chose to present an alternative method, entirely based on sub and super-solutions that come closer and closer to each other. It avoids the spectral arguments of the preceding proof, and is more flexible as there are many reaction-diffusion problems where the comparison principle and the Harnack inequality are available but the energy functionals do not exist. The reader should also be aware that there are many problems, such as many reaction-diffusion systems, where the situation is the opposite: the energy functional exists but the comparison principle is not applicable.

Before we begin, we note that the function f satisfies

$$f'(u) \le -1 \quad \text{for } |u| \ge 5/6, \qquad f'(u) \le -3/2 \quad \text{for } |u| \ge 11/12. \tag{2.6.9}$$

We will also take $R_0 > 0$ such that

$$|\phi(x)| \ge 11/12 \text{ for } |x| \ge R_0. \tag{2.6.10}$$

A bound on the level sets

The first ingredient is to prove that the level sets of $u(t,x)$ do not, indeed, go to infinity, so that the region of activity, where $u(t,x)$ is not too close to ± 1, happens, essentially, in a compact set. This crucial step had already been identified by Fife and McLeod, and we reproduce here their argument. The idea is to squish $u(t,x)$ between two different translates of ϕ, with a correction that goes to zero exponentially in fast time.

Lemma 2.6.4 *Let u_0 satisfy the assumptions of the theorem. There exist $\xi_\infty^\pm \in \mathbb{R}$, and $q_0 > 0$, such that*

$$\phi(x + \xi_\infty^-) - q_0 e^{-t} \le u(t,x) \le \phi(x + \xi_\infty^+) + q_0 e^{-t}, \tag{2.6.11}$$

for all $t \ge 0$ and $x \in \mathbb{R}$.

Proof. For the upper bound, we are going to devise two functions $\xi^+(t)$ and $q(t)$ such that

$$\overline{u}(t,x) = \phi(x + \xi^+(t)) + q(t) \tag{2.6.12}$$

is a super-solution to (2.6.1), with an increasing but bounded function $\xi^+(t)$, and an exponentially decreasing function $q(t) = q_0 \exp(-t)$. One would also construct, in a similar way, a sub-solution of the form

$$\underline{u}(t,x) = \phi(x + \xi^-(t)) - q(t), \tag{2.6.13}$$

possibly increasing q a little, with a decreasing but bounded function $\xi^-(t)$.

Let us denote

$$N[u] = \partial_t u - u_{xx} - f(u). \tag{2.6.14}$$

Now, with $\overline{u}(t,x)$ as in (2.6.12), we have

$$N[\overline{u}] = \dot{q} + \dot{\xi}^+\phi'(\zeta) - f(\phi(\zeta)+q) + f(\phi(\zeta)), \tag{2.6.15}$$

with $\zeta = x + \xi^+(t)$. Our goal is to choose $\xi^+(t)$ and $q(t)$ so that

$$N[\overline{u}] \geq 0, \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}, \tag{2.6.16}$$

so that $\bar{u}(t,x)$ is a super-solution to (2.6.1). We will consider separately the regions $|\zeta| \leq R_0$ and $|\zeta| \geq R_0$.

Step 1. The region $|\zeta| \geq R_0$. First, we have

$$\phi(\zeta) + q(t) \geq 11/12 \text{ for } \zeta \geq R_0,$$

as $q(t) \geq 0$. If we assume that $q(0) \leq 1/12$ and make sure that $q(t)$ is decreasing in time, then we also have

$$\phi(\zeta) + q \leq -5/6 \text{ for } \zeta \leq -R_0.$$

We have, therefore, as long as $\xi^+(t)$ is increasing, using (2.6.9):

$$N[\overline{u}] \geq \dot{q} - f(\phi(\zeta)+q) + f(\phi) \geq \dot{q} + q, \text{ for } |\zeta| \geq R_0. \tag{2.6.17}$$

It suffices, therefore, to choose

$$q(t) = q(0)e^{-t}, \tag{2.6.18}$$

with $q(0) \leq 1/12$, and an increasing $\xi^+(t)$, to ensure that

$$N[\overline{u}] \geq 0, \quad \text{for all } t \geq 0 \text{ and } |\zeta| \geq R_0. \tag{2.6.19}$$

Step 2. The region $|\zeta| \leq R_0$. This time, we have to choose $\xi^+(t)$ properly. We write

$$N[\overline{u}] \geq \dot{q} + \dot{\xi}^+\phi'(\zeta) - M_f q, \quad M_f = \|f'\|_{L^\infty}, \tag{2.6.20}$$

and choose

$$\dot{\xi}^+ = \frac{1}{k_0}\big(-\dot{q} + M_f q\big), \quad k_0 = \inf_{|\zeta|\leq R_0} \phi'(\zeta), \tag{2.6.21}$$

to ensure that the right side of (2.6.20) is non-negative. Using expression (2.6.18) for $q(t)$, we obtain

$$\xi^+(t) = \xi^+(0) + \frac{q(0)}{k_0}(1+M_f)(1-e^{-t}). \tag{2.6.22}$$

To summarize, with the above choices of $q(t)$ and $\xi^+(t)$, we know that $\overline{u}$ satisfies (2.6.16).

It remains to choose $q(0)$ and $\xi^+(0)$ so that $\overline{u}(t,x)$ is actually above $u(t,x)$ – as we have already established (2.6.16), the comparison principle tells us that we only need to verify that

$$\overline{u}(0,x) \geq u_0(x), \quad \text{for all } x \in \mathbb{R}. \tag{2.6.23}$$

Because u_0 tends to ± 1 at $\pm\infty$, there exists ξ_0^+ (possibly quite large), and $q_0 \in (0, 1/12)$ such that

$$u_0(x) \leq \phi(x + \xi_0^+) + q_0. \tag{2.6.24}$$

Thus, it is enough to choose $q(0) = q_0$, $\zeta^+(0) = \zeta_0^+$. $\square$

Exercise 2.6.5 Follow the same strategy to construct a sub-solution $\underline{u}(t,x)$ as in (2.6.13).

Lemma 2.6.4 traps nicely the level sets of u. But will this imply convergence to a steady solution, or will the level sets of $u(t,x)$ oscillate inside a bounded set? First, let us restate our findings in a more precise way. We have shown the following

Corollary 2.6.6 *Assume that we have*

$$\phi(x+\xi_0^-)-q_0\le u_0(x)\le\phi(x+\xi_0^+)+q_0, \tag{2.6.25}$$

with $0\le q_0\le 1/12$. *Then, we have*

$$\phi(x+\xi^-(t))-q(t)\le u_0(x)\le\phi(x+\xi^+(t))+q(t). \tag{2.6.26}$$

with $q(t)=q_0e^{-t}$, *and*

$$\xi^+(t)=\xi_0^+ +\frac{q_0}{k_0}(1+M_f)(1-e^{-t}),\quad \xi^-(t)=\xi_0^- -\frac{q_0}{k_0}(1+M_f)(1-e^{-t}). \tag{2.6.27}$$

One issue here is that the gap between $\xi^+(t)$ and $\xi^-(t)$ is not decreasing in time but rather increasing – the opposite of what we want! Our goal is to show that we can actually choose $\xi^+(t)$ and $\xi^-(t)$ in (2.6.26) so that the "sub-solution/super-solution gap" $\xi^+(t)-\xi^-(t)$ would decrease to zero as $t\to+\infty$ – this will prove convergence of the solution to a translate of ϕ. The mechanism to decrease this difference will be kindly provided by the strong maximum principle. The idea is to iteratively trap the solutions, at an increasing sequence of times, between translates of ϕ_0, that will come closer and closer to each other, thus implying the convergence. However, as there will be some computations, it is worth explaining beforehand what the main idea is, and which difficulties we will see.

Let us consider for the moment a slightly better situation than in Lemma 2.6.4 – assume that $u_0(x)$ is actually trapped between $\phi(x+\xi_0^-)$ and $\phi(x+\xi_0^+)$, without the need for an additional term $q(t)$:

$$\phi(x+\xi_0^-)\le u_0(x)\le\phi(x+\xi_0^+). \tag{2.6.28}$$

Then, $u(t,x)$ is at a positive distance from one of the two translates, on compact sets, at least for $0\le t\le 1$, say, $\phi(x+\xi_0^+)$. This is where the strong maximum principle strikes: at $t=1$, it will make the infimum of $\phi(x+\xi_0^+)-u(t,x)$ strictly positive, at least on a large compact set. We would like to think that then we may translate $\phi(x+\xi_0^+)$ to the right a little, decreasing ξ_0^+, while keeping it above $u(1,x)$. The catch is that, potentially, the tail of $u(1,x)$ – that we do not control very well at the moment – might go over $\phi(x+\xi)$, as soon as ξ is just a little smaller than ξ_0^+. Let us ignore this, and assume that magically we have

$$\phi(x+\xi_0^-)\le u(1,x)\le\phi(x+\xi_1^+), \tag{2.6.29}$$

with

$$\xi_1^+=\xi_0^+-\delta(\xi_0^+-\xi_0^-), \tag{2.6.30}$$

with some $\delta>0$. If we believe in this scenario, we might just as well hope that the situation may be iterated: at the time $t=n$, we have

$$\phi(x+\xi_n^-)\le u(n,x)\le\phi(x+\xi_n^+), \tag{2.6.31}$$

with

$$\xi^+_{n+1} - \xi^-_{n+1} \le (1-\delta)(\xi^+_n - \xi^-_n). \tag{2.6.32}$$

This would imply a geometric decay of $\xi^+_n - \xi^-_n$ to zero, which, in turn, would imply the exponential convergence of $u(t,x)$ to a translate of ϕ.

The gap in the previous argument is, of course, in our lack of control of the tail of $u(t,x)$ that prevents us from being sure that (2.6.29), with ξ^+_1 as in (2.6.30), holds everywhere on $\mathbb{R}$ rather than on a compact set. There is no way we can simply ignore it: we will see in Chapter **??** that the dynamics of many respectable equations is controlled exactly by the tail of its solutions. Such will not be the case here, but we will have to go through the pain of controlling the tail of u at every step. This leads to the somewhat heavy proof that follows. However, there is essentially no other idea than what we have just explained, the rest are just technical embellishments. The reader should also recall that we have already encountered a tool for the tail-control in the Allen-Cahn equation: Corollary 1.4.12 in Chapter 1 served exactly that purpose in the proof of Theorem 1.4.8. We are going to use something very similar here.

The proof of Theorem 2.6.2

As promised, the strategy is a refinement of the proof of Lemma 2.6.4. We will construct a sequence of sub-solutions $\underline{u}_n$ and super-solutions $\overline{u}_n$ defined for $t \ge T_n$, such that

$$\underline{u}_n(t,x) \le u(t,x) \le \overline{u}_n(t,x) \text{ for } t \ge T_n. \tag{2.6.33}$$

Here, $T_n \to +\infty$ is a sequence of times with

$$T_n + T \le T_{n+1} \le T_n + 2T, \tag{2.6.34}$$

and the time step $T > 0$ to be specified later on. The sub- and super-solutions will be of the familiar form (2.6.26)-(2.6.27):

$$\underline{u}_n(t,x) = \phi(x+\xi^-_n(t)) - q_n e^{-(t-T_n)}, \quad \overline{u}_n(t,x) = \phi(x+\xi^+_n(t)) + q_n e^{-(t-T_n)}, \quad t \ge T_n, \tag{2.6.35}$$

with $\xi^\pm_n(t)$ as in (2.6.27):

$$\xi^+_n(t) = \xi^+_n + \frac{q_n}{k_0}(1+M_f)(1-e^{-(t-T_n)}), \quad \xi^-_n(t) = \xi^-_n - \frac{q_n}{k_0}(1+M_f)(1-e^{-(t-T_n)}). \tag{2.6.36}$$

The reader has surely noticed a slight abuse of notation: we denote by $\xi^\pm_n$ the values of $\xi^\pm_n(t)$ at the time $t = T_n$. This allows us to avoid introducing further notation, and we hope it does not cause too much confusion.

Our plan is to switch from one pair of sub- and super-solutions to another at the times T_n, and improve the difference in the two shifts at the "switching" times, to ensure that

$$\xi^+_{n+1} - \xi^-_{n+1} \le (1-\delta)(\xi^+_n - \xi^-_n), \tag{2.6.37}$$

with some small but fixed constant $\delta > 0$ such that

$$e^{-T} \le c_T\delta \le \frac{1}{4}. \tag{2.6.38}$$

The constant c_T will also be chosen very small in the end – one should think of (2.6.38) as the requirement that the time step T is very large. This is natural: we can only hope to improve on the difference $\xi_n^+ - \xi_n^-$, as in (2.6.37), after a very large time step T. The shifts can be chosen so that they are uniformly bounded:

$$|\xi_n^\pm| \le M, \tag{2.6.39}$$

with a sufficiently large M – this follows from the bounds on the level sets of $u(t,x)$ that we have already obtained. As far as q_n are concerned, we will ask that

$$0 \le q_n \le c_q \delta(\xi_n^+ - \xi_n^-), \tag{2.6.40}$$

with another small constant c_q to be determined. Note that at $t = 0$ we may ensure that q_0 satisfies (2.6.40) simply by taking ξ_0^+ sufficiently positive and ξ_0^- sufficiently negative.

As we have uniform bounds on the location of the level sets of $u(t,x)$, and the shifts $\xi_n^\pm$ will be chosen uniformly bounded, as in (2.6.39), after possibly increasing R_0 in (2.6.10), we can ensure that

$$\phi(x + \xi_n^\pm(t)) \ge 11/12, \;\; u(t,x) \ge 11/12, \quad \text{for } x \ge R_0 \text{ and } t \ge T_n, \tag{2.6.41}$$

and

$$-1 < \phi(x + \xi_n^\pm(t)) \le 11/12, \quad -1 < u(t,x) \le -11/12, \quad \text{for } x \le -R_0 \text{ and } t \ge T_n, \tag{2.6.42}$$

which implies

$$f'(\phi(x + \xi_n^\pm(t))) \le -1, \quad f'(u(t,x)) \le -1, \text{ for } |x| \ge R_0 \text{ and } t \ge T_n. \tag{2.6.43}$$

Let us now assume that at the time $t = T_n$ we have the inequality

$$\phi(x + \xi_n^-) - q_n \le u(T_n, x) \le \phi(x + \xi_n^+) + q_n, \tag{2.6.44}$$

wth the shift q_n that satisfies (2.6.40). Our goal is to find a time $T_{n+1} \in [T_n + T, T_n + 2T]$, and the new shifts $\xi_{n+1}^\pm$ and q_{n+1}, so that (2.6.44) holds with n replaced by $n+1$ and the new gap $\xi_{n+1}^+ - \xi_{n+1}^-$ satisfies (2.6.37). We will consider two different cases.

Case 1: the solution gets close to the super-solution. Let us first assume that there is a time $\tau_n \in [T_n + T, T_n + 2T]$ such that the solution $u(\tau_n, x)$ is "very close" to the super-solution $\overline{u}_n(\tau_n, x)$ on the interval $\{|x| \le R_0 + 1\}$. More precisely, we assume that

$$\sup_{|x| \le R_0 + 1} \Big(\overline{u}_n(\tau_n, x) - u(\tau_n, x) \Big) \le \delta(\xi_n^+ - \xi_n^-). \tag{2.6.45}$$

We will show that in this case we may take $T_{n+1} = \tau_n$, and set

$$\xi_{n+1}^+ = \xi_n^+(\tau_n), \quad \xi_{n+1}^- = \xi_n^- + (\xi_n^+(\tau_n) - \xi_n^+) + \delta(\xi_n^+ - \xi_n^-), \tag{2.6.46}$$

as long as δ is sufficiently small, making sure that

$$\xi_{n+1}^+ - \xi_{n+1}^- = (1 - \delta)(\xi_n^+ - \xi_n^-), \tag{2.6.47}$$

and also choose q_{n+1} so that

$$q_{n+1} = c_q\delta(\xi_{n+1}^+ - \xi_{n+1}^-). \tag{2.6.48}$$

As far as the super-solution is concerned, we note that

$$\begin{aligned} u(\tau_n, x) &\le \phi(x + \xi_n^+(\tau_n)) + q_n e^{-(t-T_n)} \le \phi(x + \xi_n^+(\tau_n)) + c_q\delta(\xi_n^+ - \xi_n^-)e^{-T} \\ &\le \phi(x + \xi_n^+(\tau_n)) + q_{n+1}, \end{aligned} \tag{2.6.49}$$

for all $x \in \mathbb{R}$, provided that T is sufficiently large, independent of n.

For the sub-solution, we first look at what happens for $|x| \le R_0 + 1$ and use (2.6.45):

$$u(\tau_n, x) \ge \phi(x + \xi_n^+(\tau_n)) + q_n e^{-(\tau_n - T_n)} - \delta(\xi_n^+ - \xi_n^-), \quad \text{for all } |x| \le R_0 + 1. \tag{2.6.50}$$

Thus, for $|x| \le R_0 + 1$ we have

$$u(\tau_n, x) \ge \phi(x + \xi_n^+(\tau_n)) - \delta(\xi_n^+ - \xi_n^-) \ge \phi(x + \xi_n^+ - C_R\delta(\xi_n^+ - \xi_n^-)) \ge \phi(x + \xi_{n+1}^-), \tag{2.6.51}$$

with the constant C_R that depends on R_0, as long as $\delta > 0$ is sufficiently small.

It remains to look at $|x| \ge R_0 + 1$. To this end, recall that

$$u(\tau_n, x) \ge \phi(x + \xi_n^-(\tau_n)) - q_n e^{-(\tau_n - T_n)}, \text{ for all } x \in \mathbb{R}, \tag{2.6.52}$$

so that, as follows from the definition of $\xi_n^-(t)$, we have

$$u(\tau_n, x) \ge \phi(x + \xi_n^- - Cq_n) - q_n e^{-2T}, \text{ for all } x \in \mathbb{R}. \tag{2.6.53}$$

Observe that, as $\phi(x)$ is approaching ± 1 as $x \to \pm\infty$ exponentially fast, there exist $\omega > 0$ and $C > 0$ such that, taking into account (2.6.40) we can write for $|x| \ge R_0 + 1$:

$$\begin{aligned} \phi(x + \xi_n^- - Cq_n) &\ge \phi(x + \xi_n^- + (\xi_n^+(\tau_n) - \xi_n^+) + \delta(\xi_n^+ - \xi_n^-)) - C\delta e^{-\omega R_0}(\xi_n^+ - \xi_n^-) \\ &\ge \phi(x + \xi_{n+1}^-) - q_{n+1}, \end{aligned} \tag{2.6.54}$$

as long as R_0 is large enough. Here, we have used ξ_{n+1}^- and q_{n+1}^- as in (2.6.46) and (2.6.48). We conclude that

$$u(\tau_n, x) \ge \phi(x + \xi_{n+1}^-) - q_{n+1}, \text{ for } |x| \ge R_0 + 1. \tag{2.6.55}$$

Summarizing, if (2.6.45) holds, we set $T_{n+1} = \tau_n$, define the new shifts $\xi_{n+1}^\pm$ as in (2.6.46) and (2.6.48), which ensures that the "shift gap" is decreased by a fixed factor, so that (2.6.47) holds, and we can restart the argument at $t = T_{n+1}$, because

$$\phi(x + \xi_{n+1}^-) - q_{n+1} \le u(T_{n+1}, x) \le \phi(x + \xi_{n+1}^+) + q_{n+1}, \text{ for all } x \in \mathbb{R}. \tag{2.6.56}$$

Of course, if at some time $\tau_n \in [T_n + T, T_n + 2T]$ we have, instead of (2.6.45) that

$$\sup_{|x| \le R_0 + 1} \big(u(\tau_n, x) - \underline{u}(\tau_n, x)\big) \le \delta(\xi_n^+ - \xi_n^-), \tag{2.6.57}$$

then we could repeat the above argument essentially verbatim, using the fact that now the solution is very close to the sub-solution on a very large interval.

Case 2: the solution and the super-solution are never too close. Next, let us assume that for all $t \in [T_n + T, T_n + 2T]$, we have

$$\sup_{|x| \le R_0+1} \big(\overline{u}_n(t,x) - u(t,x)\big) \ge \delta(\xi_n^+ - \xi_n^-). \tag{2.6.58}$$

Because $\xi_n^+(t)$ is increasing, we have, for all $|x| \le R_0 + 1$ and $t \in [T_n + T, T_n + 2T]$:

$$\overline{u}_n(t,x) \le \phi(x + \xi_n^+(T_n + 2T)) + q_n e^{-T} \le \phi(x + \xi_n^+(T_n + 2T) + q_n e^{-T}\rho_0), \tag{2.6.59}$$

with

$$\rho_0 = \big(\inf_{|x| \le R_0+M+10} \phi'(x)\big)^{-1}. \tag{2.6.60}$$

Here, M is the constant in the upper bound (2.6.39) for $\xi_n^{\pm}$. Note that by choosing T sufficiently large we can make sure that the argument in ϕ in the right side of (2.6.59) is within the range of the infimum in (2.6.60). The function

$$w_n(t,x) = \phi(x + \xi_n^+(T_n + 2T) + q_n e^{-T}\rho_0) - u(t,x).$$

that appears in the right side of (2.6.59) solves a linear parabolic equation

$$\partial_t w_n - \partial_{xx} w_n + a_n(t,x) w_n = 0, \tag{2.6.61}$$

with the coefficient a_n that is bounded in n, t and x:

$$a_n(t,x) = -\frac{f(\phi(x + \xi_n^+(T_n + 2T) + q_n e^{-T}\rho_0)) - f(u(t,x))}{\phi(x + \xi_n^+(T_n + 2T) + q_n e^{-T}\rho_0) - u(t,x)}. \tag{2.6.62}$$

It follows from assumption (2.6.58) and (2.6.59) that

$$\sup_{|x| \le R_0+1} w_n(t,x) \ge \delta(\xi_n^+ - \xi_n^-), \quad \text{for all } t \in [T_n + T, T_n + 2T], \tag{2.6.63}$$

but in order to improve the shift, we would like to have not the supremum but the infimum in the above inequality. And here the Harnack inequality comes to the rescue: we will use Theorem 2.5.13 for the intervals $|x| \le R_0 + 1$ and $|x| \le R_0$. For that, we need to make sure that at least a fraction of the supremum in (2.6.63) is attained on $[-R_0, R_0]$: there exists k_1 so that

$$\sup_{|x| \le R_0} w_n(t,x) \ge k_1 \delta(\xi_n^+ - \xi_n^-), \quad \text{for all } T_n + T \le t \le T_n + 2T. \tag{2.6.64}$$

However, if there is a time $T_n + T \le s_n \le T_n + 2T$ such that

$$\sup_{|x| \le R_0} w_n(s_n,x) \le \frac{\delta}{2}(\xi_n^+ - \xi_n^-), \tag{2.6.65}$$

then we have

$$\bar{u}(s_n,x) - u(s_n,x) \le \frac{\delta}{2}(\xi_n^+ - \xi_n^-) \text{ for all } |x| \le R_0. \tag{2.6.66}$$

This is the situation we faced in Case 1, and we can proceed as in that case. Thus, we may assume that

$$\sup_{|x|\le R_0} w_n(t,x) \ge \frac{\delta}{2}(\xi_n^+ - \xi_n^-) \text{ for all } T_n + T \le t \le T_n + 2T. \tag{2.6.67}$$

In that case, we may apply the Harnack inequality of Theorem 2.5.13 to (2.6.61) on the intervals $|x| \le R_0 + 1$ and $|x| \le R_0$: there exists a Harnack constant h_{R_0} that is independent of T, such that

$$w_n(t,x) \ge h_{R_0}\delta(\xi_n^+ - \xi_n^-), \quad \text{for all } t \in [T_n + T + 1, T_n + 2T] \text{ and } |x| \le R_0. \tag{2.6.68}$$

Exercise 2.6.7 Show that, as a consequence, we can find $\rho_1 > 0$ that depends on R_0 but not on n such that for $|x| \le R_0$ and $T_n + T + 1 \le t \le T_n + 2T$, we have

$$\widetilde{w}_n(t,x) = \phi\big(x + \xi_n^+(T_n + 2T) + \rho_0 e^{-T} q_n - \rho_1 h_{R_0}\delta(\xi_n^+ - \xi_n^-)\big) - u(t,x) \ge 0. \tag{2.6.69}$$

Let us now worry about what $\widetilde{w}_n$ does for $|x| \ge R_0$. In this range, the function $\widetilde{w}_n$ solves another linear equation of the form

$$\partial_t \widetilde{w}_n - \partial_{xx}\widetilde{w}_n + \widetilde{a}_n(t,x)\widetilde{w}_n = 0, \tag{2.6.70}$$

with $\widetilde{a}_n(t,x) \ge 1$ that is an appropriate modification of the expression for $a_n(t,x)$ in (2.6.62). In addition, at the boundary $|x| = R_0$, we have $\widetilde{w}_n(t,x) \ge 0$, and at the time $t = T_n + T$, we have an estimate of the form

$$\widetilde{w}_n(T_n + T, x) \ge -K(\xi_n^+ - \xi_n^-), \quad |x| \ge R_0. \tag{2.6.71}$$

Exercise 2.6.8 What did we use to get (2.6.71)?

Therefore, the maximum principle applied to (2.6.70) implies that

$$\widetilde{w}_n(T_n + 2T, x) \ge -Ke^{-T}(\xi_n^+ - \xi_n^-), \quad |x| \ge R_0. \tag{2.6.72}$$

We now set $T_{n+1} = T_n + 2T$. The previous argument shows that we have

$$u(T_{n+1}, x) \le \phi\big(x + \xi_n^+(T_{n+1}) + \rho_0 e^{-T} q_n - \rho_1 h_{R_0}\delta(\xi_n^+ - \xi_n^-)\big) + q_{n+1}, \tag{2.6.73}$$

with

$$0 \le q_{n+1} \le Ke^{-T}(\xi_n^+ - \xi_n^-). \tag{2.6.74}$$

In addition, we still have the lower bound:

$$u(T_n + 2T) \ge \phi(x + \xi_n^-(T_{n+1})) - e^{-T} q_n. \tag{2.6.75}$$

It only remains to define $\xi_{n+1}^\pm$ and q_{n+1} properly, to convert (2.6.73) and (2.6.75) into the form required to restart the iteration process. We take

$$q_{n+1} = \max(e^{-T} q_n, Ke^{-T}(\xi_n^+ - \xi_n^-)), \quad \xi_{n+1}^- = \xi_n^-(T_{n+1}), \tag{2.6.76}$$

and

$$\xi_{n+1}^+ = \xi_n^+(T_{n+1}) + \rho_0 e^{-T} q_n - h_{R_0}\rho_1\delta(\xi_n^+ - \xi_n^-). \tag{2.6.77}$$

It is easy to see that assumption (2.6.40) holds for q_{n+1} provided we take T sufficiently large, so that

$$e^{-T} \ll c_q. \tag{2.6.78}$$

The main point to verify is that the contraction in (2.6.37) does happen with the above choice. We recall (2.6.36):

$$\xi_n^+(T_{n+1}) = \xi_n^+ + \frac{q_n}{k_0}(1+M_f)(1-e^{-2T}), \quad \xi_n^-(T_{n+1}) = \xi_n^- - \frac{q_n}{k_0}(1+M_f)(1-e^{-2T}). \tag{2.6.79}$$

Hence, in order to ensure that

$$\xi_{n+1}^+ - \xi_{n+1}^- \le (1 - \frac{h_{R_0}\rho_1\delta}{2})(\xi_n^+ - \xi_n^-), \tag{2.6.80}$$

it suffices to make sure that the term $h_{R_0}\rho_1\delta(\xi_n^+ - \xi_n^-)$ dominates all the other multiples of $\delta(\xi_n^+ - \xi_n^-)$ in the expression for the difference $\xi_{n+1}^+ - \xi_{n+1}^-$ that come with the opposite sign. However, all such terms are multiples of q_n, thus it suffices to make sure that the constant c_q is small, which, in turn, can be accomplished by taking T sufficiently large. This completes the proof. □

Spreading in an unbalanced Allen-Cahn equation

Let us now discuss, informally, what one would expect, from the physical considerations, to happen to the solution of the initial value problem if the balance condition (2.6.5) fails, that is,

$$\int_{-1}^{1} f(u)du \neq 0. \tag{2.6.81}$$

To be concrete, let us consider the nonlinearity $f(u)$ of the form

$$f(u) = (u+1)(u+a)(1-u), \tag{2.6.82}$$

with $a \in (0,1)$. so that $u = \pm 1$ are still the two stable solutions of the ODE

$$\dot{u} = f(u),$$

but instead of (2.6.5) we have

$$\int_{-1}^{1} f(u)du > 0.$$

As an indication of what happens we give the reader the following exercises. They are by no means short but they can all be done with the tools of this section, and we strongly recommend them to a reader interested in understanding this material well.

Exercise 2.6.9 To start, show that for $f(u)$ given by (2.6.82), we can find a special solution $u(t,x)$ of the Allen-Cahn equation (2.6.1):

$$u_t = u_{xx} + f(u), \tag{2.6.83}$$

of the form

$$u(t,x)=\psi(x+ct), \tag{2.6.84}$$

with $c>0$ and a function $\psi(x)$ that satisfies

$$c\psi'=\psi''+f(\psi), \tag{2.6.85}$$

together with the boundary condition

$$\psi(x)\to\pm 1, \quad \text{as } x\to\pm\infty. \tag{2.6.86}$$

Solutions of the form (2.6.84) are known as traveling waves. Show that such c is unique, and ψ is unique up to a translation: if $\psi_1(x)$ is another solution of (2.6.85)-(2.6.86) with c replaced c_1, then $c=c_1$ and there exists $x_1\in\mathbb{R}$ such that $\psi_1(x)=\psi(x+x_1)$.

Exercise 2.6.10 Try to modify the proof of Lemma 2.6.4 to show that if $u(t,x)$ is the solution of the Allen-Cahn equation (2.6.83) with an initial condition $u_0(x)$ that satisfies (2.6.7):

$$u_0(x)\to\pm 1, \quad \text{as } x\to\pm\infty, \tag{2.6.87}$$

then we have

$$u(t,x)\to 1 \text{ as } t\to+\infty, \text{ for each } x\in\mathbb{R} \text{ fixed.} \tag{2.6.88}$$

It should be helpful to use the traveling wave solution to construct a sub-solution that will force (2.6.88). Thus, in the "unbalanced" case, the "more stable" of the two states $u=-1$ and $u=+1$ wins in the long time limit. Show that the convergence in (2.6.88) is not uniform in $x\in\mathbb{R}$.

Exercise 2.6.11 Let $u(t,x)$ be a solution of (2.6.83) with an initial condition $u_0(x)$ that satisfies (2.6.87). Show that for any $c'<c$ and $x\in\mathbb{R}$ fixed, we have

$$\lim_{t\to+\infty} u(t,x-c't)=1, \tag{2.6.89}$$

and for any $c'>c$ and $x\in\mathbb{R}$ fixed, we have

$$\lim_{t\to+\infty} u(t,x-c't)=-1. \tag{2.6.90}$$

Exercise 2.6.12 Let $u(t,x)$ be a solution of (2.6.83) with an initial condition $u_0(x)$ that satisfies (2.6.87). Show that there exists $x_0\in\mathbb{R}$ (which depends on u_0) so that for all $x\in\mathbb{R}$ fixed we have

$$\lim_{t\to+\infty} u(t,x-ct)=\psi(x+x_0). \tag{2.6.91}$$

2.7 The principal eigenvalue for elliptic operators and the Krein-Rutman theorem

One consequence of the strong maximum principle is the existence of a positive eigenfunction for an elliptic operator in a bounded domain with the Dirichlet or Neumann boundary conditions. Such eigenfunction necessarily corresponds to the eigenvalue with the smallest real part. A slightly different way to put it is that the strong maximum principle makes the Krein-Rutman Theorem applicable, which in turn, implies the existence of such eigenfunction. In this section, we will prove this theorem in the context of parabolic operators with time periodic coefficients. We then deduce, in an easy way, some standard properties of the principal elliptic eigenvalue.

2.7.1 The periodic principal eigenvalue

The maximum principle for elliptic and parabolic problems has a beautiful connection to the eigenvalue problems, which also allows to extend it to operators with a zero-order term. We will first consider the periodic eigenvalue problems, that is, elliptic equations where the coefficients are 1-periodic in every direction in $\mathbb{R}^n$, and the sought for solutions are all 1-periodic in $\mathbb{R}^n$. It would, of course, be easy to deduce, by dilating the coordinates, the same results for coefficients with general periods $T_1, \dots, T_n$ in the directions $e_1, \dots, e_n$. We will consider operators of the form

$$Lu(x) = -\Delta u + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{2.7.1}$$

with bounded, smooth and 1-periodic coefficients $b_j(x)$ and $c(x)$. We could also consider more general operators of the form

$$Lu(x) = -a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u,$$

with uniformly elliptic (and 1-periodic) , and regular coefficients a_{ij},with the help of the elliptic regularity theory. This will not, however, be needed for our purposes. In order to avoid repeating that the coefficients and the solutions are 1-periodic, we will just say that $x \in \mathbb{T}^n$, the n-dimensional unit torus.

The key spectral property of the operator L comes from the comparison principle. To this end, let us recall the Krein-Rutman theorem. It says that if M is a compact operator in a strongly ordered Banach space X (that is, there is a solid cone K which serves for defining an order relation: $u \le v$ iff $v - u \in K$), that preserves K: $Mu \in K$ for all $u \in K$, and maps the boundary of K into its interior, then M has an eigenfunction ϕ that lies in this cone:

$$M\phi = \lambda\phi. \tag{2.7.2}$$

Moreover, the corresponding eigenvalue λ has the largest real part of all eigenvalues of the operator M. The classical reference [50] has a nice and clear presentation of this theorem but one can find it in other textbooks, as well.

How can we apply this theorem to the elliptic operators? The operator L given by (2.7.1) is not compact, nor does it preserve any interesting cone. However, let us assume momentarily that $c(x)$ is continuous and $c(x) > 0$ for all $x \in \mathbb{T}^n$. Then the problem

$$Lu = f, \quad x \in \mathbb{T}^n \tag{2.7.3}$$

has a unique solution, and, in addition, if $f(x) \ge 0$ and $f \not\equiv 0$, then $u(x) > 0$ for all $x \in \mathbb{T}^n$. Indeed, let $v(t, x)$ be the solution of the initial value problem

$$v_t + Lv = 0, \;\; t > 0, \; x \in \mathbb{T}^n, \tag{2.7.4}$$

with $v(0, x) = f(x)$. The comparison principle implies a uniform upper bound

$$|v(t, x)| \le e^{-\bar{c}t}\|f\|_{L^\infty}, \tag{2.7.5}$$

with

$$\bar{c} = \inf_{x \in \mathbb{T}^n} c(x) > 0. \tag{2.7.6}$$

This allows us to define

$$u(x) = \int_0^\infty v(t, x)x. \tag{2.7.7}$$

Exercise 2.7.1 Verify that if $c(x) > 0$ for all $x \in \mathbb{T}^n$, then $u(x)$ given by (2.7.7) is a solution to (2.7.3). Use the maximum principle to show that (2.7.3) has a unique solution.

This means that we may define the inverse operator $M = L^{-1}$. This operator preserves the cone of the positive functions, and maps its boundary (non-negative functions that vanish somewhere in Ω) into its interior – this is a consequence of the strong maximum principle that holds if $c(x) > 0$. In addition, M is a compact operator from $C(\mathbb{T}^n)$ to itself. Hence, the inverse operator satisfies the assumptions of the Krein-Rutman theorem.

Exercise 2.7.2 Compactness of the inverse M follows from the elliptic regularity estimates. One way to convince yourself of this fact is to consult Evans [60]. Another way is to go back to Theorem 2.4.12, use it to obtain the Hölder regularity estimates on $v(t, x)$, and translate them in terms of $u(x)$ to show that, if f is continuous, then ∇u is α-Hölder continuous, for all $\alpha \in (0, 1)$. The Arzela-Ascoli theorem implies then compactness of M. Hint: be careful about the regularity of $v(t, x)$ as $t \downarrow 0$.

Thus, there exists a positive function f and $\mu \in \mathbb{R}$ so that the function $u = \mu f$ satisfies (2.7.3). Positivity of f implies that the solution of (2.7.3) is also positive, hence $\mu > 0$. As μ is the eigenvalue of L^{-1} with the largest real part, $\lambda = \mu^{-1}$ is the eigenvalue of L with the smallest real part. In particular, it follows that all eigenvalues λ_k of the operator L have a positive real part.

If the assumption $c(x) \geq 0$ does not hold, we may take $K > \|c\|_{L^\infty}$, and consider the operator

$$L'u = Lu + Ku.$$

The zero-order coefficient of L' is

$$c'(x) = c(x) + K \geq 0.$$

Hence, we may apply the previous argument to the operator L' and conclude that L' has an eigenvalue μ_1 that corresponds to a positive eigenfunction, and has the smallest real part among all eigenvalues of L'. The same is true for the operator L, with the eigenvalue

$$\lambda_1 = \mu_1 - K.$$

We say that λ_1 is the principal periodic eigenvalue of the operator L.

2.7.2 The Krein-Rutman theorem: the periodic parabolic case

As promised, we will prove the Krein-Rutman Theorem in the context of the periodic eigenvalue problems. Our starting point will be a slightly more general problem with time-periodic coefficients:

$$u_t - \Delta u + b_j(t, x)\frac{\partial u}{\partial x_j} + c(t, x)u = 0, \qquad x \in \mathbb{T}^n. \tag{2.7.8}$$

Here, the coefficients $b_j(t,x)$ and $c(t,x)$ are smooth, 1-periodic in x and T-periodic in t. Let $u(t,x)$ be the solution of the Cauchy problem for (2.7.8), with a 1-periodic, continuous initial condition

$$u(t,x) = u_0(x). \tag{2.7.9}$$

We define the "time T" operator S_T as

$$[S_T u_0](x) = u(T,x). \tag{2.7.10}$$

Exercise 2.7.3 Use the results of Section 2.4 to show that S_T is compact operator on $C(\mathbb{T}^n)$ that preserves the cone of positive functions.

We are going to prove the Krein-Rutman Theorem for S_T first.

Theorem 2.7.4 *The operator S_T has an eigenvalue $\bar{\mu} > 0$ that corresponds to a positive eigenfunction $\phi_1(x) > 0$. The eigenvalue $\bar{\mu}$ is simple: the only solutions of*

$$(S_T - \bar{\mu})u = 0, \qquad x \in \mathbb{T}^n$$

are multiples of ϕ_1. If μ is another (possibly non-real) eigenvalue of S_T, then $|\mu| < \bar{\mu}$.

Proof. Let us pick any positive function $\phi_0 \in C(\mathbb{T}^n)$, set $\psi_0 = \phi_0/\|\phi_0\|_{L^\infty}$, and consider the iterative sequence (ϕ_n, ψ_n):

$$\phi_{n+1} = S_T\psi_n, \qquad \psi_{n+1} = \frac{\phi_{n+1}}{\|\phi_{n+1}\|_{L^\infty}}.$$

Note that, because ϕ_0 is positive, both ϕ_n and ψ_n are positive for all n, by the strong maximum principle. For every n, let μ_n be the smallest μ such that

$$\phi_{n+1}(x) \le \mu\psi_n(x), \quad \text{for all } x \in \mathbb{T}^n. \tag{2.7.11}$$

Note that (2.7.11) holds for large μ, because each of the ϕ_n is positive, hence the smallest such μ exists. It is also clear that $\mu_n \ge 0$. We claim that the sequence μ_n is non-increasing. To see that, we apply the operator S_T to both sides of the inequality (2.7.11) with $\mu = \mu_n$, written as

$$S_T\psi_n(x) \le \mu_n\psi_n(x), \quad \text{for all } x \in \mathbb{T}^n. \tag{2.7.12}$$

and use the fact that S_T preserves positivity, to get

$$(S_T \circ S_T)\psi_n(x) \le \mu_n S_T\psi_n(x), \quad \text{for all } x \in \mathbb{T}^n, \tag{2.7.13}$$

which is

$$S_T\phi_{n+1}(x) \le \mu_n\phi_{n+1}(x), \quad \text{for all } x \in \mathbb{T}^n. \tag{2.7.14}$$

Dividing both sides by $\|\phi_{n+1}\|_{L^\infty}$. we see that

$$S_T\psi_{n+1}(x) \le \mu_n\psi_{n+1}(x), \quad \text{for all } x \in \mathbb{T}^n, \tag{2.7.15}$$

hence

$$\phi_{n+2}(x) \le \mu_n\psi_{n+1}(x), \quad \text{for all } x \in \mathbb{T}^n. \tag{2.7.16}$$

It follows that $\mu_{n+1} \le \mu_n$.

Thus, μ_n converges to a limit $\bar{\mu}$.

Exercise 2.7.5 Show that, up to an extraction of a subsequence, the sequence ψ_n converges to a limit ψ_∞, with $\|\psi_\infty\|_{L^\infty} = 1$.

The corresponding subsequence ϕ_{n_k} converges to $\phi_\infty = S_T\psi_\infty$, by the continuity of S_T. And we have, by (2.7.11):

$$S_T\psi_\infty \leq \bar{\mu}\psi_\infty. \tag{2.7.17}$$

If we have the equality in (2.7.17):

$$S_T\psi_\infty(x) = \bar{\mu}\psi_\infty(x) \text{ for all } x \in \mathbb{T}^n, \tag{2.7.18}$$

then ψ_∞ is a positive eigenfunction of S_T corresponding to the eigenvalue $\bar{\mu}$. If, on the other hand, we have

$$S_T\psi_\infty(x) < \bar{\mu}\psi_\infty(x), \text{ on an open set } U \subset \mathbb{T}^n, \tag{2.7.19}$$

they we may use the fact that S_T maps the boundary of the cone of non-negative functions into its interior. In other words, we use the strong maximum principle here. Applying S_T to both sides of (2.7.17) gives then:

$$S_T\phi_\infty < \bar{\mu}\phi_\infty \text{ for all } x \in \mathbb{T}^n. \tag{2.7.20}$$

This contradicts, for large n, the minimality of μ_n. Thus, (2.7.19) is impossible, and $\bar{\mu}$ is the sought for eigenvalue. We set, from now on, $\phi_1 = \psi_\infty$:

$$S_T\phi_1 = \bar{\mu}\phi_1, \quad \phi_1(x) > 0 \text{ for all } x \in \mathbb{T}^n. \tag{2.7.21}$$

Exercise 2.7.6 So far, we have shown that $\bar{\mu} \geq 0$. Why do we know that, actually, $\bar{\mu} > 0$?

Let ϕ be an eigenfunction of S_T that is not a multiple of ϕ_1, corresponding to an eigenvalue μ:

$$S_T\phi = \mu\phi.$$

Let us first assume that μ is real, and so is the eigenfunction ϕ. If $\mu \geq 0$, after multiplying ϕ by an appropriate factor, we may assume without loss of generality that $\phi_1(x) \geq \phi(x)$ for all $x \in \mathbb{T}^n$, $\phi_1 \not\equiv \phi$, and there exists $x_0 \in \mathbb{T}^n$ such that $\phi_1(x_0) = \phi(x_0)$. The strong comparison principle implies that then

$$S_T\phi_1(x) > S_T\phi(x) \quad \text{for all } x \in \mathbb{T}^n.$$

It follows, in particular, that

$$\bar{\mu}\phi_1(x_0) > \mu\phi(x_0),$$

hence $\bar{\mu} > \mu \geq 0$, as $\phi_1(x_0) = \phi(x_0) > 0$. This argument also shows that $\bar{\mu}$ is a simple eigenvalue.

If $\mu < 0$, then we can choose ϕ (after multiplying it by a, possibly negative, constant) so that, first,

$$\phi_1(x) \geq \phi(x), \quad \phi(x) \geq -\phi_1(x), \quad \text{for all } x \in \mathbb{T}^n, \tag{2.7.22}$$

and there exists $x_0 \in \mathbb{T}^n$ such that

$$\phi(x_0) = \phi_1(x_0).$$

Applying S_T to the second inequality in (2.7.22) gives, in particular,

$$\mu\phi(x_0) > -\bar{\mu}\phi_1(x_0), \tag{2.7.23}$$

thus $\bar{\mu} > -\mu$. In both cases, we see that $|\mu| < \bar{\mu}$.

Exercise 2.7.7 Use a similar consideration for the case when μ is complex. In that case, it helps to write the corresponding eigenfunction as

$$\phi = u + iv,$$

and consider the action of S_T on the span of u and v, using the same comparison idea. Show that $|\mu| < \bar{\mu}$. If in doubt, consult [50].

This completes the proof of Theorem 2.7.4. □

2.7.3 Back to the principal periodic elliptic eigenvalue

Consider now again the operator L given by (2.7.1):

$$Lu(x) = -\Delta u + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{2.7.24}$$

with bounded, smooth and 1-periodic coefficients $b_j(x)$ and $c(x)$. One consequence of Theorem 2.7.4 is the analogous result for the principal periodic eigenvalue for L. We will also refer to the following as the Krein-Rutman theorem.

Theorem 2.7.8 *The operator L has a unique eigenvalue λ_1 associated to a positive function ϕ_1. Moreover, each eigenvalue of L has a real part strictly larger than λ_1.*

Proof. The operator L falls, of course, in the realm of Theorem 2.7.4, since its time-independent coefficients are T-periodic for all $T > 0$. We are also going to use the formula

$$L\phi = -\lim_{t\downarrow 0}\frac{S_t\phi - \phi}{t}, \tag{2.7.25}$$

for smooth $\phi(x)$, with the limit in the sense of uniform convergence. This is nothing but an expression of the fact that the function $u(t,x) = [S_t\phi](x)$ is the solution of

$$u_t + Lu = 0, \tag{2.7.26}$$

with the initial condition $u(0,x) = \phi(x)$, and if ϕ is smooth, then (2.7.26) holds also at $t = 0$.

Given $n \in \mathbb{N}$, let $\bar{\mu}_n$ be the principal eigenvalue of the operator $S_{1/n}$, with the principal eigenfunction $\phi_n > 0$:

$$S_{1/n}\phi_n = \bar{\mu}_n\phi_n,$$

normalized so that $\|\phi_n\|_\infty = 1$.

Exercise 2.7.9 Show that

$$\lim_{n\to\infty}\bar{\mu}_n = 1$$

directly, without using (2.7.27) below.

As $(S_{1/n})^n = S_1$ for all n, we conclude that ϕ_n is a positive eigenfunction of S_1 with the eigenvalue $(\bar{\mu}_n)^n$. By the uniqueness of the positive eigenfunction, we have

$$\bar{\mu}_n = (\bar{\mu}_1)^{1/n}, \qquad \phi_n = \phi_1. \tag{2.7.27}$$

Note that, by the parabolic regularity, ϕ_1 is infinitely smooth, simply because it is a multiple of $S_1\phi_1$, which is infinitely smooth. Hence, (2.7.25) applies to ϕ_1, and

$$L\phi_1 = -\lim_{n\to+\infty} n(S_{1/n} - I)\phi_1 = -\lim_{n\to+\infty} n(\bar{\mu}_1^{1/n} - 1)\phi_1 = -(\log \bar{\mu}_1)\phi_1.$$

We have thus proved the existence of an eigenvalue $\lambda_1 = -\log \bar{\mu}_1$ of L that corresponds to a positive eigenfunction. It is easy to see that if

$$L\phi = \lambda\phi,$$

then

$$S_1\phi = e^{-\lambda}\phi.$$

It follows that L can have only one eigenvalue corresponding to a positive eigenfunction. As we know that all eigenvalues μ of S_1 satisfy $|\mu| < \bar{\mu}_1$, we conclude that λ_1 is the eigenvalue of L with the smallest real part. $\Box$

If L is symmetric – that is, it has the form

$$Lu = -\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c(x)u, \tag{2.7.28}$$

with $a_{ij} = a_{ji}$, then the first eigenvalue is given by the minimization over $H^1(\mathbb{T}^n)$ of the Rayleigh quotient

$$\lambda_1 = \inf_{u\in H^1(\mathbb{T}^n)} \frac{\displaystyle\int_{\mathbb{T}^n} (a_{ij}(x)(\partial_i u)(\partial_j u) + c(x)u^2(x))dx}{\displaystyle\int_{\mathbb{T}^n} u^2(x)dx}. \tag{2.7.29}$$

The existence and uniqueness (up to a factor) of the minimizer is a classical exercise that we do not reproduce here. As for the positivity of the minimizer, we notice that, if ϕ is a minimizer of the Rayleigh quotient, then $|\phi_1|$ is also a minimizer, thus the unique minimizer is a positive function.

The Dirichlet principal eigenvalue, related issues

We have so far talked about the principal eigenvalue for spatially periodic elliptic problems. This discussion applies equally well to problems in bounded domains, with the Dirichlet or Neumann boundary conditions. In the rest of this book, we will often encounter the Dirichlet problems, so let us explain this situation. Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, and consider our favorite elliptic operator

$$Lu = -\Delta u + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{2.7.30}$$

with smooth coefficients $b_j(x)$ and $c(x)$. One could easily look at the more general problem

$$Lu = -a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{2.7.31}$$

with essentially identical results, as long as the matrix $a_{ij}(x)$ is uniformly elliptic – we will avoid this just to keep the notation simpler. We are interested in the eigenvalue problem

$$\begin{aligned} Lu &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad\quad \text{on } \partial\Omega, \end{aligned} \tag{2.7.32}$$

and, in particular, in the existence of a positive eigenfunction $\phi > 0$ in Ω. The strategy will be as in the periodic case, to look at the initial value problem

$$\begin{aligned} &u_t - \Delta u + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u = 0, \quad t > 0,\ x \in \Omega, \\ &u = 0, \qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ x \in \partial\Omega, \\ &u(0,x) = u_0(x). \end{aligned} \tag{2.7.33}$$

The coefficients b_j and c are smooth in (t,x) and T-periodic in t. Again, we set

$$(S_T u_0)(x) = u(T,x).$$

The main difference with the periodic case is that, here, the cone of continuous functions which are positive in Ω and vanish on $\partial\Omega$ has an empty interior, so we can not repeat verbatim the proof of the Krein-Rutman theorem for the operators on $\mathbb{T}^n$.

Exercise 2.7.10 Revisit the proof of the Krein-Rutman theorem in that case and identify the place where the proof would fail for the Dirichlet boundary conditions.

What will save the day is the strong maximum principle and the Hopf Lemma. We are not going to fully repeat the proof of Theorems 2.7.4 and 2.7.8, but we are going to prove a key proposition that an interested reader can use to prove the Krein-Rutman theorem for the Dirichlet problem.

Proposition 2.7.11 *Assume $u_0 \in C^1(\overline{\Omega})$ – that is, u_0 has derivatives that are continuous up to $\partial\Omega$, and that $u_0 > 0$ in Ω, and both $u_0 = 0$ and $\partial_\nu u_0 < 0$ on $\partial\Omega$. Then there is $\mu_1 > 0$ defined by the formula*

$$\mu_1 = \inf\{\mu > 0 :\ S_T u_0 \le \mu u_0\}. \tag{2.7.34}$$

Moreover, if $\mu_2 > 0$ is defined as

$$\mu_2 = \inf\{\mu > 0 :\ (S_T \circ S_T)u_0 \le \mu S_T u_0\}, \tag{2.7.35}$$

then either $\mu_1 > \mu_2$, or $\mu_1 = \mu_2$, and in the latter case $(S_T \circ S_T)u_0 \equiv \mu_2 S_T u_0$.

Proof. For the first claim, the existence of the infimum in (2.7.34), we simply note that

$$\mu u_0 \ge S_T u_0,$$

as soon as $\mu > 0$ is large enough, because $\partial_\nu u_0 < 0$ on $\partial\Omega$, $u_0 > 0$ in Ω, and $S_T u_0$ is a smooth function up to the boundary. As for the second item, let us first observe that

$$u(t+T,x) \leq \mu_1 u(t,x), \tag{2.7.36}$$

for any $t > 0$, by the maximum principle. Let us assume that

$$u(2T,x) \not\equiv \mu_1 u(T,x). \tag{2.7.37}$$

Then the maximum principle implies that

$$u(2T,x) < \mu_1 u(T,x) \text{ for all } x \in \Omega. \tag{2.7.38}$$

As

$$\max_{x\in\bar\Omega}[u(2T,x) - \mu_1 u(T,x)] = 0,$$

the parabolic Hopf lemma, together with (2.7.36) and (2.7.37), implies the existence of $\delta > 0$ such that

$$\partial_\nu(u(2T,x) - \mu_1 u(T,x)) \geq \delta > 0, \quad \text{for all } x \in \partial\Omega. \tag{2.7.39}$$

It follows that for $\varepsilon > 0$ sufficiently small, we have

$$u(2T,x) - \mu_1 u(T,x) \leq -\frac{\delta}{2} d(x,\partial\Omega) \qquad \text{for } x \in \Omega \text{ such that } d(x,\partial\Omega) < \varepsilon.$$

On the other hand, once again, the strong maximum principle precludes a touching point between $u(2T,x)$ and $\mu_1 u(T,x)$ inside

$$\overline{\Omega}_\varepsilon = \{x \in \Omega : \ d(x,\partial\Omega) \geq \varepsilon\}.$$

Therefore, there exists δ_1 such that

$$u(2T,x) - \mu_1 u(T,x) \leq -\delta_1, \qquad \text{for all } x \in \overline{\Omega}_\varepsilon.$$

We deduce that there is a – possibly very small – constant $c > 0$ such that

$$u(2T,x) - \mu_1 u(T,x) \leq -cd(x,\partial\Omega) \quad \text{in } \Omega.$$

However, $u(T,x)$ is controlled from above by $Cd(x,\partial\Omega)$, for a possibly large constant $C > 0$. All in all, we have

$$u(2T,x) \leq (\mu_1 - \frac{c}{C})u(T,x),$$

hence (2.7.37) implies that $\mu_2 < \mu_1$, which proves the second claim of the proposition. □

Exercise 2.7.12 Deduce from Proposition 2.7.11 the versions of Theorems 2.7.4 and 2.7.8 for operators S_T and L, this time with the Dirichlet boundary conditions.

Thus, the eigenvalue problem (2.7.32), has a principal eigenvalue that enjoys all the properties we have proved in the periodic one: it has the least real part among all eigenvalues, and is the only eigenvalue associated to a positive eigenfunction.

Exercise 2.7.13 Assume that L is symmetric; it has the form

$$Lu = -\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + c(x)u \tag{2.7.40}$$

Then, the principal eigenvalue is given by the minimization of the Rayleigh quotient over the Sobolev space $H_0^1(\Omega)$:

$$\lambda_1 = \inf_{u\in H_0^1(\Omega),\quad \|u\|_{L^2}=1} \int_\Omega \Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + c(x)u^2(x)\Big)dx. \tag{2.7.41}$$

Exercise 2.7.14 Adapt the preceding discussion to prove the existence of a principal eigenvalue to the Nemann eigenvalue problem

$$\begin{aligned} &Lu = \lambda u, \qquad x \in \Omega, \\ &\partial_\nu u = 0, \qquad x \in \partial\Omega. \end{aligned} \tag{2.7.42}$$

2.7.4 The principal eigenvalue and the comparison principle

Let us now connect the principal eigenvalue and the comparison principle. Since we are at the moment dealing with the Dirichet problems, let us remain in this context. There would be nothing significantly different about the periodic problems.

The principal eigenfunction $\phi_1 > 0$, solution of

$$L\phi_1 = \lambda_1\phi_1, \text{ in } \Omega, \tag{2.7.43}$$

$$\phi_1 = 0 \text{ on } \partial\Omega, \tag{2.7.44}$$

with

$$Lu = -\Delta u + b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{2.7.45}$$

in particular, provides a special solution

$$\psi(t,x) = e^{-\lambda_1 t}\phi_1(x) \tag{2.7.46}$$

for the linear parabolic problem

$$\begin{aligned} &\psi_t + L\psi = 0, \quad t > 0, x \in \Omega \\ &\psi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.7.47}$$

Consider then the Cauchy problem

$$\begin{aligned} &v_t + Lv = 0, \quad t > 0, x \in \Omega \\ &v = 0 \text{ on } \partial\Omega, \\ &v(0,x) = g(x), \quad x \in \Omega, \end{aligned} \tag{2.7.48}$$

with a smooth bounded function $g(x)$ that vanishes at the boundary $\partial\Omega$. We can find a constant $M > 0$ so that

$$-M\phi_1(x) \le g(x) \le M\phi_1(x), \quad \text{for all } x \in \Omega.$$

The comparison principle then implies that for all $t > 0$ we have a bound

$$-M\phi_1(x)e^{-\lambda_1 t} \le v(t,x) \le M\phi_1(x)e^{-\lambda_1 t}, \quad \text{for all } x \in \Omega, \tag{2.7.49}$$

which is very useful, especially if $\lambda_1 > 0$. The assumption that the initial condition g vanishes at the boundary $\partial\Omega$ is not necessary but removes the technical step of having to show that even if $g(x)$ does not vanish on the boundary, then for any positive time $t_0 > 0$ we can find a constant C_0 so that $|v(t_0,x)| \le C_0\phi_1(x)$. This leads to the bound (2.7.49) for all $t > t_0$.

Let us now apply the above considerations to the solutions of the elliptic problem

$$\begin{aligned} &Lu = g(x), \text{ in } \Omega, \\ &u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.7.50}$$

with a non-negative function $g(x)$. When can we conclude that the solution $u(x)$ is also non-negative? The solution of (2.7.50) can be formally written as

$$u(x) = \int_0^\infty v(t,x)dt. \tag{2.7.51}$$

Here, the function $v(t,x)$ satisfies the Cauchy problem (2.7.48). If the principal eigenvalue λ_1 of the operator L is positive, then the integral (2.7.51) converges for all $x \in \Omega$ because of the estimates (2.7.49), and the solution of (2.7.50) is, indeed, given by (2.7.51). On the other hand, if $g(x) \ge 0$ and $g(x) \not\equiv 0$, then the parabolic comparison principle implies that $v(t,x) > 0$ for all $t > 0$ and all $x \in \Omega$. It follows that $u(x) > 0$ in Ω.

Therefore, we have proved the following theorem that succinctly relates the notions of the principal eigenvalue and the comparison principle.

Theorem 2.7.15 *If the principal eigenvalue of the operator L is positive then solutions of the elliptic equation (2.7.50) satisfy the comparison principle: $u(x) > 0$ in Ω if $g(x) \ge 0$ in Ω and $g(x) \not\equiv 0$.*

This theorem allows to look at the maximum principle in narrow domains introduced in the previous chapter from a slightly different point of view: the narrowness of the domain implies that the principal eigenvalue of L is positive no matter what the sign of the free coefficient $c(x)$ is. This is because the "size" of the second order term in L increases as the domain narrows, while the "size" of the zero-order term does not change. Therefore, in a sufficiently narrow domain the principal eigenvalue of L will be positive (recall that the required narrowness does depend on the size of $c(x)$). A similar philosophy applies to the maximum principle for the domains of a small volume.

We conclude this topic with another characterization of the principal eigenvalue of an elliptic operator in a bounded domain, which we leave as an (important) exercise for the reader. Let us define

$$\mu_1(\Omega) = \sup\{\lambda : \ \exists\phi \in C^2(\Omega) \cap C^1(\bar{\Omega}), \ \phi > 0 \text{ and } (L-\lambda)\phi \ge 0 \text{ in } \Omega\}, \tag{2.7.52}$$

and

$$\mu_1'(\Omega) = \inf\{\lambda : \ \exists \phi \in C^2(\Omega) \cap C^1(\bar{\Omega}), \ \phi = 0 \text{ on } \partial\Omega, \ \phi > 0 \text{ in } \Omega, \text{ and } (L - \lambda)\phi \leq 0 \text{ in } \Omega\}. \tag{2.7.53}$$

Exercise 2.7.16 *Let L be an elliptic operator in a smooth bounded domain Ω, and let λ_1 be the principal eigenvalue of the operator L, and $\mu_1(\Omega)$ and $\mu_1'(\Omega)$ be as above. Show that*

$$\lambda_1 = \mu_1(\Omega) = \mu_1'(\Omega). \tag{2.7.54}$$

As a hint, say, for the equality $\lambda_1 = \mu_1(\Omega)$, we suggest, assuming existence of some $\lambda > \lambda_1$ and $\phi > 0$ such that

$$(L - \lambda)\phi \geq 0,$$

to consider the Cauchy problem

$$u_t + (L - \lambda)u = 0, \quad \text{in } \Omega$$

with the initial data $u(0, x) = \phi(x)$, and with the Dirichlet boundary condition $u(t, x) = 0$ for $t > 0$ and $x \in \partial\Omega$. One should prove two things: first, that $u_t(t, x) \leq 0$ for all $t > 0$, and, second, that there exists some constant $C > 0$ so that

$$u(t, x) \geq C\phi_1(x)e^{(\lambda - \lambda_1)t},$$

where ϕ_1 is the principal Dirichlet eigenfunction of L. This will lead to a contradiction. The second equality in (2.7.54) is proved in a similar way.

2.8 The long time behavior for viscous Hamilton-Jacobi equations

A (once again, to our taste, rather striking) application of the principal elliptic eigenvalue is a study of the long time behavior of the solutions to the viscous Hamilton-Jacobi equations, that we now present. This problem falls in the same class as in Section 2.6, where we proved, essentially with the sole aid of the strong maximum principle and the Harnack inequality, the convergence of the solutions of the Cauchy problem for the Allen-Cahn equations to a translate of a stationary solution. The main difference is that now we will have to fight a little to show the existence of a steady state, while the long time convergence will be relatively effortless. We are interested in the large time behaviour of the solutions $u(t, x)$ to the Cauchy problem for

$$u_t - \Delta u = H(x, \nabla u), \qquad t > 0, \quad x \in \mathbb{R}^n. \tag{2.8.1}$$

This is an equation of the form (2.4.45) that we have considered in Section 2.4.3, and we make the same assumptions on the nonlinearity, that we now denote by H, the standard notation in the theory of the Hamilton-Jacobi equations, as in that section. We assume that H is smooth and 1-periodic in x. We also make the Lipschitz assumption on the function $H(x, p)$: there exists $C > 0$ so that

$$|H(x, p_1) - H(x, p_2)| \leq C|p_1 - p_2|, \text{ for all } x, p_1, p_2 \in \mathbb{R}^n. \tag{2.8.2}$$

In addition, we assume that H is growing linearly in p at infinity: there exist $\alpha > 0$ and $\beta > 0$ so that

$$0 < \alpha \le \liminf_{|p|\to+\infty} \frac{H(x,p)}{|p|} \le \limsup_{|p|\to+\infty} \frac{H(x,p)}{|p|} \le \beta < +\infty, \text{ uniformly in } x \in \mathbb{T}^n. \tag{2.8.3}$$

One consequence of (2.8.3) is that there exist $C_{1,2} > 0$ so that

$$C_1(1 + |p|) \le H(x,p) \le C_2(1 + |p|), \text{ for all } x \in \mathbb{T}^n \text{ and } p \in \mathbb{R}^n. \tag{2.8.4}$$

As we have seen in Section 2.4.3, these assumptions ensure the existence of a unique smooth 1-periodic solution $u(t,x)$ to (2.8.1) supplemented by a continuous, 1-periodic initial condition $u_0(x)$. In order to discuss its long time behavior, we need to introduce a special class of solutions of (2.8.1).

Theorem 2.8.1 *Under the above assumptions, there exists a unique $m \in \mathbb{R}$ so that* (2.8.1) *has solutions (that we will call the wave solutions) of the form*

$$w(t,x) = mt + \phi(x), \tag{2.8.5}$$

with a 1-periodic function $\phi(x)$. The profile $\phi(x)$ is unique up to an additive constant: if $w_1(t,x)$ and $w_2(t,x)$ are two such solutions then there exists $c \in \mathbb{R}$ so that $\phi_1(x) - \phi_2(x) \equiv c$ for all $x \in \mathbb{T}^n$.

The large time behaviour of $u(t,x)$ is summarized in the next theorem.

Theorem 2.8.2 *Let $u(t,x)$ be the solution of the Cauchy problem for* (2.8.1) *with a continuous 1-periodic initial condition u_0. There is a wave solution $w(t,x)$ of the form* (2.8.5)*, a constant $\omega > 0$ that does not depend on u_0 and $C_0 > 0$ that depends on u_0 such that*

$$|u(t,x) - w(t,x)| \le C_0 e^{-\omega t}, \tag{2.8.6}$$

for all $t \ge 0$ and $x \in \mathbb{T}^n$.

We will first prove the existence part of Theorem 2.8.1, and that will occupy most of the rest of this section, while its uniqueness part and the convergence claim of Theorem 2.8.2 will be proved together rather quickly in the end. Plugging the ansatz (2.8.5) into (2.8.1) and integrating over $\mathbb{T}^n$ gives

$$m = \int_{\mathbb{T}^n} H(x, \nabla\phi)dx. \tag{2.8.7}$$

The equation for ϕ can, therefore, be written as

$$-\Delta\phi = H(x,\nabla\phi) - \int_{\mathbb{T}^n} H(x,\nabla\phi)dx, \tag{2.8.8}$$

and this will be the starting point of our analysis.

2.8.1 Existence of a wave solution

Outline of the proof

Let us first outline how we will prove the existence of a wave solution. We are going to use the inverse function theorem, and, as this strategy is typical for the existence proofs for many nonlinear PDEs, it is worth sketching out the general plan, even if without stating all the details. Instead of just looking at (2.8.8), we consider a family of equations

$$-\Delta\phi_\sigma = H_\sigma(x,\nabla\phi_\sigma) - \int_{\mathbb{T}^n} H_\sigma(x,\nabla\phi_\sigma)dx, \tag{2.8.9}$$

with the Hamiltonians

$$H_\sigma(x,p) = (1-\sigma)H_0(x,p) + \sigma H(x,p), \tag{2.8.10}$$

parametrized by $\sigma \in [0,1]$. We start with $H_0(x,p)$ for which we know that (2.8.8) has a solution. In our case, we can take

$$H_0(x,p) = \sqrt{1+|p|^2} - 1,$$

so that $\phi_0(x) \equiv 0$ is a solution to (2.8.8). We end with

$$H_1(x,p) = H(x,p). \tag{2.8.11}$$

The goal is show that (2.8.9) has a solution for all $\sigma \in [0,1]$ and not just for $\sigma = 0$ by showing that the set Σ of σ for which (2.8.9) has a solution is both open and closed in $[0,1]$.

Showing that Σ is closed requires a priori bounds on the solution ϕ_σ of (2.8.9) that would both be uniform in σ and ensure the compactness of the sequence ϕ_{σ_n} of solutions of (2.8.9) as $\sigma_n \to \sigma$ in a suitable function space. The estimates should be strong enough to ensure that the limit ϕ_σ is a solution to (2.8.9).

In order to show that Σ is open, one relies on the inverse function theorem. Let us assume that (2.8.9) has a solution $\phi_\sigma(x)$ for some $\sigma \in [0,1]$. In order to show that (2.8.9) has a solution for $\sigma + \varepsilon$, with a sufficiently small ε, we are led to consider the linearized problem

$$-\Delta h - \frac{\partial H_\sigma(x,\nabla\phi_\sigma)}{\partial p_j}\frac{\partial h}{\partial x_j} + \int_{\mathbb{T}^n} \frac{\partial H_\sigma(z,\nabla\phi_\sigma)}{\partial p_j}\frac{\partial h(z)}{\partial z_j}dz = f, \tag{2.8.12}$$

with

$$f(x) = H(x,\nabla\phi_\sigma) - H_0(x,\nabla\phi_\sigma) - \int_{\mathbb{T}^n} H(z,\nabla\phi(z))dz + \int_{\mathbb{T}^n} H_0(z,\nabla\phi(z))dz. \tag{2.8.13}$$

The inverse function theorem guarantees existence of the solution $\phi_{\sigma+\varepsilon}$, provided that the linearized operator in the left side of (2.8.12) is invertible, with the norm of the inverse a priori bounded in σ. This will show that the set Σ of $\sigma \in [0,1]$ for which the solution to (2.8.9) exists is open.

The bounds on the operator that maps $f \to h$ in (2.8.12) also require the a priori bounds on ϕ_σ. Thus, both proving that Σ is open and that it is closed require us to prove the a priori uniform bounds on ϕ. Therefore, our first step will be to assume that a solution $\phi_\sigma(x)$

to (2.8.9) exists and obtain a priori bounds on ϕ_σ. Note that if ϕ_σ is a solution to (2.8.9), then $\phi_\sigma + k$ is also a solution for any $k \in \mathbb{R}$. Thus, it is more natural to obtain a priori bounds on $\nabla\phi_\sigma$ than on ϕ itself, and then normalize the solution so that $\phi_\sigma(0) = 0$ to ensure that ϕ_σ is bounded.

It is important to observe that the Hamiltonians $H_\sigma(x,p)$ obey the same Lipschitz bound, and estimate (2.8.4) holds with the same $C_{1,2} > 0$ for all $\sigma \in [0,1]$, The key bound to prove will be to show that there exists a constant $K > 0$ that depends only on the Lipschitz constant of H and the two constants in the linear growth estimate (2.8.4) such that any solution to (2.8.8) satisfies

$$\|\nabla\phi\|_{L^\infty(\mathbb{T}^n)} \leq K. \tag{2.8.14}$$

We stress that this bound will be obtained not just for one Hamiltonian but for all Hamiltonians with the same Lipschitz constant that satisfy (2.8.4) with the same $C_{1,2} > 0$. The estimate (2.8.14) will be sufficient to apply the argument we have outlined above.

An a priori L^1-bound on the gradient

For simplicity, we will drop the subscript σ in the proof whenever possible. Before establishing (2.8.14), let us first prove that there exists a constant $C > 0$ such that any solution $\phi(x)$ of (2.8.8) satisfies

$$\int_{\mathbb{T}^n} H(x, \nabla\phi)dx \leq C. \tag{2.8.15}$$

Because of the lower bound in (2.8.3), this is equivalent to an a priori L^1 bound on $|\nabla\phi|$:

$$\int_{\mathbb{T}^n} |\nabla\phi(x)|dx \leq C, \tag{2.8.16}$$

with a possibly different $C > 0$. To prove (2.8.15), we will rely on the following ingredient, which comes from the Krein-Rutman theorem – and this is one of the reasons why it is quite suitable to put this example here. For an $\mathbb{R}^n$-valued function $v(x)$ we denote the divergence of $v(x)$ by

$$\nabla \cdot v = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}.$$

Lemma 2.8.3 *Let $b(x)$ be a smooth vector field over $\mathbb{T}^n$. The linear equation*

$$-\Delta e + \nabla \cdot (eb) = 0, \qquad x \in \mathbb{T}^n, \tag{2.8.17}$$

has a unique solution $e_1^*(x)$ normalized so that

$$\|e_1^*\|_{L^\infty} = 1, \tag{2.8.18}$$

and such that $e_1^* > 0$ on $\mathbb{T}^n$. Moreover, for all $\alpha \in (0,1)$, the function e_1^* is α-Hölder continuous, with the α-Hölder norm bounded by a universal constant depending only on $\|b\|_{L^\infty(\mathbb{T}^n)}$.

A key point here is that the Hölder regularity of the solution only depends on the L^∞-norm of $b(x)$ but not on its smoothness or any of its derivatives – this is typical for equations in the divergence form, and we will see much more of this in Chapter **??**. This is very different from what we have seen so far in this chapter: we have always relied on the assumption that the coefficients are smooth, and the Hölder bounds for the solutions depended on the regularity of the coefficients. A very remarkable fact is that for equations in the divergence form, such as (2.8.17), one may often obtain bounds on the regularity of the solutions that depend only on the L^∞-norm of the coefficients but not on their smoothness. Such bounds are much harder to get for equations in the non-divergence form.

Let us first see why this lemma implies (2.8.15). An immediate consequence of the normalization (2.8.18) and the claim about the Hölder norm of e_1^*, together with the positivity of e_1^* is that

$$\int_{\mathbb{T}^n} e_1^*(x)dx \geq C > 0, \tag{2.8.19}$$

with a constant $C > 0$ that depends only on $\|b\|_{L^\infty}$. Now, given a solution $\phi(x)$ of (2.8.8), set

$$b_j(x) = \int_0^1 \partial_{p_j} H(x, \sigma\nabla\phi(x))d\sigma, \tag{2.8.20}$$

so that

$$b_j(x)\frac{\partial\phi}{\partial x_j} = H(x, \nabla\phi) - H(x, 0), \tag{2.8.21}$$

and (2.8.8) can be re-stated as

$$-\Delta\phi - b_j(x)\frac{\partial\phi}{\partial x_j} = H(x, 0) - \int_{\mathbb{T}^n} H(x, \nabla\phi)dx. \tag{2.8.22}$$

Note that while $b(x)$ does depend on $\nabla\phi$, the L^∞-norm of $b(x)$ depends only on the Lipschitz constant of the function $H(x, p)$ in the p-variable. Let now e_1^* be given by Lemma 2.8.3, with the above $b(x)$. Multiplying (2.8.22) by e_1^* and integrating over $\mathbb{T}^n$ yields

$$0 = \int_{\mathbb{T}^n} e_1^*(x)H(x, 0)dx - \Big(\int_{\mathbb{T}^n} e_1^*(x)dx\Big)\Big(\int_{\mathbb{T}^n} H(x, \nabla\phi)dx\Big), \tag{2.8.23}$$

hence

$$\int_{\mathbb{T}^n} H(x, \nabla\phi)dx \leq \Big(\int_{\mathbb{T}^n} e_1^*(x)dx\Big)^{-1} \int_{\mathbb{T}^n} e_1^*(x)H(x, 0)dx, \tag{2.8.24}$$

and (2.8.16) follows from (2.8.19).

Proof of Lemma 2.8.3

Let us denote

$$L\phi = -\Delta\phi - b_j(x)\frac{\partial\phi}{\partial x_j}. \tag{2.8.25}$$

The constant functions are the principal periodic eigenfunctions of L and zero is the principal eigenvalue:

$$L1 = 0. \tag{2.8.26}$$

Thus, by Theorem 2.7.8, the operator L has no other eigenvalue with a non-positive real part, which entails the same result for the operator

$$L^*\phi = -\Delta\phi + \nabla \cdot (b(x)\phi).$$

In particular, zero is the principal eigenvalue of L^*, associated to a positive eigenfunction $e_1^*(x) > 0$:

$$L^* e_1^* = 0, \quad \text{for all } x \in \mathbb{T}^n,$$

and we can normalize e_1^* so that that (2.8.18) holds. Thus, existence of $e_1^*(x)$ is the easy part of the proof.

The challenge is, of course, to bound the Hölder norms of e_1^* in terms of $\|b\|_{L^\infty(\mathbb{T}^n)}$ only. We would like to use a representation formula, as we already did many times in this chapter. In other words, we would like to treat the term $\nabla \cdot (e_1^* b)$ as a force, and convolve it with the fundamental solution of the Laplace equation in $\mathbb{R}^n$. For that, we need various quantities to be sufficiently integrable, so we first localize the equation, and then write a representation formula. This is very similar to the proof of the interior regularity estimates that we have mentioned very briefly in Section 2.5 – see Exercise 2.5.11. We recommend the reader to go back to this Section after finishing the current proof, and attempt this exercise again, setting $a_{ij}(t,x) = \delta_{ij}$ in (2.5.46) for simplicity.

Let $\Gamma(x)$ be a nonnegative smooth cut-off function such that $\Gamma(x) \equiv 1$ for $x \in [-2,2]^n$ and $\Gamma(x) \equiv 0$ outside $(-3,3)^n$. The function $v(x) = \Gamma(x) e_1^*(x)$ satisfies

$$-\Delta v = -2\nabla\Gamma \cdot \nabla e_1^* - e_1^* \Delta\Gamma - \Gamma \nabla \cdot (e_1^* b), \quad x \in \mathbb{R}^n. \tag{2.8.27}$$

Remember that e_1^* is bounded in L^∞, thus so is v. As we will see, nothing should be feared from the cumbersome quantities like $\Delta\Gamma$ or $\nabla\Gamma$. We concentrate on the space dimensions $n \geq 2$, leaving $n = 1$ as an exercise. Let $E(x)$ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$: the solution of

$$-\Delta u = f, \quad x \in \mathbb{R}^n, \tag{2.8.28}$$

is given by

$$u(x) = \int_{\mathbb{R}^n} E(x-y) u(y) dy. \tag{2.8.29}$$

Then we have

$$v(x) = \int_{\mathbb{R}^n} E(x-y)\Big[-2\nabla\Gamma(y) \cdot \nabla e_1^*(y) - e_1^*(y)\Delta\Gamma(y) - \Gamma(y)\nabla \cdot (e_1^*(y) b(y))\Big] dy. \tag{2.8.30}$$

After an integration by parts, we obtain

$$v(x) = \int_{\mathbb{R}^n} \Big((\nabla E(x-y) \cdot \nabla\Gamma(y)) e_1^*(y) + E(x-y) e_1^*(y) \Delta\Gamma(y) + \nabla (E(x-y)\Gamma(y)) \cdot b(y) e_1^*(y) \Big) dy. \tag{2.8.31}$$

The key point is that no derivatives of $b(x)$ or $e_1^*(x)$ appear in the right side of (2.8.31) – this is important as the only a priori information that we have on these functions is that they are bounded in L^∞. Thus, the main point is to prove that integrals of the form

$$P(x) = \int_{\mathbb{R}^n} E(x-y) G(y) dy, \tag{2.8.32}$$

with a bounded and compactly supported function $G(x)$, and

$$I(x) = \int_{\mathbb{R}^n} \nabla E(x-y) \cdot F(y) dy, \tag{2.8.33}$$

with a bounded and compactly supported vector-valued function $F:\ \mathbb{R}^n \mapsto \mathbb{R}^n$, are α-Hölder continuous for all $\alpha \in (0,1)$, with the Hölder constants depending only on α and the L^∞-norms of F and G. Both F and G are supported inside the cube $[-3,3]^n$. We will only consider the integral $I(x)$, as this would also show that $\nabla P(x)$ is α-Hölder. Using the expression

$$\nabla E(z) = c_n \frac{z}{|z|^n},$$

we see that

$$|I(x) - I(x')| \le c_n \|F\|_{L^\infty} K(x,x'), \tag{2.8.34}$$

with

$$K(x,x') = \int_{(-3,3)^n} \left| \frac{x-y}{|x-y|^n} - \frac{x'-y}{|x'-y|^n} \right| dy. \tag{2.8.35}$$

Pick now $\alpha \in (0,1)$. We estimate K by splitting the integration domain into two disjoint pieces:

$$A_x = \{y \in (-3,3)^n :\ |x-y| \le |x-x'|^\alpha\},\quad B_x = \{y \in (-3,3)^n :\ |x-y| > |x-x'|^\alpha\},$$

and denote by $K_A(x,x')$ and $K_B(x,x')$ the contribution to $K(x,x')$ by the integration over each of these two regions. To avoid some unnecessary trouble, we assume that $|x-x'| \le l_\alpha$, with l_α such that

$$3l \le l^\alpha \ \text{ for all } l \in [0, l_\alpha]. \tag{2.8.36}$$

With this choice, we have

$$|x'-y| \le |x'-x| + |x-y| \le 2|x-x'|^\alpha \quad \text{if } y \in A_x, \tag{2.8.37}$$

and

$$|x'-y| \ge |x-y| - |x'-x| \ge 2|x-x'| \quad \text{if } y \in B_x. \tag{2.8.38}$$

It follows that

$$K_A(x,x') \le \int_{|x-y| \le |x-x'|^\alpha} \frac{dy}{|x-y|^{n-1}} + \int_{|x'-y| \le 2|x-x'|^\alpha} \frac{dy}{|x'-y|^{n-1}} \le C|x-x'|^\alpha. \tag{2.8.39}$$

To estimate K_B, we write

$$\left| \frac{x-y}{|x-y|^n} - \frac{x'-y}{|x'-y|^n} \right| \le C|x-x'| \int_0^1 \frac{d\sigma}{|x_\sigma - y|^n}, \quad x_\sigma = \sigma x + (1-\sigma)x'. \tag{2.8.40}$$

Note that for all $y \in B_x$ we have

$$|x_\sigma - y| \ge |x-y| - |x - x_\sigma| \ge |x-x'|^\alpha - |x-x'| \ge 2|x'-x|,$$

and $|y| \le 3\sqrt{n}$, hence

$$\begin{aligned} K_B(x,x') &\le |x-x'| \int_0^1 d\sigma \int_{B_x} \frac{dy}{|x_\sigma - y|^n} \le |x-x'| \int_0^1 d\sigma \int_{|x_\sigma - y| \ge |x-x'|} \frac{\chi(|y| \le 3\sqrt{n}) dy}{|x_\sigma - y|^n} \\ &\le C|x-x'| \log|x-x'|, \end{aligned} \tag{2.8.41}$$

which implies the uniform α-Hölder bound for $I(x)$, for all $\alpha \in (0,1)$. $\Box$

An a priori L^∞ bound on the gradient

So far, we have obtained an a priori L^1-bound for the gradient of any solution ϕ to (2.8.8). Now, we improve this estimate to an L^∞ bound.

Proposition 2.8.4 *There is $C > 0$, universal, such that any solution ϕ of (2.8.8) satisfies*

$$\|\nabla\phi\|_{L^\infty(\mathbb{T}^n)} \leq C. \tag{2.8.42}$$

As a consequence, if ϕ is normalized such that $\phi(0) = 0$, then we also have $\|\phi\|_{L^\infty(\mathbb{T}^n)} \leq C$.

Proof. We borrow the strategy in the proof of Lemma 2.8.3. Let ϕ be a solution of (2.8.8) such that $\phi(0) = 0$. The only estimate we have so far is an L^1-bound for $\nabla\phi$ – the idea is to estimate $\|\phi\|_{L^\infty(\mathbb{T}^n)}$ and $\|\nabla\phi\|_{L^\infty(\mathbb{T})}$ solely from this quantity and the equation. Let $\Gamma(x)$ be as in the preceding proof: a nonnegative smooth function equal to 1 in $[-2,2]^n$ and to zero outside $(-3,3)^n$, and set $\psi(x) = \Gamma(x)\phi(x)$. The function $\psi(x)$ satisfies an equation similar to (2.8.27):

$$-\Delta\psi = -2\nabla\Gamma\cdot\nabla\phi - \phi\Delta\Gamma + F(x), \quad x \in \mathbb{R}^n, \tag{2.8.43}$$

with

$$F(x) = \Gamma(x)\Big[H(x,\nabla\phi(x)) - \int_{\mathbb{T}^n} H(z,\nabla\phi(z))dz\Big]. \tag{2.8.44}$$

The only a priori information we have about $F(x)$ and the term $\nabla\Gamma\cdot\nabla\phi(x)$ so far is that they are supported inside $[-3,3]^n$ and are uniformly bounded in $L^1(\mathbb{R}^n)$. It helps to combine them:

$$G(x) = F(x) - 2\nabla\Gamma(x)\cdot\nabla\phi(x), \tag{2.8.45}$$

with $G(x)$ supported inside $[-3,3]^n$, and

$$\int_{\mathbb{R}^n} |G(x)|dx \leq C. \tag{2.8.46}$$

We also know that

$$|G(x)| \leq C(1 + |\nabla\phi(x)|. \tag{2.8.47}$$

Then, we write

$$\psi(x) = \int_{\mathbb{R}^n} E(x-y)[G(y) - \phi(y)\Delta\Gamma(y)]dy. \tag{2.8.48}$$

Differentiating in x gives

$$\nabla\psi(x) = \int_{\mathbb{R}^n} \nabla E(x-y)[G(y) - \phi(y)\Delta\Gamma(y)\big]dy. \tag{2.8.49}$$

The function $\nabla E(x-y)$ has an integrable singularity at $y = x$, of the order $|x-y|^{-n+1}$ and is bounded everywhere else. Thus, for all $\varepsilon > 0$ we have, with the help of (2.8.45) and (2.8.47):

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} G(y)\nabla E(x-y)dy\Big| &\leq \Big|\int_{|x-y|\leq\varepsilon} G(y)\nabla E(x-y)dy\Big| + \Big|\int_{|x-y|\geq\varepsilon} G(y)\nabla E(x-y)dy\Big| \\ &\leq C(1 + \|\nabla\phi\|_{L^\infty})\int_{|x-y|\leq\varepsilon}\frac{dy}{|x-y|^{n-1}} + \varepsilon^{-n+1}\int_{|x-y|\geq\varepsilon}|G(y)|dy \\ &\leq C\varepsilon(1 + \|\nabla\phi\|_{L^\infty}) + C\varepsilon^{1-n}.\end{aligned} \tag{2.8.50}$$

The integral in (2.8.49) also contains a factor of ϕ, whereas our bounds so far deal with $\nabla\phi$. However, we may assume without loss of generality that $\phi(0) = 0$, and then

$$\phi(y) = \int_0^1 y \cdot \nabla\phi(sy)ds = \int_0^\varepsilon y \cdot \nabla\phi(sy)ds + \int_\varepsilon^1 y \cdot \nabla\phi(sy)ds,$$

so that

$$|\phi(y)| \leq \|\nabla\phi\|_{L^\infty}, \tag{2.8.51}$$

and

$$\begin{aligned}\int_{\mathbb{T}^n} |\phi(y)|dy &\leq C\varepsilon\|\nabla\phi\|_{L^\infty} + \int_\varepsilon^1 \int_{\mathbb{T}^n} |y||\nabla\phi(sy)|dyds\\ &\leq C\varepsilon\|\nabla\phi\|_{L^\infty} + C\int_\varepsilon^1 \int_{s^{-1}\mathbb{T}^n} |\nabla\phi(y)|dy\frac{ds}{s} \leq C\varepsilon\|\nabla\phi\|_{L^\infty} + C\int_\varepsilon^1 \frac{ds}{s^{1+n}}\\ &\leq C\varepsilon\|\nabla\phi\|_{L^\infty} + C\varepsilon^{-n}.\end{aligned} \tag{2.8.52}$$

We used above the a priori bound (2.8.16) on $\|\nabla\phi\|_{L^1(\mathbb{T}^n)}$. Combining (2.8.51) and (2.8.52), we obtain, as in (2.8.50):

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \phi(y)\Delta\Gamma(y)\nabla E(x-y)dy\right| \leq \int_{|x-y|\leq\varepsilon} |\phi(y)||\Delta\Gamma(y)|\nabla E(x-y)|dy\\ &+\int_{|x-y|\geq\varepsilon} |\phi(y)||\Delta\Gamma(y)||\nabla E(x-y)|dy \leq C\varepsilon\|\nabla\phi\|_{L^\infty} + C\varepsilon^{1-2n}.\end{aligned} \tag{2.8.53}$$

Now, because $\Gamma \equiv 1$ in $[-2,2]^n$ and ϕ is 1-periodic, we have

$$\|\nabla\phi\|_{L^\infty(\mathbb{T}^n)} = \|\nabla(\Gamma\phi)\|_{L^\infty([-1,1]^n)} \leq \|\nabla(\Gamma\phi)\|_{L^\infty([-3,3]^n)} = \|\nabla\psi\|_{L^\infty}. \tag{2.8.54}$$

Together with the previous estimates, this implies

$$\|\nabla\phi\|_{L^\infty} \leq C\varepsilon\|\nabla\phi\|_\infty + C_\varepsilon, \tag{2.8.55}$$

with a universal constant $C > 0$ and C_ε that does depend on ε. Now, the proof is concluded by taking $\varepsilon > 0$ small enough. □

Going back to the equation (2.8.8) for ϕ:

$$-\Delta\phi = H(x,\nabla\phi) - \int_{\mathbb{T}^n} H(x,\nabla\phi)dx, \tag{2.8.56}$$

the reader should do the following exercise.

Exercise 2.8.5 Use the L^∞-bound on $\nabla\phi$ in Proposition 2.8.4 to deduce from (2.8.56) that, under the assumption that $H(x,p)$ is smooth in both variables x and p, the function $\phi(x)$ is, actually, infinitely differentiable, with all its derivatives of order n bounded by a priori constants C_n that do not depend on ϕ.

The linearized problem

We need one last ingredient to finish the proof of the existence part of Theorem 2.8.1: to set-up an application of the inverse function theorem. Let ϕ be a solution to (2.8.8) and let us consider the linearized problem, with an unknown h:

$$-\Delta h - \partial_{p_j} H(x,\nabla\phi)\partial_{x_j} h + \int_{\mathbb{T}^n} \partial_{p_j} H(y,\nabla\phi)\partial_{x_j} h(y)dy = f \quad x \in \mathbb{T}^n. \tag{2.8.57}$$

We assume that $f \in C^{1,\alpha}(\mathbb{T}^n)$ for some $\alpha \in (0,1)$, and f has zero mean over $\mathbb{T}^n$:

$$\int_{\mathbb{T}^n} f(x)dx = 0.$$

Proposition 2.8.6 *Equation* (2.8.57) *has a unique solution* $h \in C^{3,\alpha}(\mathbb{T}^n)$ *with zero mean. The mapping* $f \mapsto h$ *is continuous from the set of* $C^{1,\alpha}$ *functions with zero mean to* $C^{3,\alpha}(\mathbb{T}^n)$.

Proof. The Laplacian is a one-to-one map between the set of $C^{m+2,\alpha}$ functions with zero mean and the set of $C^{m,\alpha}(\mathbb{T}^n)$ functions with zero mean, for any $m \in \mathbb{N}$. Thus, we may talk about its inverse that we denote by $(-\Delta)^{-1}$. Equation (2.8.57) is thus equivalent to

$$(I + K)h = (-\Delta)^{-1} f, \tag{2.8.58}$$

with the operator

$$Kh = (-\Delta)^{-1}\Big(-\partial_{p_j} H(x,\nabla\phi)\partial_{x_j} h + \int_{\mathbb{T}^n} \partial_{p_j} H(y,\nabla\phi)\partial_{x_j} h(y)dy\Big). \tag{2.8.59}$$

Exercise 2.8.7 Show that K is a compact operator on the set of functions in $C^{1,\alpha}(\mathbb{T}^n)$ with zero mean.

The problem has been now reduced to showing that the only solution of

$$(I + K)h = 0 \tag{2.8.60}$$

with $h \in C^{1,\alpha}(\mathbb{T}^n)$ with zero mean is $h \equiv 0$. Note that (2.8.60) simply says that h is a solution of (2.8.57) with $f \equiv 0$. Let $e_1^* > 0$ be given by Lemma 2.8.3, with

$$b_j(x) = -\partial_{p_j} H(x,\nabla\phi). \tag{2.8.61}$$

That is, e_1^* is the positive solution of the equation

$$-\Delta e_1^* + \nabla\cdot(e_1^* b) = 0, \tag{2.8.62}$$

normalized so that $\|e_1^*\|_{L^\infty(\mathbb{T}^n)} = 1$. The uniform Lipschitz bound on $H(x,p)$ in the p-variable implies that $b(x)$ is in $L^\infty(\mathbb{T}^n)$, and thus Lemma 2.8.3 can be applied. Multiplying (2.8.57) with $f = 0$ by e_1^* and integrating gives, as $e_1^* > 0$:

$$\int_{\mathbb{T}^n} \partial_{p_j} H(y,\nabla\phi)\partial_{x_j} h(y)dy = 0.$$

But then, the equation for h becomes simply

$$-\Delta h + b_j(x)\partial_{x_j} h = 0, \quad x \in \mathbb{T}^n,$$

which entails that h is constant, by the Krein-Rutman theorem. Because h has zero mean, we get $h \equiv 0$. $\Box$

Exercise 2.8.8 Let $H_0(x,p)$ satisfy the assumptions of Theorem 2.8.2, and assume that equation (2.8.16), with $H = H_0$, has a solution ϕ_0. Consider $H_1(x,p) \in C^\infty(\mathbb{T} \times \mathbb{R}^n)$. Prove, with the aid of Propositions 2.8.4 and 2.8.6, and the implicit function theorem, the existence of $R_0 > 0$ and $\varepsilon_0 > 0$ such that if

$$|H_1(x,p)| \le \varepsilon, \quad \text{for } x \in \mathbb{T}^n \text{ and } |p| \le R_0, \tag{2.8.63}$$

then equation (2.8.16) with $H = H_0 + H_1$ has a solution ϕ.

Existence of the solution

We finally prove the existence part of Theorem 2.8.1. Consider $H(x,p)$ satisfying the assumptions of the theorem. Let us set

$$H_0(x,p) = \sqrt{1+|p|^2} - 1,$$

and

$$H_\sigma(x,p) = H_0(x,p) + \sigma(H(x,p) - H_0(x,p)),$$

so that $H_1(x,p) = H(x,p)$. Consider the set

$$\Sigma = \{\sigma \in [0,1] : \ \text{equation (2.8.16), with } H = H_\sigma, \text{ has a solution.}\}$$

We already know that Σ is non empty, because $0 \in \Sigma$: indeed, $\phi_0(x) \equiv 0$ is a solution to (2.8.16) with $H(x,p) = H_0(x,p)$. Thus, if we show that Σ is both open and closed in $[0,1]$, this will imply that $\Sigma = [0,1]$, and, in particular, establish the existence of a solution to (2.8.16) for $H_1(x,p) = H(x,p)$.

Now that we know that the linearized problem is invertible, the openness of Σ is a direct consequence of the inverse function theorem. Closedness of Σ is not too difficult to see either: consider a sequence $\sigma_n \in [0,1]$ converging to $\bar\sigma \in [0,1]$, and let ϕ_n be a solution to (2.8.16) with $H(x,p) = H_{\sigma_n}(x,p)$, normalized so that

$$\phi_n(0) = 0. \tag{2.8.64}$$

Proposition 2.8.4 implies that

$$\|\nabla\phi_n\|_{L^\infty(\mathbb{T}^n)} \le C,$$

and thus

$$\|H(x,\nabla\phi_n)\|_{L^\infty} \le C.$$

However, this means that ϕ_n solve an equation of the form

$$-\Delta\phi_n = F_n(x), \quad x \in \mathbb{T}^n, \tag{2.8.65}$$

with a uniformly bounded function

$$F_n(x) = H(x,\nabla\phi_n) - \int_{\mathbb{T}^n} H(z,\nabla\phi_n(z))dz. \tag{2.8.66}$$

It follows that that ϕ_n is bounded in $C^{1,\alpha}(\mathbb{T}^n)$, for all $\alpha \in [0,1)$:

$$\|\phi_n\|_{C^{1,\alpha}(\mathbb{T}^n)} \le C. \tag{2.8.67}$$

But this implies, in turn, that the functions $F_n(x)$ in (2.8.66) are also uniformly bounded in $C^{1,\alpha}$, hence ϕ_n are uniformly bounded in $C^{2,\alpha}(\mathbb{T}^n)$:

$$\|\phi_n\|_{C^{2,\alpha}(\mathbb{T}^n)} \leq C. \tag{2.8.68}$$

Now, the Arzela-Ascoli theorem implies that a subsequence ϕ_{n_k} will converge in $C^2(\mathbb{T}^n)$ to a function $\bar{\phi}$, which is a solution to (2.8.16) with $H = H_{\bar{\sigma}}$. Thus, $\sigma_\infty \in \Sigma$, and Σ is closed. This finishes the proof of the existence part of the theorem.

2.8.2 Long time convergence and uniqueness of the wave solutions

We will now prove simultaneously the claim of the uniqueness of m and of the profile $\phi(x)$ in Theorem 2.8.1, and the long time convergence for the solutions of the Cauchy problem stated in Theorem 2.8.2.

Let $u(t,x)$ be the solution of (2.8.1)

$$u_t = \Delta u + H(x, \nabla u), \quad t > 0,\ x \in \mathbb{T}^n, \tag{2.8.69}$$

with $u(0,x) = u_0(x) \in C(\mathbb{T}^n)$. We also take a solution $\phi(x)$ of

$$-\Delta\phi - H(x, \nabla\phi) = m. \tag{2.8.70}$$

We wish to prove that there exists $\bar{k} \in \mathbb{R}$ so that $u(t,x) - mt$ is attracted exponentially to $\phi(x) + \bar{k}$:

$$|u(t,x) - mt - \bar{k} - \phi(x)| \leq Ce^{-\omega t}, \tag{2.8.71}$$

with some $C > 0$ and $\omega > 0$. The idea is the same as in the proof of Theorem 2.6.2, but the situation here is much simpler: we do not have any tail to control, because we are now considering the problem for $x \in \mathbb{T}^n$. Actually, the present setting realizes what would be the dream scenario for the Allen-Cahn equation.

We may assume that $m = 0$, just by setting

$$H'(x,p) = H(x,p) - m,$$

and dropping the prime. Let ϕ be any solution of (2.8.70), and set

$$k_0^- = \sup\{k : \quad u(t,x) \geq \phi(x) + k \text{ for all } x \in \mathbb{T}^n\},$$

and

$$k_0^+ = \inf\{k : \quad u(t,x) \leq \phi(x) + k \text{ for all } x \in \mathbb{T}^n.\}$$

Because $\phi(x) - k_0^\pm$ solve (2.8.70), we have, by the maximum principle:

$$\phi(x) + k_0^- \leq u(t,x) \leq \phi(x) + k_0^+, \text{ for all } t \geq 0 \text{ and } x \in \mathbb{T}^n.$$

Now, for all $p \in \mathbb{N}$, let us set

$$k_p^- = \sup\{k : \quad u(t=p,x) \geq \phi(x) + k \text{ for all } x \in \mathbb{T}^n\} = \inf_{x \in \mathbb{T}^n}[u(t=p,x) - \phi(x)], \tag{2.8.72}$$

and

$$k_p^+ = \inf\{k : \quad u(t=p,x) \le \phi(x) + k \text{ for all } x \in \mathbb{T}^n\} = \sup_{x\in\mathbb{T}^n}[u(t=p,x) - \phi(x)]. \tag{2.8.73}$$

The maximum principle implies that the sequence k_p^- is increasing, whereas k_p^+ is decreasing. The theorem will be proved if we manage to show that

$$0 \le k_p^+ - k_p^- \le Ca^p, \quad \text{for all } p \ge 0, \tag{2.8.74}$$

with some $C \in \mathbb{R}$ and $a \in (0,1)$. However, this is easy: the function

$$w(t,x) = u(t,x) - \phi(x) - k_p^-$$

is nonnegative for $t \ge p$, and solves an equation of the form

$$\partial_t w - \Delta w + b_j(t,x)\partial_{x_j} w = 0, \qquad t > p, \quad x \in \mathbb{T}^n, \tag{2.8.75}$$

with a Lipschitz drift $b(t,x)$ such that

$$b(t,x)\cdot[\nabla u(t,x) - \nabla\phi(x)] = H(x,\nabla u) - H(x,\nabla\phi(x)).$$

In particular, we know that

$$\sup_{x\in\mathbb{T}^n} w(p,x) \ge \sup_{x\in\mathbb{T}^n} w(t,x), \qquad \text{for all } t \ge p. \tag{2.8.76}$$

The Harnack inequality implies that there exists $q_0 > 0$ such that

$$\inf_{x\in\mathbb{T}^n} w(p+1,x) \ge q_0 \sup_{x\in\mathbb{T}^n} w(p,x). \tag{2.8.77}$$

Using (2.8.72) and (2.8.73), we may rewrite this inequality as

$$k_{p+1}^- - k_p^- \ge q_0(k_p^+ - k_p^-), \tag{2.8.78}$$

which implies

$$k_{p+1}^+ - k_{p+1}^- \le k_p^+ - k_p^- - q_0(k_p^+ - k_p^-) \le (1-q_0)(k_p^+ - k_p^-). \tag{2.8.79}$$

This implies the geometric decay as in (2.8.74), hence the theorem. Note that the constant

$$a = 1 - q_0$$

comes from the Harnack inequality and does not depend on the initial condition u_0. □

Exercise 2.8.9 Why does the uniqueness of m and of the profile $\phi(x)$ follow?

Exercise 2.8.10 There is a certain recklessness in the way we have applied the Harnack inequality. We have proved the Harnack inequality in Theorem 2.5.13 for a fixed smooth drift $b(t,x)$. Here, we use it a family of drifts $b(t,x)$ that depend on $u(t,x)$ and $\phi(x)$ – how do we know that the constant q_0 does not degenerate to zero? Hint: revisit the proof of Theorem 2.5.13 and show that the bounds we have on $b(t,x)$ are sufficient to bound q_0 from below.

2.9 The inviscid Hamilton-Jacobi equations

In this section, we will consider the Hamilton-Jacobi equations

$$u_t + H(x, \nabla u) = 0 \tag{2.9.1}$$

on the unit torus $\mathbb{T}^n \subset \mathbb{R}^n$. Note that here, unlike in the viscous Hamilton-Jacobi equations we have considered so far, the diffusion coefficient vanishes. One may thus question why we consider it in the chapter on the diffusion equations – the answer is to emphasize the new difficulties and new phenomena that one encounters in the absence of diffusion. Another possible answer is that, philosophically, solutions to (2.9.1) behave very much like the solutions of

$$u_t^\varepsilon + H(x, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon, \tag{2.9.2}$$

with a small diffusivity $\varepsilon > 0$. Most of the techniques we have introduced so far deteriorate badly when the diffusion coefficient is small. We will see here that, actually, some of the bounds may survive, because they are helped by the nonlinear Hamiltonian $H(x, \nabla u)$. Obviously, not every nonlinearity is beneficial: for example, solutions of the rather benign looking advection equation

$$u_t + b(x) \cdot \nabla u(x) = 0, \tag{2.9.3}$$

are no better than the initial condition $u_0(x) = u(0, x)$, no matter how smooth the drift $b(x)$ is. Therefore, we will have to restrict ourselves to some class of Hamiltonians $H(x, p)$ that do help regularize the problem.

As in the viscous case, we will be interested both in the Cauchy problem, that is, (2.9.1) supplemented with an initial condition

$$u(0, x) = u_0(x), \tag{2.9.4}$$

and in a stationary version of (2.9.1):

$$H(x, \nabla u) = c, \quad x \in \mathbb{T}^n. \tag{2.9.5}$$

After what we have done in Section 2.8, it should be clear to the reader why (2.9.5) has a constant c in the right side – solutions of (2.9.5) lead to the wave solutions for the time-dependent problem (2.9.1). As in the viscous case, we will prove that under reasonable assumptions, solutions of (2.9.5) exist only for a unique value of c which has no reason to be equal to zero. Thus, the "standard" steady equation

$$H(x, \nabla u) = 0$$

typically would have no solutions. Alas, even though the speed c is unique, we will lose the uniqueness of the profile of the study solutions – unlike in the diffusive case, (2.9.5) may have non-unique solutions, even up to a translation. This is a major difference with the diffusive Hamilton-Jacobi equations, and one point we would like to emphasize in this section. However, we need to understand first what we mean by a solution to (2.9.1) or (2.9.5), and this will take some time.

A reader familiar with the theory of conservation laws, would see immediately the connection between them and the Hamilton-Jacobi equations: in one dimension, $n = 1$, differentiating (2.9.1) in x, we get a conservation law for $v = u_x$:

$$v_t + (H(x, v))_x = 0. \tag{2.9.6}$$

The basic conservation laws theory tells us that it is reasonable to expect that $v(t, x)$ becomes discontinuous in x at a finite time t. However, an entropy solution $v(t, x)$ to (2.9.6) will remain uniformly bounded in time. This means that the function $u(t, x)$ will fail to be C^1 but will remain Lipschitz. In agreement with this intuition, it is well known that, for a smooth initial condition u_0 on $\mathbb{T}^n$, the Cauchy problem (2.9.1), (2.9.4) has a unique *local* smooth solution. That is, there exists a time $t_0 > 0$, which depends on u_0, such that (2.9.1) has a C^1 solution $u(t, x)$ on the time interval $[0, t_0]$ such that $u(0, x) = u_0(x)$. However, this solution is not global in time: in general, it is impossible to extend it in a smooth fashion to $t = +\infty$. This is described very nicely in [60].

On the other hand, if we relax the constraint "u is C^1", and replace it by "u is Lipschitz", and require (2.9.1) and (2.9.4) to hold almost everywhere, there are, in general, several solutions to the Cauchy problem. This parallels the fact that the weak solutions to the conservation laws are not unique – for uniqueness, one must require that the weak solution satisfies the entropy condition. See, for instance, [100] for a discussion of these issues. A natural question is, therefore, to know if an additional condition, less stringent than the C^1-regularity, but stronger than the mere Lipschitz regularity, enables us to select a unique solution to the Cauchy problem – as the notion of the entropy solutions does for the conservation laws.

The above considerations have motivated the introduction, by Crandall and Lions [48], at the beginning of the 80's, of the notion of a *viscosity solution* to (2.9.1). The idea is to select, among all the solutions of (2.9.1), "the one that has a physical meaning" – though understanding the connection to physics may require some thought from the reader. Being weaker than the notion of a classical solution, it introduces new difficulties to the existence and uniqueness issues. Note that even if there is a unique viscosity solution to the Cauchy problem (2.9.1), (2.9.4), the stationary equation (2.9.5) has no reason to have a unique steady solution – unlike what we have seen in the diffusive situation of the previous section.

As a concluding remark to the introduction, we must mention that we will by no means do justice to a very rich subject in this short section, an interested reader can, and should happily delve into the sea of excellent papers on the Hamilton-Jacobi equations.

2.9.1 Viscosity solutions

Here, we present the basic notions of the viscosity solutions for the first order Hamilton-Jacobi equations, and prove a uniqueness result which is typical in this theory. The reader interested in all the subtleties of the theory may enjoy reading Barles [7], or Lions [100].

The definition of a viscosity solution

Let us begin with more general equations than (2.9.1) – we will restrict the assumptions as the theory develops. Consider the Cauchy problem

$$u_t + F(x, u, \nabla u) = 0, \quad t > 0, \ x \in \mathbb{T}^n, \tag{2.9.7}$$

with a continuous initial condition $u(0,x) = u_0(x)$, and $F \in C(\mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R})$.

In order to motivate the notion of a viscosity solution, one takes the point of view that the smooth solutions of the regularized problem

$$u_t^\varepsilon + F(x, u^\varepsilon, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon \tag{2.9.8}$$

are a good approximation to $u(t,x)$. Note that existence of the solution of the Cauchy problem for (2.9.8) for $\varepsilon > 0$ is not an issue – we have already seen how this can be proved. Hence, a natural attempt would be to pass to the limit $\varepsilon \downarrow 0$ in (2.9.8). This, however, is too blunt to succeed in general. To motivate a different route, instead, consider a smooth sub-solution of (2.9.8):

$$u_t + F(x, u, \nabla u) \le \varepsilon \Delta u. \tag{2.9.9}$$

Let us take a smooth function $\phi(t,x)$ such that the difference $\phi - u$ attains its minimum at a point (t_0, x_0). One may simply think of the case when $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \ge u(t,x)$ elsewhere. Then, at this point we have

$$u_t(t_0,x_0) = \phi_t(t_0,x_0), \quad \nabla\phi(t_0,x_0) = \nabla u(t_0,x_0),$$

and

$$D^2\phi(t_0,x_0) \ge D^2 u(t_0,x_0),$$

in the sense of the quadratic forms. It follows that

$$\begin{aligned} &\phi_t(t_0,x_0) + F(x_0, u(t_0,x_0), \nabla\phi(t_0,x_0)) - \varepsilon\Delta\phi(t_0,x_0) \\ &\le u_t(t_0,x_0) + F(x_0, u(t_0,x_0), \nabla u(t_0,x_0)) - \varepsilon \Delta u(t_0,x_0) \le 0. \end{aligned} \tag{2.9.10}$$

In other words, if u is a smooth sub-solution, and ϕ is a smooth function that touches u at (t_0,x_0) from above, then ϕ is a sub-solution to our equation at this point.

In a similar vein, if $u(t,x)$ is a smooth super-solution to the regularized problem:

$$u_t + F(x, u, \nabla u) \ge \varepsilon \Delta u, \tag{2.9.11}$$

we consider a smooth function $\phi(t,x)$ such that the difference $\phi - u$ attains its maximum at a point (t_0,x_0). Again, we may assume without loss of generality that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \le u(t,x)$ elsewhere. Then, at this point we have

$$\phi_t(t_0,x_0) + F(x_0, u(t_0,x_0), \nabla\phi(t_0,x_0)) - \varepsilon\Delta\phi(t_0,x_0) \ge 0. \tag{2.9.12}$$

In other words, if u is a smooth super-solution, and ϕ is a smooth function that touches u at (t_0,x_0) from below, then ϕ is a super-solution to our equation at this point.

These two observations lead to the following definition.

Definition 2.9.1 *A continuous function $u(t,x)$ is a viscosity sub-solution to* (2.9.7) *if, for all test functions $\phi \in C^1([0,+\infty) \times \mathbb{T}^n)$ and all $(t_0,x_0) \in (0,+\infty) \times \mathbb{T}^n$ such that (t_0,x_0) is a local minimum for $\phi - u$, we have:*

$$\phi_t(t_0,x_0) + F(x_0, u(t_0,x_0), \nabla\phi(t_0,x_0)) \le 0. \tag{2.9.13}$$

Furthermore, a continuous function $u(t,x)$ is a viscosity super-solution to (2.9.7) *if, for all test functions $\phi \in C^1((0,+\infty)\times\mathbb{T}^n)$ and all $(t_0,x_0) \in (0,+\infty)\times\mathbb{T}^n$ such that the point (t_0,x_0) is a local maximum for the difference $\phi - u$, we have:*

$$\phi_t(t_0,x_0) + F(x_0, u(t_0,x_0), \nabla\phi(t_0,x_0)) \geq 0. \tag{2.9.14}$$

Finally, $u(t,x)$ is a viscosity solution to (2.9.7) *if it is both a viscosity sub-solution and a viscosity super-solution to* (2.9.7).

Definition 2.9.1 trivially extends to steady equations of the type

$$F(x,u,\nabla u) = 0 \text{ on } \mathbb{T}^n.$$

Exercise 2.9.2 Show that a C^1 solution to (2.9.7) is a viscosity solution. Also show that the maximum of two viscosity subsolutions is a viscosity subsolution, and the minimum of two viscosity supersolutions is a viscosity supersolution.

The following exercise may help the reader gain some intuition.

Exercise 2.9.3 Consider the Hamilton-Jacobi equation

$$u_t + u_x^2 = 0, \quad x \in \mathbb{R}, \tag{2.9.15}$$

with a zigzag initial condition $u_0(x) = u(0,x)$:

$$u_0(x) = \begin{cases} x, & 0 \leq x \leq 1/2, \\ 1-x, & 1/2 \leq x \leq 1, \end{cases} \tag{2.9.16}$$

extended periodically to $\mathbb{R}$. How will the solution $u(t,x)$ of the Cauchy problem look like? Where will it be smooth, and where will it be just Lipschitz? Hint; it may help to do this in at least three ways: (1) use the definition of the viscosity solution, (2) use the notion of the entropy solution for the Burgers' equation for $v(t,x) = u_x(t,x)$, and (3) add the term εu_{xx} to the right side of (2.9.15), us the Hopf-Cole transformation $z(t,x) = \exp(u(t,x)/\varepsilon)$, solve the linear problem for $z(t,x)$ and then pass to the limit $\varepsilon \to 0$.

The reader may justly wonder whether such a seemingly weak definition has any selective power – can it possibly ensure uniqueness of the solution? This is the case, and we give below, without proof, a list of some basic properties of the viscosity solutions to (2.9.7), as exercises to the reader. These exercises are not as easy as Exercise 2.9.2, but the hints below should be helpful.

Exercise 2.9.4 (Stability) Let F_j be a sequence of functions in $C(\mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n)$, which converges locally uniformly to $F \in C(\mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n)$. Let u_j be a viscosity solution to (2.9.7) with $F = F_j$, and assume that u_j converges locally uniformly to $u \in C([0,+\infty),\mathbb{T}^n)$. Show that then u is a viscosity solution to (2.9.7). Hint: this is not difficult.

The above exercise is extremely important: it shows that, somewhat similar to the weak solutions, we do not need to check the convergence of the derivatives of u_j to the derivatives of u to know that u is a viscosity solution – this is an extremely useful property to have.

Exercise 2.9.5 Let u be a locally Lipschitz viscosity solution to (2.9.7). Then it satisfies (2.9.7) almost everywhere. Hint: if u is Lipschitz, then u is differentiable almost everywhere. Prove that, at a point of differentiability (t_0, x_0), one may construct a C^1 test function $\phi(t,x)$ such that (t_0, x_0) is a local maximum (respectively, a local minimum) of $\phi - u$. If you have no idea of how to do it, see [48].

Exercise 2.9.6 (The maximum principle) Assume that $F(x,u,p) = H(x,p)$, with a continuous function H that satisfies the following (coercivity) property:

$$\lim_{|p|\to+\infty} H(x,p) = +\infty, \quad \text{uniformly in } x \in \mathbb{T}^n. \tag{2.9.17}$$

Let $u_1(t,x)$ and $u_2(t,x)$ be the viscosity solutions for (2.9.7) with the initial conditions u_{10} and u_{20} such that $u_{10}(x) \le u_{20}(x)$ for all $x \in \mathbb{T}^n$. Show that then $u_1(t,x) \le u_2(t,x)$ for all $t \ge 0$ and $x \in \mathbb{T}^n$. This proves the uniqueness of the viscosity solutions. Hint: try to reproduce the proof of Proposition 2.9.7 below.

Definition 2.9.1 has been introduced by Crandall and Lions in their seminal paper [48]. Let us notice one of the main advantages of the notion: Exercise 2.9.4 asserts that one may safely "pass to the limit" in equation (2.9.7), as soon as estimates on the moduli of continuity of the solutions are available rather than on the derivatives. Exercise 2.9.6 implies uniqueness of the solutions to the Cauchy problem – without, however, implying existence.

The name "viscosity solution" comes out of trying to identify a "physically meaningful" solution to (2.9.7). As we have mentioned, a natural idea is to regularize (2.9.7) by a second order dissipative term, and to solve (2.9.8):

$$u_t + F(x,u,\nabla u) = \varepsilon \Delta u. \tag{2.9.18}$$

Then one tries to pass to the limit $\varepsilon \to 0$. This can be carried out when the Hamiltonian $F(x,u,p)$ has, for instance, the form $H(x,p)$. It is possible to prove that there is a unique limiting solution and that one actually ends up with a nonlinear semigroup. In particular, one may show that, if we take this notion of solution as a definition, there are uniqueness and contraction properties analogous to above – see [100] for further details. We will see below, in the proof of the Lions-Papanicolaou-Varadhan theorem how that can be done in one simple example. Taking (2.9.18) as a definition is, however, not intrinsic: there is always the danger that the solution depends on the underlying regularization (why regularize with the Laplacian?), and Definition 2.9.1 bypasses this philosophical question, much like the notion of an entropy solution does this for the conservation laws. Let us finally note that the notion of a viscosity solution has turned out to be especially relevant to the second order elliptic and parabolic equations – especially those fully nonlinear with respect to the Hessian of the solution. There have been spectacular developments, which are out of the scope of this chapter.

Warning. For the rest of this section, a solution of (2.9.1) or (2.9.5) will always be meant in the viscosity sense.

Uniqueness of the viscosity solutions

One of the main issues of the theory of the viscosity solutions is uniqueness. Let us give the simplest uniqueness result, and prove it by the method of doubling of variables. This argument appears in almost all uniqueness proofs, in more or less elaborate forms.

Proposition 2.9.7 *Assume that the Hamiltonian $H(x,p)$ is continuous in all its variables, and satisfies the coercivity assumption* (2.9.17). *Consider the equation*

$$H(x, \nabla u) + u = 0, \qquad x \in \mathbb{T}^n. \tag{2.9.19}$$

Let $\underline{u}$ and $\overline{u}$ be, respectively, a viscosity sub- and a super-solution to (2.9.1)*, then $\underline{u} \le \overline{u}$.*

Proof. Assume for a moment that both $\underline{u}$ and $\overline{u}$ are C^1-functions. If x_0 is a maximum of $\underline{u}-\overline{u}$ we have,

$$H(x_0, \nabla \underline{u}(x_0)) + \overline{u}(x_0) \ge 0, \tag{2.9.20}$$

as $\overline{u}$ is a super-solution, and $\underline{u}$ can be considered a test function, since it is differentiable. On the other hand, $\overline{u} - \underline{u}$ attains its minimum at the same point x_0, and, as $\underline{u}$ is a sub-solution, and $\overline{u}$ can serve as a test function, we have

$$H(x_0, \nabla \overline{u}(x_0)) + \underline{u}(x_0) \le 0. \tag{2.9.21}$$

As x_0 is a minimum of $\overline{u} - \underline{u}$, and $\underline{u}$ and $\overline{u}$ are differentiable, we have $\nabla \overline{u}(x_0) = \nabla \underline{u}(x_0)$, whence (2.9.20) and (2.9.21) imply

$$\underline{u}(x_0) \le \overline{u}(x_0).$$

Once again, as $\overline{u} - \underline{u}$ attains its minimum at x_0, we conclude that $\overline{u}(x) \ge \underline{u}(x)$ for all $x \in \mathbb{T}^n$ if both of these functions are in $C^1(\mathbb{T}^n)$.

Unfortunately, we only know that $\underline{u}$ and $\overline{u}$ are continuous, so we can not use the argument above unless we know, in addition, that they are both C^1-functions. In the general case, we use the method of doubling the variables. Let us define, for all $\varepsilon > 0$, the penalization

$$u_\varepsilon(x,y) = \overline{u}(x) - \underline{u}(y) + \frac{|x-y|^2}{2\varepsilon^2}$$

and let $(x_\varepsilon, y_\varepsilon)$ be a minimum for $u_\varepsilon(x,y)$.

Exercise 2.9.8 Show that

$$\lim_{\varepsilon \to 0} |x_\varepsilon - y_\varepsilon| = 0.$$

and that the family $(x_\varepsilon, y_\varepsilon)$ converges, as $\varepsilon \to 0$, up to a subsequence, to a point (x_0, x_0), where x_0 is a minimum to $\overline{u}(x) - \underline{u}(x)$.

Consider the function

$$\phi(x) = \underline{u}(y_\varepsilon) - \frac{|x-y_\varepsilon|^2}{2\varepsilon^2},$$

as a (smooth) function of the variable x. The difference

$$\phi(x) - \overline{u}(x) = -u_\varepsilon(x, y_\varepsilon)$$

attains its maximum, as a function of x, at the point $x = x_\varepsilon$. As $\overline{u}(x)$ is a super-solution, we have

$$H(x_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon^2}) + \overline{u}(x_\varepsilon) \geq 0. \tag{2.9.22}$$

Next, we apply the sub-solution part of Definition 2.9.13 to the test function

$$\psi(y) = \overline{u}(x_\varepsilon) + \frac{|x_\varepsilon - y|^2}{2\varepsilon^2}.$$

The difference

$$\psi(y) - \underline{u}(y) = \overline{u}(x_\varepsilon) + \frac{|x_\varepsilon - y|^2}{2\varepsilon^2} - \underline{u}(y) = u_\varepsilon(x_\varepsilon, y)$$

attains its minimum at $y = y_\varepsilon$, hence

$$H(y_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon^2}) + \underline{u}(y_\varepsilon) \leq 0; \tag{2.9.23}$$

The coercivity of the Hamiltonian and (2.9.23), together with the boundedness of $\underline{u}_\varepsilon$, imply that $|x_\varepsilon - y_\varepsilon|/\varepsilon^2$ is bounded, uniformly in ε. Hence, as $|x_\varepsilon - y_\varepsilon| \to 0$, it follows that

$$H(y_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon^2}) = H(x_\varepsilon, \frac{y_\varepsilon - x_\varepsilon}{\varepsilon^2}) + o(1), \text{ as } \varepsilon \to 0.$$

Subtracting (2.9.23) from (2.9.22), we obtain

$$\overline{u}(x_\varepsilon) - \underline{u}(y_\varepsilon) \geq o(1), \text{ as } \varepsilon \to 0.$$

Sending $\varepsilon \to 0$ implies

$$\overline{u}(x_0) - \underline{u}(x_0) \geq 0,$$

and, as x_0 is the minimum of $\overline{u} - \underline{u}$, the proof is complete. □

An immediate consequence is that (2.9.19) has at most one solution.

2.9.2 Steady solutions

We will now look for the wave solutions of (2.9.1) of the form

$$-ct + u(x),$$

with a constant $c \in \mathbb{R}$, as we did in the viscous case. Such function u solves

$$H(x, \nabla u) = c, \quad x \in \mathbb{T}^n. \tag{2.9.24}$$

Let us point out that (2.9.24) may have solutions for at most one c. Indeed, assume there exist $c_1 \neq c_2$, such that (2.9.24) has a solution u_1 for $c = c_1$ and another solution u_2 for $c = c_2$. Let $K > 0$ be such that

$$u_1(x) - K \leq u_2(x) \leq u_1(x) + K, \quad \text{for all } x \in \mathbb{T}^n.$$

The functions $-c_1t + u_1(x) \pm K$ and $-c_2t + u_2(x)$ solve the Cauchy problem (2.9.1) with the respective initial conditions $u_1(x) \pm K$ and $u_2(x)$. By the maximum principle (Exercise 2.9.6), we have

$$-c_1t + u_1(x) - K \leq -c_2t + u_2(x) \leq -c_1t + u_1(x) + K, \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{T}^n.$$

This is a contradiction since $c_1 \neq c_2$, and the functions u_1 and u_2 are bounded.

The main result of this section is the following theorem, due to Lions, Papanicolaou, Varadhan [101], that asserts the existence of a constant c for which (2.9.24) has a solution.

Theorem 2.9.9 *Assume that $H(x,p)$ is continuous, uniformly Lipschitz:*

$$|H(x,p_1) - H(x,p_2)| \leq C|p_1 - p_2|, \text{ for all } x \in \mathbb{T}^n, \text{ and } p_1, p_2 \in \mathbb{R}^n, \tag{2.9.25}$$

the coercivity condition (2.9.17) *holds, and*

$$|\nabla_x H(x,p)| \leq C(1 + |p|), \text{ for all } x \in \mathbb{T}^n, \text{ and } p \in \mathbb{R}^n. \tag{2.9.26}$$

There is a unique $c \in \mathbb{R}$ for which (2.9.24) *has a solution.*

It is important to point out that the periodicity assumption in x on the Hamiltonian is indispensable – for instance, when $H(x,p)$ is a random function (in x) on $\mathbb{R}^n \times \mathbb{R}^n$, the situation is totally different – an interested reader should consult the literature on stochastic homogenization of the Hamilton-Jacobi equations, a research area that is active and evolving at the moment of this writing.

The homogenization connection

Before proceeding with the proof of the Lions-Papanicolaou-Varadhan theorem, let us explain how the steady equation (2.9.24) appears in the context of periodic homogenization, which was probably the main motivation behind this theorem. We can not possibly do justice to the area of homogenization here – an interested reader should explore the huge literature on the subject, with the book [120] by G. Pavliotis and A. Stuart providing a good starting point. Let us just briefly illustrate the general setting on the example of the periodic Hamilton-Jacobi equations. Consider the Cauchy problem

$$u^\varepsilon_t + H(x, \nabla u^\varepsilon) = 0, \tag{2.9.27}$$

in the whole space $x \in \mathbb{R}^n$ (and not on the torus). We assume that the initial condition is slowly varying and large:

$$u^\varepsilon(0,x) = \varepsilon^{-1} u_0(\varepsilon x). \tag{2.9.28}$$

The general issue of homogenization is how the "microscopic" variations in the Hamiltonian that varies on the scale $O(1)$ affect the evolution of the initial condition that varies on the "macroscopic" scale $O(\varepsilon^{-1})$. The goal is to describe the evolution on purely "macroscopic" terms, and extract an effective macroscopic problem that approximates the full microscopic problem well. This allows to avoid, say, in numerical simulations, modeling the microscopic variations of the Hamiltonian, and do the simulations on the macroscopic scale – a huge advantage in the engineering problems. It also happens that from the purely mathematical view point, homogenization is also an extremely rich subject.

This general philosophy translates into the following strategy. As the initial condition in (2.9.28) is slowly varying, one should observe the solution on a macroscopic spatial scale, in the slow variable $y = \varepsilon x$. Since $u^\varepsilon(0, x)$ is also very large itself, of the size $O(\varepsilon^{-1})$, it is appropriate to rescale it down. In other words, instead of looking at $u^\varepsilon(t, x)$ directly, we would represent it as

$$u^\varepsilon(t, x) = \varepsilon^{-1} w^\varepsilon(t, \varepsilon x),$$

and consider the evolution of $w^\varepsilon(t, y)$, which satisfies

$$w^\varepsilon_t + \varepsilon H(\frac{y}{\varepsilon}, \nabla w^\varepsilon) = 0, \qquad (2.9.29)$$

with the initial condition $w^\varepsilon(0, y) = u_0(y)$ that is now independent of ε. However, we see that w^ε evolves very slowly in t – its time derivative is of the size $O(\varepsilon)$. Hence, we need to wait a long time until it changes. To remedy this, we introduce a long time scale of the size $t = O(\varepsilon^{-1})$. In other words, we write

$$w^\varepsilon(t, y) = v^\varepsilon(\varepsilon t, y).$$

In the new variables the problem takes the form

$$v^\varepsilon_s + H\Big(\frac{y}{\varepsilon}, \nabla v^\varepsilon\Big) = 0, \quad y \in \mathbb{R}^n, \quad s > 0, \qquad (2.9.30)$$

with the initial condition $v^\varepsilon(0, y) = u_0(y)$.

It seems that we have merely shifted the difficulty – we used to have ε in the initial condition in (2.9.28) while now we have it appear in the equation itself – the Hamiltonian depends on y/ε. However, it turns out that we may now find an ε-independent problem that has a spatially uniform Hamiltonian that provides a good approximation to (2.9.30). The reason this is possible is that we have chosen the correct temporal and spatial scales to track the evolution of the solution.

Here is how one finds the approximating problem. Let us seek the solution in the form of an asymptotic expansion

$$v^\varepsilon(s, y) = \bar{v}(s, y) + \varepsilon v_1(s, y, \frac{y}{\varepsilon}) + \varepsilon^2 v_2(s, y, \frac{y}{\varepsilon}) + \dots \qquad (2.9.31)$$

The functions $v_j(s, y, z)$ are assumed to be periodic in the "fast" variable z. Inserting this expansion into (2.9.30), we obtain in the leading order in ε:

$$\bar{v}_s(s, y) + H\Big(\frac{y}{\varepsilon}, \nabla_y \bar{v}(s, y) + \nabla_z v_1(s, y, \frac{y}{\varepsilon})\Big) = 0. \qquad (2.9.32)$$

As is standard in such multiple scale expansions, we consider (2.9.32) as

$$\bar{v}_s(s, y) + H(z, \nabla_y \bar{v}(s, y) + \nabla_z v_1(s, y, z)) = 0, \qquad (2.9.33)$$

an equation for v_1 as a function of the fast variable $z \in \mathbb{T}^n$, for each $s > 0$ and $y \in \mathbb{R}^n$ fixed. In other words, for each pair of the "macroscopic" variables s and y we consider a microscopic problem in the z-variable. In the area of numerical analysis, one would call this "sub-grid modeling": the variables t and x live on the macroscopic grid, and the z-variable lives on the microscopic sub-grid.

The function $\bar{v}(s,y)$ will then be found from the solvability condition for (2.9.32). Indeed, the terms $\bar{v}_s(s,y)$ and $\nabla_y \bar{v}(s,y)$ in (2.9.33) do not depend on the fast variable z and should be treated as constants – we solve (2.9.33) independently for each s and y. Let us then, for each fixed $p \in \mathbb{R}^n$, consider the problem

$$H(z, p + \nabla_z w) = \bar{H}(p), \tag{2.9.34}$$

posed on the torus $z \in \mathbb{T}^n$, for an unknown function $w(z)$. Here, $\bar{H}(p)$ is the unique constant (that depends on p), whose existence is guaranteed by the Lions-Papanicolaou-Varadhan theorem, for which the equation

$$H(z, p + \nabla_z w) = c, \tag{2.9.35}$$

has a solution. Then, the solvability condition for (2.9.33) is that the function $\bar{v}(s,y)$ satisfies the homogenized (or effective) equation

$$\bar{v}_s + \bar{H}(\nabla_y \bar{v}) = 0, \quad \bar{v}(0,y) = u_0(y), \quad s > 0,\ y \in \mathbb{R}^n, \tag{2.9.36}$$

and the function $\bar{H}(p)$ is called the effective, or homogenized Hamiltonian. Note that the effective Hamiltonian does not depend on the spatial variable – the "small scale" variations are averaged out via the above homogenization procedure. The point is that the solution $v^\varepsilon(s,y)$ of (2.9.30), an equation with highly oscillatory coefficients is well approximated by $\bar{v}(s,y)$, the solution of (2.9.36), an equation with uniform coefficients, that is much simpler to study analytically or solve numerically.

Thus, the existence and uniqueness of the constant c for which solution of the steady equation (2.9.35) exists, is directly related to the homogenization (long time behavior) of the solutions of the Cauchy problem (2.9.27) with slowly varying initial conditions as it provides the corresponding effective Hamiltonian. Unfortunately, there is a catch: very little is known in general on how the effective Hamiltonian $\bar{H}(p)$ depends on the original Hamiltonian $H(x,p)$, except for some very generic properties.

The proof of the Lions-Papanicolaou-Varadhan theorem

As we have already proved uniqueness of the constant c, it only remains to prove its existence. We will use the viscosity solution to the auxiliary problem

$$H(x, \nabla u^\varepsilon) + \varepsilon u^\varepsilon = 0, \quad x \in \mathbb{T}^n, \tag{2.9.37}$$

with $\varepsilon > 0$. We have already shown that this problem has at most one solution. Let us for the moment accept that the solution exists and show how one can finish the proof from here. Then, we will come back to the construction of a solution to (2.9.37). Our task is to pass to the limit $\varepsilon \downarrow 0$ in (2.9.37).

Exercise 2.9.10 Show that for all $\varepsilon > 0$, the solution $u^\varepsilon(x)$ of (2.9.37) satisfies

$$-\frac{\|H(\cdot,0)\|_{L^\infty}}{\varepsilon} \le u^\varepsilon(x) \le \frac{\|H(\cdot,0)\|_{L^\infty}}{\varepsilon}, \tag{2.9.38}$$

for all $x \in \mathbb{T}^n$.

Note that the fact that $u^\varepsilon(x)$ is of the size ε^{-1} is not a fluke of the estimate – we will see that $\varepsilon u^\varepsilon(x)$ converges as $\varepsilon \downarrow 0$ to a constant limit c that will be the speed. In order to pass to the limit $\varepsilon \downarrow 0$ in (2.9.37), we need a modulus of continuity estimate that does not depend on $\varepsilon \in (0,1)$.

Lemma 2.9.11 *There is $C > 0$ independent of ε such that $|\mathrm{Lip}\, u^\varepsilon| \le C$.*

Proof. Again, we use the doubling of the independent variables. Fix $x \in \mathbb{T}^n$ and, for $K > 0$, consider the function

$$\zeta(y) = u^\varepsilon(y) - u^\varepsilon(x) - K|y - x|.$$

Let $\hat{x}$ be a maximum of $\zeta(y)$ (the point $\hat{x}$ depends on x). If $\hat{x} = x$ for all $x \in \mathbb{T}^n$, we have

$$u^\varepsilon(y) - u^\varepsilon(x) \le K|x - y|,$$

for all $x, y \in \mathbb{T}^n$, which implies that u^ε is Lipschitz. If there exists some x such that $\hat{x} \ne x$, then the function

$$\psi(y) = u^\varepsilon(x) + K|y - x|$$

is, in a vicinity of the point $y = \hat{x}$, an admissible test function, as a function of y. Moreover, the difference

$$\psi(y) - u^\varepsilon(y) = -\zeta(y)$$

attains its minimum at $y = \hat{x}$. The sub-solution condition (2.9.13) at this point gives:

$$H\Big(\hat{x}, K\frac{\hat{x} - x}{|\hat{x} - x|}\Big) + \varepsilon u^\varepsilon(\hat{x}) \le 0.$$

As $\varepsilon u^\varepsilon(x)$ is bounded by $\|H(\cdot, 0)\|_{L^\infty}$, the coercivity condition (2.9.17) implies the existence of a constant $C > 0$ independent of ε such that $K \le C$. Therefore, if we take $K = 2C$, we must have $\hat{x} = x$ for all $x \in \mathbb{T}^n$, which implies

$$u(y) - u(x) - 2C|y - x| \le 0.$$

The points x and y being arbitrary, this finishes the proof. □

In order to finish the proof of Theorem 2.9.9, denote by $\langle u^\varepsilon \rangle$ the mean of u^ε over $\mathbb{T}^n$, and set

$$v^\varepsilon = u^\varepsilon - \langle u^\varepsilon \rangle.$$

This function satisfies

$$H(x, \nabla v^\varepsilon) + \varepsilon \langle u^\varepsilon \rangle + \varepsilon v^\varepsilon = 0.$$

Because of Lemma 2.9.11, the family v^ε converges uniformly, up to a subsequence, to a function $v \in C(\mathbb{T}^n)$, and $\varepsilon v^\varepsilon \to 0$. The bound (2.9.38) implies that the family $\varepsilon \langle u^\varepsilon \rangle$ is bounded. We may, therefore, assume its convergence (along a subsequence) to a constant denoted by $-c$. By the stability result in Exercise 2.9.4, v is a viscosity solution of

$$H(x, \nabla v) = c. \tag{2.9.39}$$

This finishes the proof of Theorem 2.9.9 except for the construction of a solution to (2.9.37).

Existence of the solution to the auxiliary problem

Let us now construct a solution to (2.9.37):

$$H(x,\nabla u)+\varepsilon u=0. \tag{2.9.40}$$

We will do this in the most pedestrian way possible. We take a function $f\in C(\mathbb{T}^n)$, and consider an approximation problem

$$-\delta\Delta u^{\gamma,\delta}+H(x,\nabla u^{\gamma,\delta})+\varepsilon u^{\gamma,\delta}=f_\gamma(x),\quad x\in\mathbb{T}^n, \tag{2.9.41}$$

with $\varepsilon>0$ and $\gamma>0$, and

$$f_\gamma=G_\gamma\star f. \tag{2.9.42}$$

Here, G_γ is a smooth approximation of a δ-function:

$$G_\gamma(x)=\gamma^{-n}G(x),\;\; G(x)\geq 0,\;\; \int_{\mathbb{R}^n}G(x)dx=1,$$

so that $f_\gamma(x)$ is smooth, and $f_\gamma\to f$ in $C(\mathbb{T}^n)$. It is straightforward to adapt what we have done in Section 2.4.3 for the time-dependent problems to show that (2.9.41) admits a solution $u^{\gamma,\delta}$ for each $\gamma>0$ and $\delta>0$. The difficulty is to pass to the limit $\delta\downarrow 0$, followed by $\gamma\downarrow 0$ to construct in the limit a viscosity solution to (2.9.40).

We claim that there exists $M>0$ so that if $\varepsilon>M$ then $u^{\gamma,\delta}$ obeys a gradient bound

$$|\nabla u^{\gamma,\delta}(x)|\leq C_\gamma \text{ for all } x\in\mathbb{T}^n. \tag{2.9.43}$$

To see that, let us look at the point x_0 where $|\nabla u^{\gamma,\delta}(x)|^2$ attains its maximum. Note that (we drop the super-scripts γ and δ for the moment)

$$\frac{\partial}{\partial x_i}(|\nabla u|^2)=2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

so that

$$\begin{aligned}\Delta(|\nabla u|^2)&=2\sum_{i,j=1}^n\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2+2\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial\Delta u}{\partial x_j}=2\sum_{i,j=1}^n\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2+\frac{2\varepsilon}{\delta}|\nabla u|^2\\&+\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial H(x,\nabla u)}{\partial x_j}+\frac{2}{\delta}\sum_{k,j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial H(x,\nabla u)}{\partial p_k}\frac{\partial^2 u}{\partial x_j\partial x_k}-\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial f_\gamma}{\partial x_j}\\&=2\sum_{i,j=1}^n\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2+\frac{2\varepsilon}{\delta}|\nabla u|^2+\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial H(x,\nabla u)}{\partial x_j}+\frac{\varepsilon}{\delta}\sum_{k=1}^n\frac{\partial H(x,\nabla u)}{\partial p_k}\frac{\partial|\nabla u|^2}{\partial x_k}\\&-\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial f_\gamma}{\partial x_j}.\end{aligned}$$

Thus, at x_0 we have

$$0\geq\Delta(|\nabla u|^2)(x_0)=2\sum_{i,j=1}^n\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2+\frac{2\varepsilon}{\delta}|\nabla u|^2+\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial H(x,\nabla u)}{\partial x_j}-\frac{2}{\delta}\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial f_\gamma}{\partial x_j}. \tag{2.9.44}$$

Let us recall the gradinet bound (2.9.26) on $H(x,p)$:

$$|\nabla_x H(x,p)| \le C(1+|p|). \tag{2.9.45}$$

Exercise 2.9.12 Use (2.9.44) and (2.9.45) to show that there exists $M>0$, independent of $\gamma>0$, so that if $\varepsilon > M$, then $u^{\gamma,\delta}$ obeys an a priori bound (2.9.43) with a constant $C_\gamma>0$ that may depend on γ but not on δ or ε. Show also that there exists a constant $C'_\varepsilon>0$ that depends on $\varepsilon>0$ but not on $\gamma>0$ such that

$$|u^{\gamma,\delta}(x)| \le C'_\varepsilon \text{ for all } x\in\mathbb{T}^n. \tag{2.9.46}$$

The Lipschitz bound (2.9.43) and (2.9.46) show that, after passing to a subsequence $\delta_k\to 0$, the family $u^{\gamma,\delta_k}(x)$ converges uniformly in $x\in\mathbb{T}^n$, to a function $u^\gamma(x)$.

Exercise 2.9.13 Show that $u^\gamma(x)$ is the viscosity solution to

$$H(x,\nabla u^\gamma)+\varepsilon u^\gamma = f_\gamma(x), \quad x\in\mathbb{T}^n. \tag{2.9.47}$$

Hint: Exercise 2.9.4 and its solution should be helpful here.

So far, we have constructed a solution to the γ-regularized problem (2.9.47) for $\varepsilon>0$ sufficiently large – this seems very far from what we want since we plan to send ε to zero eventually but the end is not that far. The next step is to send $\gamma\to 0$.

Exercise 2.9.14 Mimic the proof of Lemma 2.9.11 to show that $u^\gamma(x)$ are uniformly Lipschitz: there exists a constant $C>0$ independent of $\gamma\in(0,1)$ such that $|\text{Lip } u^\gamma|\le C$. Note that you can not use the derivatives of f_γ in x as these may blow up as $\gamma\downarrow 0$ – we only know that $f\in C(\mathbb{T}^n)$.

This exercise shows that u^{γ_k} converges, along as subsequence $\gamma_k\downarrow 0$, uniformly in $x\in\mathbb{T}^n$, to a limit $u(x)\in C(\mathbb{T}^n)$ that obeys the same uniform Lipschitz bound in Exercise 2.9.14. Invoking again the stability result of Exercise 2.9.4 shows that $u(x)$ is the unique viscosity solution to

$$H(x,\nabla u)+\varepsilon u = f(x), \quad x\in\mathbb{T}^n. \tag{2.9.48}$$

The final task is to remove the restriction $\varepsilon\ge M$. Let us take $\varepsilon<M$, and re-write (2.9.48) as

$$H(x,\nabla u)+Mu = (M-\varepsilon)u, \quad x\in\mathbb{T}^n. \tag{2.9.49}$$

Consider the following map $\mathcal{S}$: given a function $v(x)\in C(\mathbb{T}^n)$, let $u=\mathcal{S}v$ be the solution of

$$H(x,\nabla u)+Mu = (M-\varepsilon)v, \quad x\in\mathbb{T}^n. \tag{2.9.50}$$

We claim that $\mathcal{S}$ is a contraction in $C(\mathbb{T}^n)$. Indeed, given $v_1,v_2\in C(\mathbb{T}^n)$, let us go back to the corresponding δ,γ-problems:

$$-\delta\Delta u_1^{\gamma,\delta}+H(x,\nabla u_1^{\gamma,\delta})+Mu_1^{\gamma,\delta} = (M-\varepsilon)v_{1,\gamma}, \quad x\in\mathbb{T}^n, \tag{2.9.51}$$

and

$$-\delta\Delta u_2^{\gamma,\delta}+H(x,\nabla u_2^{\gamma,\delta})+Mu_2^{\gamma,\delta} = (M-\varepsilon)v_{2,\gamma}, \quad x\in\mathbb{T}^n. \tag{2.9.52}$$

Assume that the difference

$$w = u_1^{\gamma,\delta} - u_2^{\gamma,\delta}$$

attains its maximum at a point x_0. The function w satisfies

$$-\delta\Delta w + H(x, \nabla u_1^{\gamma,\delta}) - H(x, \nabla u_2^{\gamma,\delta}) + Mw = (M - \varepsilon)(v_{1,\gamma} - v_{2,\gamma}), \quad x \in \mathbb{T}^n. \tag{2.9.53}$$

Evaluating this at $x = x_0$, we see that

$$-\delta\Delta w(x_0) + Mw(x_0) = (M - \varepsilon)(v_{1,\gamma}(x_0) - v_{2,\gamma}(x_0)), \quad x \in \mathbb{T}^n, \tag{2.9.54}$$

hence

$$w(x_0) \le \frac{M - \varepsilon}{M} \|v_{1,\gamma} - v_{2,\gamma}\|_{C(\mathbb{T}^n)}.$$

Using an identical computation for the minimum, we conclude that

$$\|u_1^{\gamma,\delta} - u_2^{\gamma,\delta}\|_{C(\mathbb{T}^n)} \le \frac{M - \varepsilon}{M} \|v_{1,\gamma} - v_{2,\gamma}\|_{C(\mathbb{T}^n)}. \tag{2.9.55}$$

Passing to the limit $\delta \downarrow 0$ and $\gamma \downarrow 0$, we obtain

$$\|u_1 - u_2\|_{C(\mathbb{T}^n)} \le \frac{M - \varepsilon}{M} \|v_1 - v_2\|_{C(\mathbb{T}^n)}, \tag{2.9.56}$$

hence $\mathcal{S}$ is a contraction on $C(\mathbb{T}^n)$. Thus, this map has a fixed point, which is the viscosity solution of

$$H(x, \nabla u) + \varepsilon u = 0, \quad x \in \mathbb{T}^n. \tag{2.9.57}$$

This completes the proof.

Non-uniqueness of the steady solutions

Once the correct c has been identified, one may wonder about the uniqueness of the solution for equation (2.9.24). Clearly, if u is a solution, $u + q$ is also a solution for every constant q. However, uniqueness modulo constants is also not true. Consider a very simple example

$$|u'| = f(x), \quad x \in \mathbb{T}^1. \tag{2.9.58}$$

Assume that $f \in C^1(\mathbb{T}^1)$ is 1/2-periodic, satisfies

$$f(x) > 0 \text{ on } (0, 1/2) \cup (1/2, 1), \text{ and } f(0) = f(1/2) = f(1) = 0.$$

and is symmetric with respect to $x = 1/4$ (and thus $x = 3/4$). Let u_1 and u_2 be 1-periodic and be defined, over a period, as follows:

$$u_1(x) = \begin{cases} \displaystyle\int_0^x f(y)\,dy & 0 \le x \le \frac{1}{2} \\ \displaystyle\int_x^1 f(y)\,dy & \frac{1}{2} \le x \le 1 \end{cases} \qquad u_2(x) = \begin{cases} \displaystyle\int_0^x f(y)\,dy & 0 \le x \le \frac{1}{4} \\ \displaystyle\int_x^{1/2} f(y)\,dy & \frac{1}{4} \le x \le \frac{1}{2} \\ u_2 \text{ is } \frac{1}{2}\text{-periodic.} \end{cases}$$

Exercise 2.9.15 Verify that both u_1 and u_2 are viscosity solutions of (2.9.58), and u_2 cannot be obtained from u_1 by the addition a constant. Pay attention to what happens at $x = 1/4$ and $x = 3/4$ with $u_2(x)$. Why can't you construct a solution that would have a corner at a minimum rather than the maximum?

A very remarkable study of the non-uniqueness may be found in Lions [100] for a multi-dimensional generalization of (2.9.58), that is,

$$|\nabla u| = f(x), \quad x \in \Omega$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and f is nonnegtive and vanishes only at a finite number of points. The zero set of f is shown to play an important role: essentially, imposing u at those points ensures uniqueness – but not always existence.

Chapter 3

The two dimensional Euler equations

3.1 The derivation of the Euler equations

In this chapter, we will study some of the very basic questions concerning the behavior of the solutions to the two-dimensional incompressible Euler equations of the fluid mechanics. These equations describe the flow of an incompressible, inviscid fluid, and were first derived by Leonhard Euler in 1755 [59] – and appear to be the second PDE ever written! The first one was the wave equation, published by D'Alambert just eight years earlier [49].

The incompressibility constraint

Let us first explain how the Euler equations are derived, either in two or three dimensions. Each fluid particle is following a trajectory governed by the fluid velocity $u(t,x)$:

$$\frac{d\Phi_t(x)}{dt} = u(t,\Phi_t(x)), \quad \Phi_t(x) = x. \tag{3.1.1}$$

Here, x is the starting position of the particle at the time $t = 0$, and is sometimes called "the label", and the inverse map $A_t : \Phi_t(x) \to x$ is called the "back-to-the-labels" map. If the flow $u(t,x)$ is sufficiently smooth, the forward map $x \to \Phi_t(x)$ should preserve the total mass, as no fluid particles are created or destroyed. In addition, we will assume that the fluid density ρ is a constant – physically, this means that the fluid is incompressible. Then, mass preservation is equivalent to the conservation of the volume. That is, if $V_0 \subset \mathbb{R}^d$ ($d = 2,3$) is an initial volume, then the set

$$V(t) = \{\Phi_t(x) : \ x \in V_0\},$$

which is the image of V_0 under the evolution by the flow, should have the same volume as V_0. In order to quantify this property, let us define the Jacobian of the map $x \to \Phi_t(x)$:

$$J(t,x) = \det\left(\frac{\partial \Phi_t^i(x)}{\partial x_j}\right).$$

Volume preservation is equivalent to the condition

$$J(t,x) \equiv 1. \tag{3.1.2}$$

As $\Phi_0(x) = x$, we have $J(0,x) \equiv 1$, hence (3.1.2) can be restated as

$$\frac{dJ}{dt} \equiv 0.$$

The full matrix of the derivatives

$$H_{ij}(t,x) = \frac{\partial \Phi_t^i(x)}{\partial x_j}$$

obeys the evolution equation

$$\frac{dH_{ij}}{dt} = \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k}\frac{\partial \Phi_t^k}{\partial x_j}, \tag{3.1.3}$$

obtained by differentiating (3.1.1) with respect to the labels x_j. That is, we have, in the matrix form

$$\frac{dH}{dt} = (\nabla u)H, \tag{3.1.4}$$

with

$$(\nabla u)_{ik} = \frac{\partial u_i}{\partial x_k}.$$

In order to find dJ/dt from (3.1.3), we consider the evolution of a general $n \times n$ matrix $A_{ij}(t)$ and decompose, for each $i = 1,\ldots,n$ fixed:

$$\text{det}A = \sum_{j=1}^{n} (-1)^{i+j} M_{ij} A_{ij}.$$

Here, M_{ij} are the minors of the matrix A. Note that, for all $1 \le j' \le n$, the minors $M_{ij'}$ do not depend on the matrix element A_{ij}, hence

$$\frac{\partial}{\partial A_{ij}}(\det A) = (-1)^{i+j} M_{ij}.$$

We conclude that

$$\frac{d}{dt}(\det A) = \sum_{i,j=1}^{n} \frac{\partial}{\partial A_{ij}}(\det A)\frac{dA_{ij}}{dt} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij}\frac{dA_{ij}}{dt}.$$

Recall also that the inverse matrix A^{-1} has the elements

$$(A^{-1})_{ij} = \frac{1}{\det A}(-1)^{i+j} M_{ji},$$

meaning that

$$\sum_{j=1}^{n} (-1)^{j+i} M_{ij} A_{kj} = (\det A)\delta_{ik}.$$

We apply now this consideration to the matrix $H_{ij}(t,x)$ and obtain

$$\frac{dJ}{dt} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij} \frac{d}{dt}\Big(\frac{\partial \Phi_t^i(t,x)}{\partial x_j}\Big),$$

and

$$J\delta_{ik} = \sum_{j=1}^{n} (-1)^{j+i} M_{ij} \frac{\partial \Phi_t^k}{\partial x_j}. \tag{3.1.5}$$

Here, M_{ij} are the minors of the matrix H_{ij}. As

$$\frac{d}{dt}\Big(\frac{\partial \Phi_t^i(t,x)}{\partial x_j}\Big) = \frac{\partial}{\partial x_j}(u_i(t,\Phi_t(x))) = \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k}\frac{\partial \Phi_t^k}{\partial x_j},$$

we get

$$\frac{dJ}{dt} = \sum_{i,j,k=1}^{n} (-1)^{i+j} M_{ij} \frac{\partial u_i}{\partial x_k}\frac{\partial \Phi_t^k}{\partial x_j} = \sum_{i,k=1}^{n} \frac{\partial u_i}{\partial x_k} J\delta_{ik} = J(\nabla \cdot u). \tag{3.1.6}$$

Thus, preservation of the volume is equivalent to the incompressibility condition:

$$\nabla \cdot u = 0. \tag{3.1.7}$$

Exercise 3.1.1 More generally, if the density is not constant, mass conservation would require that for any initial volume V_0 the fluid density $\rho(t,x)$ would satisfy

$$\frac{d}{dt}\int_{V(t)} \rho(t,x)dx = 0. \tag{3.1.8}$$

Use this condition to obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{3.1.9}$$

which reduces to (3.1.7) when the density $\rho(t,x)$ is constant in time and space.

Newton's second law in an inviscid fluid

The incompressibility condition (3.1.7) should be supplemented by an evolution equation for the fluid velocity $u(t,x)$. This will come from Newton's second law of motion, which we will, once again, derive under the assumption that the fluid density is constant. Consider a fluid volume V. If the fluid is inviscid, so that there is no "internal friction" in the fluid, the only force acting on this volume is due to the fluid pressure:

$$F = -\int_{\partial V} p\nu dS = -\int_{V} \nabla p dx, \tag{3.1.10}$$

where ∂V is the boundary of V, and ν is the outside normal to ∂V. Taking V to be an infinitesimal volume at a point $X(t)$, which moves with the fluid, Newton's second law of motion leads to the balance

$$\rho \ddot{X}(t) = -\nabla p(t, X(t)). \tag{3.1.11}$$

We may compute $\ddot{X}(t)$ starting from the trajectory equation (3.1.1):

$$\begin{aligned} \ddot{X}_j(t) &= \frac{d}{dt}(u_j(t,X(t)) = \frac{\partial u_j(t,X(t))}{\partial t} + \sum_k \dot{X}_k(t)\frac{\partial u_j(t,X(t))}{\partial x_k} \\ &= \frac{\partial u_j(t,X(t))}{\partial t} + u(t,X(t)) \cdot \nabla u_j(t,X(t)). \end{aligned} \tag{3.1.12}$$

Therefore, we have the following equation of motion:

$$\rho\Big(\frac{\partial u}{\partial t} + u \cdot \nabla u\Big) + \nabla p = 0. \tag{3.1.13}$$

The Euler equations

Equations (3.1.7) and (3.1.13) together form the system of the Euler equations for an incompressible inviscid fluid:

$$\begin{aligned} &u_t + (u \cdot \nabla)u + \nabla p = 0, \\ &\nabla \cdot u = 0. \end{aligned} \tag{3.1.14}$$

We have set $\rho = 1$ for convenience. The system (3.1.14) should be supplemented by the initial condition $u(0, x) = u_0(x)$. Moreover, if it is posed in a domain D, we also need to impose a boundary condition on the flow $u(t, x)$. If the boundary is impenetrable, then the natural boundary condition is

$$u \cdot \nu|_{\partial D} = 0, \tag{3.1.15}$$

Here, ν is the normal at the boundary ∂D. The Euler equations are also often considered in the whole space $\mathbb{R}^d$, with the decay conditions at infinity, or on a torus – which is equivalent to taking periodic initial data in $\mathbb{R}^d$. There are many great textbooks outlining the basic properties of the Euler equations – see, for example, [36], [102], [56] and [105]. Throughout this section, we will consider the 2D Euler equations in a smooth bounded domain D, or on a torus $\mathbb{T}^2$.

The Euler equations are some of the most fundamental and widely used partial differential equations. They are nonlinear and nonlocal, the latter property a consequence of the nonlocal dependence of the pressure on the fluid velocity. On the physical level, this reflects that pushing the fluid in one region produces an instantaneous pressure in a different region, because of the fluid incompressibility. Mathematically, taking the divergence of (3.1.14) and using the incompressibility of u, we obtain the Poisson equation for the pressure:

$$-\Delta p = \nabla \cdot (u \cdot \nabla u).$$

Since the well-known formulas for inversion of the Laplacian are non local, this shows the non locality of the pressure-velocity relation. This will be even more clear from the nonlocal Biot-Savart law for the vorticity formulation of the equation presented below. This explains, from the mathematical point of view, why the analysis of the Euler equations is challenging. From the intuitive point of view, anyone who observed the flow of a river, or the intricate structures of the fluid motion in a rising smoke, or a tornado, can understand that only a very rich and complex system of equations has a chance of modeling these exquisite phenomena. The solutions of the Euler equations are often very unstable, and prone to creation of small scale structures. Due to the central role of these equations in mathematical physics, a lot of studies have focused on these problems over the 250 years that have passed since their discovery. We have no hope of covering much of this research here, so after a brief overview we will focus on a few specific questions, including some very recent developments.

The vorticity formulation of the Euler equations

The theory of the existence, uniqueness and regularity of the solutions to the Euler equations is quite different in two and three spatial dimensions. In the two dimensional case, there exists a unique global in time smooth solution for smooth initial data, while for the three dimensional case an analogous result is only known locally in time. The question of the global existence of smooth solutions to the Euler equations in three dimensions is a major open problem. This difference can be illustrated on a basic level by rewriting the Euler equations in a different form. An important quantity in the fluid mechanics is the vorticity $\omega = \nabla\times u$, which describes the rotational motion of the fluid. In three dimensions, if we apply the curl operator to the system (3.1.14), we obtain the Euler equation in the vorticity form:

$$\omega_t + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \tag{3.1.16}$$

with the initial condition $\omega(0,x) = \omega_0(x)$.

Exercise 3.1.2 Use vector algebra to derive the vorticity equation (3.1.16) in three dimensions.

The vector field u can be recovered from ω via the Biot-Savart law. In order to obtain this law in $\mathbb{R}^3$, consider the (vector-valued) stream function ψ defined (in terms of the vorticity) as the solution of the Poisson equation

$$-\Delta\psi = \omega, \quad \text{in } \mathbb{R}^3. \tag{3.1.17}$$

Then, one can show via vector algebra that u is given by (see, for example, [56, 102])

$$u = \nabla\times\psi. \tag{3.1.18}$$

That is, if u and ω are related via (3.1.17) and (3.1.18), then $\omega = \nabla\times u$. Together, (3.1.17) and (3.1.18) form the Biot-Savart law which expresses the velocity u via the vorticity ω.

Exercise 3.1.3 Verify that u given by (3.1.17)-(3.1.18) satisfies $\nabla\times u = \omega$. You have to use the divergence free property of u and some vector identities (or brute force computations).

On the other hand, in the two dimensional case the term in the right side of (3.1.16) vanishes. Indeed, the solutions of the two-dimensional Euler equations can be thought of as solutions of the three-dimensional equations of the special form $(u_1(x_1,x_2), u_2(x_1,x_2), 0)$. In that case, the vorticity vector has only one non-zero component:

$$\omega = (0,0,\partial_1 u_2 - \partial_2 u_1),$$

and can be regarded as a scalar. Then, the term in the right side of (3.1.16) is simply

$$(\omega\cdot\nabla)u = \omega_3\partial_3 u,$$

but the two dimensional u does not depend on x_3. Thus, in two dimensions, the vorticity equation simplifies. We will use the notation

$$\omega = \partial_1 u_2 - \partial_2 u_1, \tag{3.1.19}$$

instead of ω_3. Given a smooth bounded domain D, let us also define the operator $(-\Delta_D)^{-1}$ as follows: given a function ψ, we denote by $\eta = (-\Delta_D)^{-1}\psi$ the unique solution of the boundary value problem

$$\begin{aligned} -\Delta\eta = \psi, \quad & \text{in } \Omega, \\ \psi = 0, \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.1.20}$$

The vorticity formulation of the two-dimensional Euler equations is the following system:

$$\partial_t\omega + (u\cdot\nabla)\omega = 0, \tag{3.1.21}$$
$$u = \nabla^\perp(-\Delta_D)^{-1}\omega, \tag{3.1.22}$$
$$\omega(0,x) = \omega_0(x),$$

where $\nabla^\perp = (\partial_2, -\partial_1)$. Note that the flow u defined by (3.1.22) automatically satisfies the boundary condition

$$u\cdot\nu = 0 \text{ on } \partial D. \tag{3.1.23}$$

This is because the gradient of the stream function

$$\psi = (-\Delta_D)^{-1}\omega, \quad u = \nabla^\perp\psi,$$

is normal to ∂D at the boundary.

Exercise 3.1.4 Verify that if $u(t,x)$ satisfies the Euler equations in two dimensions, then the vorticity $\omega(t,x)$ given by (3.1.19) satisfies (3.1.21), and $u(t,x)$ and $\omega(t,x)$ are related via (3.1.22).

The vorticity formulation of the Euler equations in two dimensions has significant consequences. As we will see, any L^p norm of the vorticity is conserved for smooth solutions of (3.1.21). In particular, $\|\omega\|_{L^\infty}$ does not change. In contrast, in three dimensions, the amplitude of vorticity can and often does grow due to the non-zero term in the right side of (3.1.16). This term is often called the vortex stretching term in the literature.

Our focus in the present chapter will be on the basic questions of existence, uniqueness, and regularity properties of the solutions to the two dimensional Euler equations. First, we will present the existence and uniqueness theory of solutions due to Yudovich [142] which works for a very natural class of initial data. We will then study the small scale formation in the smooth solutions of the 2D Euler equations, proving an upper bound for the growth of the derivatives of the solution as well as constructing examples that show that in general this upper bound is sharp. The set of techniques we will need in this chapter is a rich mix of the Fourier analysis, ODE methods, comparison principles, and all sorts of other PDE estimates.

3.2 The Yudovich theory

The Yudovich theory addresses the existence and uniqueness of the solutions to the 2D Euler equations with a bounded initial vorticity. The L^∞ class for the vorticity is very natural since it is preserved by the evolution. In addition, many phenomena in nature, such as hurricanes or tornados, feature vorticities with a very sharp variation, hence the theory of solutions with rough vorticities is not a purely mathematical issue. As we will see, if the initial condition is more regular, this regularity is reflected in the additional regularity of the solution, even though the quantitative estimates can deteriorate very quickly. Our exposition in this section roughly follows [105].

It is not immediately clear how one can define the low regularity solutions (such as L^∞) of the vorticity equation (3.1.21) since we need to take derivatives. A "canonical" way around that is to define a weak solution of a nonlinear equation via the multiplication of the equation by a test function and integration by parts, and then to try to obtain some a priori bounds and use compactness arguments to show that such weak solution exists. However, there is a more elegant (and efficient) approach for the two-dimensional Euler equations, via a reformulation of the problem that allows us to define a weak solution in a different way. Given a divergence-free flow $u(t,x)$, recall our definition of the particle trajectories $\Phi_t(x)$:

$$\frac{d\Phi_t(x)}{dt} = u(t, \Phi_t(x)), \quad \Phi_0(x) = x. \tag{3.2.1}$$

As we have seen, if u is sufficiently regular and incompressible, (3.2.1) defines a volume preserving map $x \to \Phi_t(x)$ for each t.

A direct calculation, using the method of characteristics. shows that if $\omega(t,x)$ is a smooth solution of (3.1.21), then

$$\omega(t, \Phi_t(x)) = \omega_0(x), \quad \text{thus } \omega(t,x) = \omega_0(\Phi_t^{-1}(x)). \tag{3.2.2}$$

In addition, if we denote, as before, by $G_D(x,y)$ the Green's function for the Dirichlet Laplacian in a domain D, in the sense that the solution of (3.1.20) is given by

$$\eta(x) = \int_D G_D(x,y)\psi(y)dy, \quad x \in D, \tag{3.2.3}$$

and set

$$K_D(x,y) = \nabla_x^\perp G_D(x,y), \tag{3.2.4}$$

then the Biot-Savart law in two dimensions can be written as

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)\, dy. \tag{3.2.5}$$

A classical C^1 solution of the two-dimensional Euler equations (3.1.21) satisfies the system (3.2.1), (3.2.2) and (3.2.5). On the other hand, a direct computation shows that a smooth solution of (3.2.1), (3.2.2) and (3.2.5) gives rise to a classical solution of (3.1.21). Thus, for smooth solutions the two problems are equivalent. We will generalize the notion of the solution to the 2D Euler equations by saying that a triple $(\omega, u, \Phi_t(x))$ solves the 2D Euler equations if it satisfies (3.2.1), (3.2.2) and (3.2.5). The obvious next task is to make sense of the solutions of the latter system with the only requirement that $\omega_0 \in L^\infty$. Classically, for the trajectories of (3.2.1) to be well-defined, the flow $u(t,x)$ needs to be Lipschitz in x. Thus, if it were true that if $\omega(t,x)$ is in L^∞, the Biot-Savart law would give a Lipschitz function $u(t,x)$, then it would be very reasonable to expect (3.2.1), (3.2.2) and (3.2.5) to be a well-posed system. This is not totally unreasonable – (3.1.22) indicates that u is "one derivative better than ω", but is not quite true – the regularity for $u(t,x)$ when $\omega \in L^\infty$ is slightly lower than Lipschitz. Nevertheless, we will see that this lower regularity is sufficient to define the unique trajectories of the ODE (3.2.1), making the system well-posed.

3.2.1 The regularity of the flow

In order to construct the solutions of the 2D Euler equations in the trajectory formulation (3.2.1)-(3.2.5) with the vorticity $\omega_0 \in L^\infty$, we first need to establish the regularity of the fluid flow given by (3.2.5) for a vorticity in L^∞. This question is clearly related to the regularity of the kernel $K_D(x, y)$. The following proposition summarizes some well known properties of the Dirichlet Green's function (see, for instance, [60, 80]).

Proposition 3.2.1 *If $D \subset \mathbb{R}^2$ is a domain with a smooth boundary, the Dirichlet Green's function $G_D(x, y)$ has the form*

$$G_D(x, y) = \frac{1}{2\pi} \log |x - y| + h(x, y).$$

Here, for each $y \in D$, $h(x, y)$ is a harmonic function solving

$$\Delta_x h = 0, \quad h|_{x \in \partial D} = -\frac{1}{2\pi} \log |x - y|. \tag{3.2.6}$$

We have $G_D(x, y) = G_D(y, x)$ for all $(x, y) \in D$, and $G_D(x, y) = 0$ if either x or y belongs to ∂D. In addition, we have the estimates

$$|G_D(x, y)| \leq C(D)(\log |x - y| + 1) \tag{3.2.7}$$
$$|\nabla G_D(x, y)| \leq C(D)|x - y|^{-1}, \tag{3.2.8}$$
$$|\nabla^2 G_D(x, y)| \leq C(D)|x - y|^{-2}. \tag{3.2.9}$$

The function G_D can be sometimes computed explicitly in a closed form (for example for a plane, a half-plane, a disk, a corner, see e.g. [60]), or as an infinite series (for example for a square or a rectangle, or a torus). For most domains only estimates as in the above proposition are available.

Exercise 3.2.2 Use the explicit form of the Green's function in a disk to show that the estimates in Proposition 3.2.1 are sharp.

The following lemma outlines a key regularity property of the Green's function which allows to construct unique solutions of the Euler equations for the initial vorticity in L^∞.

Lemma 3.2.3 *The kernel $K_D(x, y) = \nabla^\perp G_D(x, y)$ satisfies*

$$\int_D |K_D(x, y) - K_D(x', y)| \, dy \leq C(D)\phi(|x - x'|), \tag{3.2.10}$$

where

$$\phi(r) = \begin{cases} r(1 - \log r) & r < 1 \\ 1 & r \geq 1, \end{cases} \tag{3.2.11}$$

with a constant $C(D)$ which depends only o the domain D.

Proof. In what follows, $C(D)$ denotes constants that may depend only on the domain D, and may change from line to line. To show (3.2.10), we may assume that $r = |x - x'| < 1$, otherwise (3.2.10) follows from the simple observation that

$$|K_D(x,y)| \le C(D)|x-y|^{-1},$$

so that

$$\int_D |K_D(x,y)|dy \le C(D),$$

which implies (3.2.10) for $x, x' \in D$ such that $|x - x'| \ge 1$.

Assume now that $r < 1$ and suppose first that the interval connecting the points x and x' lies entirely inside D. Let us set

$$A = \{y \in D : \ |y - x| \le 2r\}.$$

The estimate (3.2.8) implies

$$\int_{D\cap A} |K_D(x,y) - K_D(x',y)|\, dy \le C(D) \int_{B_{2r}(x)} \frac{1}{|x-y|}\, dy \le C(D)r.$$

To bound the remainder of the integral, observe that for every y,

$$|K_D(x,y) - K_D(x',y)| \le r|\nabla K_D(x''(y),y)|, \tag{3.2.12}$$

where $x''(y)$ lies on the interval connecting x and x'. This follows from the mean value theorem and the assumption that the interval connecting x and x' lies in D. Then, by (3.2.9) and the choice of the set A, so that the distances $|x - y|$, $|x' - y|$ and $|x'' - y|$ are all comparable if $y \in A^c$, we have

$$\begin{aligned}\int_{D\cap A^c} |K_D(x,y) - K_D(x',y)|\, dy &\le C(D)r \int_{D\cap A^c} \frac{dy}{|x''(y)-y|^2} \\ &\le C(D)r \int_r^{C(D)} s^{-1}\, ds \le C(D)r(1-\log r).\end{aligned}$$

The case where the interval connecting x and x' does not lie entirely in D is similar, one just needs to replace this interval by a curve connecting x and x' with the length of the order r. We briefly sketch the argument. The following lemma can be proved by standard methods using the compactness of the domain and the regularity of the boundary, so we do not present its proof.

Lemma 3.2.4 *Fix $\varepsilon > 0$ and let $D \subset \mathbb{R}^2$ be bounded domain with a smooth boundary. Then there exists $r_0 = r_0(D, \varepsilon) > 0$ such that if $x_0 \in \partial D$, and $r \le r_0$, then $B_r(x_0) \cap \partial D$ is a curve that, by a rotation and a translation of the coordinate system, can be represented as a graph $x_2 = f(x_1)$, with $x_0 = (0,0)$. The function f is C^∞, and $f'(x_{0,1}) = 0$. Moreover, the part of the boundary ∂D within $B_r(x_0)$ lies in the narrow angle between the the lines $x_2 = \pm\varepsilon x_1$.*

With this lemma, suppose we have x and x' such that the interval connecting these points does not lie in D. It is enough to consider the case where $|x-x'| = r < r_0/2$, where r_0 is as in Lemma 3.2.4 corresponding to a sufficiently small ε. Indeed, the larger values of $|x-x'|$ can be handled by adjusting $C(D)$ in (3.2.10). Find a point $x_0 \in \partial D$ closest to x (it does not have to be unique). Note that by the assumption that the interval (x, x') crosses the boundary, we must have $|x - x_0| \le r_0/2$ and $|x' - x_0| < r_0$. Thus, both x and x' lie in the disk $B(x_0, r_0)$ where ∂D lies between the lines $x_2 = \pm\varepsilon x_1$. It is also not hard to see that x must lie on the vertical x_2-axis of a system of coordinates centered at x_0, with the horizontal x_1-axis tangent to ∂D at x_0. Since by assumption the interval between x and x' does not lie in D, we know that x' must lie in the narrow angle between the lines $x_2 = \pm\varepsilon x_1$. Otherwise, the interval (x, x') could not have crossed the boundary. Now take a curve connecting x and x' consisting of a straight vertical interval from x' to a point on one of the lines $x_2 = \pm\varepsilon x_1$ which is closest to x, and then an interval connecting this point to x. We can smooth out this curve without changing its length by much. It is easy to see that the length of this curve does not exceed $2r$ if ε is small enough. The rest of the proof goes through as before. □

Now we can state the regularity result for the fluid velocity.

Corollary 3.2.5 *The fluid velocity u satisfies*

$$\|u\|_{L^\infty} \le C(D)\|\omega\|_{L^\infty}, \tag{3.2.13}$$

and

$$|u(x) - u(x')| \le C\|\omega\|_{L^\infty}\phi(|x - x'|), \tag{3.2.14}$$

with the function $\phi(r)$ defined in (3.2.11).

Proof. By (3.2.8), we have, for any $x, y \in D$,

$$|K_D(x,y)| \le C(D)|x-y|^{-1},$$

so that

$$\left|\int_D K_D(x,y)\omega(y)\,dy\right| \le C(D)\|\omega\|_{L^\infty}\int_D \frac{1}{|x-y|}\,dy \le C(D)\|\omega\|_{L^\infty},$$

which is (3.2.13). The proof of (3.2.14) is immediate from Lemma 3.2.3, as

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)dy,$$

and we are done. □

We say that u is log-Lipschitz if it satisfies (3.2.14). We will see that this bound is in fact sharp: there are velocities that correspond to bounded vorticities which are just log-Lipschitz and in particular fail to be Lipschitz.

3.2.2 Trajectories for log-Lipschitz velocities

As the fluid velocity with an L^∞-vorticity is not necessarily Lipschitz but only log-Lipschitz, we may not use the classical results on the existence and uniqueness of the solutions of ODEs with Lipschitz velocities. Nevertheless, as we show next, the log-Lipschitz regularity is sufficient to determine the fluid trajectories uniquely.

Lemma 3.2.6 *Let D be a bounded smooth domain in $\mathbb{R}^d$. Assume that the velocity field $b(t,x)$ satisfies, for all $t \geq 0$:*

$$b \in L^\infty([0,\infty) \times \bar{D}) \;\; |b(t,x) - b(t,y)| \leq C\phi(|x-y|), \;\; b(t,x) \cdot \nu|_{\partial D} = 0. \tag{3.2.15}$$

Here, the function $\phi(r)$ is given by (3.2.11) and ν is the unit normal to ∂D at point x. Then the Cauchy problem in $\bar{D}$

$$\frac{dx}{dt} = b(t,x), \;\; x(0) = x_0, \tag{3.2.16}$$

has a unique global solution. Moreover, if $x_0 \notin \partial D$, then $x(t) \notin \partial D$ for all $t \geq 0$. If $x_0 \in \partial D$, then $x(t) \in \partial D$ for all $t \geq 0$.

Note that the log-Lipschitz regularity is border-line: the familiar example of the ODE

$$\dot{x} = x^\beta, \;\; x(0) = 0,$$

with $\beta \in (0,1)$ has two solutions: $x(t) \equiv 0$, and

$$x(t) = \frac{t^p}{p^p}, \;\; p = \frac{1}{1-\beta},$$

hence ODE's with Hölder (with an exponent smaller than one) velocities may have more than one solution. Existence of the solutions, on the other hand, does not really require the log-Lipschitz condition: uniform continuity of $b(t,x)$ and at most linear growth as $|x| \to +\infty$ would be sufficient, see e.g. [37] for the Peano existence theorem.

Proof. Let us first show the existence and uniqueness of a local solution using a version of the standard Picard iteration: set

$$x_n(t) = x_0 + \int_0^t b(s, x_{n-1}(s))\, ds, \;\; x_0(t) \equiv x_0.$$

Let us also assume first that $x_0 \in D$. Then, as usual, we have, using the log-Lipschitz property of b:

$$|x_n(t) - x_{n-1}(t)| \leq \int_0^t |b(s, x_{n-1}(s)) - b(s, x_{n-2}(s))|\, ds \leq C \int_0^t \phi(|x_{n-1}(s) - x_{n-2}(s)|)\, ds. \tag{3.2.17}$$

As the function $\phi(r)$ is concave, we have

$$\phi(r) \leq \phi(\varepsilon) + \phi'(\varepsilon)(r - \varepsilon) = \varepsilon(1 + \log \varepsilon^{-1}) + (r - \varepsilon)\log \varepsilon^{-1} = \varepsilon + r \log \varepsilon^{-1},$$

for every $\varepsilon < 1$. Using this in (3.2.17) gives

$$|x_n(t) - x_{n-1}(t)| \leq C \log(\varepsilon^{-1}) \int_0^t |x_{n-1}(s) - x_{n-2}(s)|\, ds + Ct\varepsilon.$$

Exercise 3.2.7 Use an induction argument to show that (3.2.17) implies, for any $0 \leq t \leq T$:

$$|x_n(t) - x_{n-1}(t)| \leq CT\varepsilon \sum_{k=0}^{n-2} \frac{C^k (\log \varepsilon^{-1})^k t^k}{k!} + \frac{C^{n-1} t^{n-1} (\log \varepsilon^{-1})^{n-1}}{(n-1)!} \text{sup}_{0 \leq t \leq T} |x_1(t) - x_0|. \tag{3.2.18}$$

As

$$|x_1(t) - x_0| \le Ct,$$

we have

$$|x_n(t) - x_{n-1}(t)| \le CT\varepsilon \exp(CT \log \varepsilon^{-1}) + \frac{C^n T^n (\log \varepsilon^{-1})^{n-1}}{(n-1)!},$$

for any $\varepsilon > 0$ and all $n \ge 2$, with a constant C that is independent of $\varepsilon > 0$ or n. We may now choose $\varepsilon = \exp(-n)$ and T sufficiently small so that $1 - CT > 1/2$. This leads to

$$|x_n(t) - x_{n-1}(t)| \le CT \exp(-n/2) + \frac{C^n T^n n^{n-1}}{(n-1)!}.$$

The Stirling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

implies that if T is sufficiently small (independently of n), then

$$|x_n(t) - x_{n-1}(t)| \le \alpha^n,$$

with $\alpha < 1$. Thus, $x_n(t)$ converges uniformly to a limit $x(t)$. The uniformity of the convergence implies that the limit satisfies the integral equation

$$x(t) = x_0 + \int_0^t b(s, x(s))\, ds. \tag{3.2.19}$$

We also need to choose T so that $|x(t) - x_0| \le \mathrm{dist}(x_0, \partial D)$. Taking

$$T < \|b\|_{L^\infty}^{-1} \mathrm{dist}(x_0, \partial D), \tag{3.2.20}$$

would suffice. As b is bounded, we may differentiate (3.2.19) and obtain the desired ODE

$$\frac{dx(t)}{dt} = b(t, x(t)), \quad x(0) = x_0,$$

for a.e. t on the time interval $0 \le t \le T$.

Next, we show the uniqueness of this local solution – here, the log-Lipchitz property will play a crucial role. We will prove a little more general stability estimate than needed for the uniqueness, as we will need it later. Let $\sigma > 0$ be a small number. Suppose that $x(t)$ and $y(t)$ are two different solutions of (3.2.16) with the initial data satisfying $|x_0 - y_0| < \sigma$. Set $z(t) = x(t) - y(t)$. Then, by the log-Lipschitz assumption on b, we have

$$|\dot z(t)| \le C\phi(z(t)), \ \ |z(0)| < \sigma.$$

In order to control $z(t)$, define $f_\delta(t)$ as the solution of

$$\dot f_\delta = 2C\phi(f_\delta(t)), \ \ f_\delta(0) = \delta > \sigma > 0.$$

We claim that $|z(t)| \le f_\delta(t)$ for all t. Indeed, this is true for some initial time interval, simply because $\delta > \sigma > 0$ and both $z(t)$ and $f_\delta(t)$ are continuous. Let $t_1 > 0$ be the smallest time such that $z(t_1) = f_\delta(t_1)$ (the case $z(t_1) = -f_\delta(t_1)$ is similar). At this time we have

$$\dot z(t_1) - \dot f_\delta(t_1) \le -C\phi(z(t_1)) < 0,$$

contradicting the definition of t_1. Thus, no such t_1 exists and

$$|z(t)| \le f_\delta(t) \text{ for all } t \ge 0, \text{ and all } \delta > \sigma > 0. \tag{3.2.21}$$

Now, we need an estimate on $f_\delta(t)$. Let us show that for any $t > 0$ fixed we have

$$\lim_{\delta \to 0^+} f_\delta(t) = 0. \tag{3.2.22}$$

It suffices to consider the case where δ is small and times are small enough so that $f_\delta(t) < 1$. Then we have

$$\frac{d}{dt} \log f_\delta(t) = 2C(1 - \log f_\delta(t)).$$

Solving this differential equation leads to

$$1 - \log f_\delta(t) = (1 - \log \delta) e^{-2Ct},$$

or

$$f_\delta(t) = \delta^{\exp(-2Ct)} \exp(1 - \exp(-2Ct)), \tag{3.2.23}$$

whence (3.2.22) follows. If the initial conditions for $x(t)$ and $y(t)$ are the same, so that $\sigma = 0$, then $|z(t)| \le f_\delta(t)$ for every $\delta > 0$. Then, (3.2.21) and (3.2.22) imply that $z(t) \equiv 0$, hence the solution $x(t)$ of (3.2.16) is unique.

Exercise 3.2.8 Identify the place in the uniqueness proof above, where we have used the log-Lipschitz condition on the function $b(t, x)$, that is, where the proof would have failed, for example, for $\phi(r) = r^\beta$, with $\beta \in (0, 1)$.

We now address the question of the global existence. Having constructed a local solution until a time t, we can continue to extend our local solution from t to a time $t + \Delta t$, using the local in time existence we have just proved, since $x(t)$ is inside D. However, as (3.2.20) shows, the time step Δt depends on the distance from $x(t)$ to ∂D. Thus, in order to construct a global in time solution we need to control this distance. Let us set

$$d(t) = \operatorname{dist}(x(t), \partial D),$$

with $d(0) \equiv d_0 > 0$ since $x_0 \in D$. Our goal is to get a lower bound on $d(t)$. Note first that since $b \in L^\infty$, the function $d(t)$ is Lipschitz in time. Thus, by the Rademacher theorem (see, for instance, [61]), $d'(t)$ exists almost everywhere, and

$$d(t) = d_0 + \int_0^t d'(s)\, ds.$$

We will now estimate $d'(t)$ from below at any given time t for which the local solution is defined. Consider the set

$$S = \{P \in \partial D \,||x(t) - P| = d(t).\}$$

This set depends on the time t, of course, but we will suppress this dependence in the notation. Given $\kappa > 0$, define

$$S_\kappa = \{Q \in \partial D \,|\exists P \in S, \;\; |Q - P| < \kappa.\}\,.$$

We can think of the set S_κ as the points on ∂D that are very close to the points at which the distance between $x(t)$ and ∂D is realized. Therefore, we expect these points to be important for the estimate of how the distance changes. Fix some small $\varepsilon > 0$. Let us choose κ so that if $Q \in S_\kappa$, then there exists $P \in S$ such that

$$\left| \frac{Q - x(t)}{|Q - x(t)|} - \nu_P \right| < \frac{\varepsilon}{\|b\|_{L^\infty}}. \tag{3.2.24}$$

Here, ν_P is the outside unit normal to ∂D at the point P. Such $\kappa(\varepsilon)$ exists due to the smoothness of the boundary ∂D.

Exercise 3.2.9 Assume that the boundary ∂D can be represented around the point P as a graph $\partial D = (w, g(w))$ with $P = (0,0)$ and $g'(0) = 0$. Assume that the function $g(w)$ is bounded in C^2 and find an explicit bound for κ which ensures that (3.2.24) holds.

Let us now proceed to estimate $d(s)$ for times s slightly large than t. Consider first any point $Q \in \partial D \backslash S_\kappa$. The set $\partial D \backslash S_\kappa$ is compact, and $\mathrm{dist}(x(t), Q) > d(t)$ for every $Q \in \partial D \backslash S_\kappa$. Therefore, there exists $\gamma > 0$ such that

$$|x(s) - Q| \geq d(t) + \gamma - \|b\|_{L^\infty}(s - t).$$

Thus, if

$$0 < s - t \leq \gamma \|b\|_{L^\infty}^{-1},$$

then $|x(s) - Q| \geq d(t)$ for any $Q \in \partial D \setminus S_\kappa$.

Next, suppose that $Q \in S_\kappa$. We have

$$x(s) - Q = x(t) + \int_t^s b(r, x(r))\, dr - Q,$$

so that

$$\begin{aligned}
|x(s) - Q| &\geq \left(x(t) - Q + \int_t^s b(r, x(r))\, dr \right) \cdot \left(\frac{x(t) - Q}{|x(t) - Q|} \right) \geq \\
&\geq |x(t) - Q| - \left| \int_t^s b(r, x(r)) \cdot \nu_P\, dr \right| - \|b\|_{L^\infty}(s - t)\varepsilon \|b\|_{L^\infty}^{-1} \\
&\geq |x(t) - Q| - \left| \int_t^s (b(r, x(r)) - b(r, P)) \cdot \nu_P\, dr \right| - \varepsilon(s - t) \\
&\geq d(t) - C \int_t^s \phi(|x(r) - P|) dr - \varepsilon(s - t).
\end{aligned} \tag{3.2.25}$$

Here, in the second step we used (3.2.24), in the third step we used the boundary condition

$$b(r, P) \cdot n_P = 0,$$

and in the last step we used the log-Lipschitz condition on b. Now, if s satisfies

$$0 < s - t < \frac{d(t)}{2\|b\|_{L^\infty}},$$

then

$$|x(r) - P| \leq 2d(t) \text{ for every } r \in [t, s],$$

and

$$\phi(|x(r) - P|) \leq 2\phi(d(t)).$$

Using this in (3.2.25) gives

$$d(s) \geq d(t) - (2C\phi(d(t)) + \varepsilon)(s - t)$$

for all $s > t$ sufficiently close to t. Since $\varepsilon > 0$ is arbitrary, it follows that

$$d'(t) \geq -2C\phi(d(t))$$

at every t such that the derivative exists. Solving this differential inequality, similarly to (3.2.23), we obtain

$$d(t) \geq d_0^{\exp(Ct)} \exp(1 - \exp(Ct)). \tag{3.2.26}$$

Therefore, the local solution can be continued indefinitely in time, and $x(t)$ will never arrive at ∂D if $x_0 \notin \partial D$.

It remains to consider the case where $x_0 \in \partial D$. In this case, take $x_n \in D$, $n = 1, \ldots,$ such that

$$\lim_{n \to \infty} x_n = x_0,$$

and consider the corresponding solutions $x_n(t)$. Due to the estimates (3.2.21) and (3.2.23), the sequence $x_n(t)$ is Cauchy in $C([0, T], \mathbb{R}^d)$ for any $T < \infty$. Therefore it has a limit $x(t)$ in this space, and this limit satisfies the integral form (3.2.19). We can then differentiate it in time, arriving at (3.2.16) for a.e. t. Finally, we claim that $x(t) \in \partial D$ for all times if $x_0 \in \partial D$. Indeed, suppose there exists t_0 such that $x(t_0) \notin \partial D$. Let us invert time and solve the characteristic backwards:

$$\frac{dy}{ds} = -b(t_0 - s, y(s)), \quad y(0) = x(t_0). \tag{3.2.27}$$

Then $y(s)$ and $x(t_0 - s)$ satisfy the same differential equation with log-Lipschitz coefficient, so by our previous result on uniqueness, we know that $y(s) = x(t_0 - s)$. But this means that $y(s)$ starts at $x(t_0) \in D$ and arrives at $x_0 \in \partial D$ in a finite time. This contradicts our earlier estimates that apply in the same fashion to the backwards equation (3.2.27). □

3.2.3 The approximation scheme

Let us return to our strategy of constructing a triple $(\omega, u, \Phi_t(x))$ solving (3.2.1), (3.2.2) and (3.2.5), with the initial vorticity $\omega_0 \in L^\infty$. We define an iterative sequence of approximations

$$\frac{d}{dt}\Phi_t^n(x) = u^n(t, \Phi_t^n(x)), \tag{3.2.28}$$

$$u^n(t, x) = \int_D K_D(x, y)\omega^{n-1}(t, y)\, dy, \tag{3.2.29}$$

$$\omega^n(t, x) = \omega_0((\Phi_t^n)^{-1}(x)), \tag{3.2.30}$$

with $\omega^0(t,x) \equiv \omega_0(x) \in L^\infty$ for all $t \geq 0$. Note that since the velocities u^n defined by (3.2.29) satisfy the no flow boundary conditions at ∂D, and by Corollary 3.2.5 and Lemma 3.2.6, the solutions of the trajectory equation (3.2.28) exist and are unique. Moreover, the trajectory maps $\Phi_t^n(x)$ are injective due to the uniqueness of the backward trajectories and surjective due to the global existence of these backward trajectories. Therefore, the inverse maps $(\Phi_t^n)^{-1}(x)$ in (3.2.30) are well-defined. Both the direct and the inverse trajectory maps are also continuous in x for each t on $\bar{D}$ due to the estimates (3.2.21) and (3.2.23), and map D to D and ∂D to ∂D. In fact, it follows from (3.2.23) that these maps also satisfy the Hölder regularity bounds, which we will spell out precisely in a moment.

Intuitively, each successive approximation involves solving a linear problem

$$\omega_t^n + (u^n \cdot \nabla)\omega^n = 0, \tag{3.2.31}$$

with the flow

$$u^n(t,x) = \int_D K_D(x,y)\omega^{n-1}(t,y)\,dy,$$

computed from the previous iteration. Note that each $\omega^n \in L^\infty$, with

$$\|\omega^n(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}.$$

However, one can not take (3.2.31) too literally, since we only know that ω_0 is in L^∞.

The next step is to obtain uniform bounds on the solutions of the approximation scheme that will allow us to pass to the limit $n \to \infty$ and get a solution of (3.2.1)-(3.2.5).

The Hölder regularity of the approximate trajectories

We will now obtain a uniform continuity bound on the trajectories $\Phi_t^n(x)$. Note first that Corollary 3.2.5 implies that all $u^n(t,x)$ are uniformly bounded and log-Lipschitz:

$$|u^n(t,x) - u^n(t,x')| \leq C(D)\phi(|x-x'|). \tag{3.2.32}$$

Let us first recall the following result.

Exercise 3.2.10 Let $u(t,x)$ be a uniformly Lipschitz function in x: there exists a constant C so that

$$|u(t,x) - u(t,y)| \leq C|x-y|, \text{ for all } t \geq 0 \text{ and } x,y \in \mathbb{R}^d. \tag{3.2.33}$$

Show that the solution of (3.2.28) satisfies a Lipschitz bound

$$|\Phi_t(x) - \Phi_t(y)| \leq e^{Ct}|x-y|. \tag{3.2.34}$$

In contrast to (3.2.34), we have the following Hölder estimate for the flow map when the velocity is only log-Lipschitz.

Lemma 3.2.11 *Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain, and the map $\Phi_t(x)$ is generated by a log-Lipschitz vector field $b(t,x)$ satisfying assumptions of Lemma 3.2.6. Then for every $x,y \in \bar{D}$ with $|x-y| \leq 1/2$, and while $|\Phi_t(x) - \Phi_t(y)| \leq 1/2$, we have*

$$|x-y|^{e^{Ct}} \leq |\Phi_t(x) - \Phi_t(y)| \leq |x-y|^{e^{-Ct}}. \tag{3.2.35}$$

The constant C in (3.2.35) *only depends on the constant in the log-Lipschitz bound for b.*

Of course, one can write the corresponding bounds for all $x, y \in D$ (recall that D is bounded, so $|x-y| \le C(D)$). We restrict to the $\le 1/2$ range to simplify the argument. Also note that the bound similarly applies to $\Phi_t^{-1}(x)$.

This is a rather remarkable estimate: we can show that $\Phi_t(x)$ is Hölder continuous in space for any $t \ge 0$, but the Hölder exponent deteriorates in time. This is a reflection of the complexity of the dynamics: the exponent in the upper bound in (3.2.35) tends to zero as $t \to +\infty$ because two trajectories that start very close at $t = 0$ may diverge very far at large times. On the other hand, the exponent in the lower bound in (3.2.35) grows as $t \to +\infty$ because even if at the time $t = 0$ the starting points x and y are "relatively far apart" (but with $|x - y| \le 1$), they can be extremely close at large times. This deterioration of the estimates is not an artefact of the proof – we will later see that the trajectories of the Euler equations can get extremely close at large times.

Proof. The result is of course closely related to the estimates (3.2.21) and (3.2.23). Let us fix x and y, and set $F(t) = |\Phi_t(x) - \Phi_t(y)|$. We compute

$$\left|\frac{d}{dt}F^2(t)\right| = 2\left|(\Phi_t(x)-\Phi_t(y))\cdot(b(\Phi_t(x),t)-b(\Phi_t(y),t))\right| \le 2C(D,\|\omega_0\|_{L^\infty})F(t)\phi(F(t)),$$

so that

$$|F'(t)| \le CF(t)\max(1, 1-\log F(t)).$$

Recall that we only need to consider the case when $F(t) \le 1/2$. Then, we have

$$|F'(t)| \le CF(t)\log F(t)^{-1}$$

with an adjusted constant C, which leads to

$$[\log F(0)]e^{Ct} \le \log F(t) \le [\log F(0)]e^{-Ct}.$$

The estimate (3.2.35) follows immediately from exponentiating this inequality and taking into account that $F(0) = |x-y|$.

The flow map corresponding to divergence free log-Lipschitz velocity is measure preserving

It will be also useful for us to know that the trajectory maps corresponding to log-Lipschitz vector fields are measure preserving. We have discussed that if u is smooth and $\nabla \cdot u = 0$, then the associated trajectories map is measure preserving. However, this argument does not apply directly when the vector field $u(t,x)$ is just log-Lipschitz in the spatial variable. Indeed, we only have the weakening in time Hölder estimates for the trajectories map. Hence, taking its derivatives and their products to study the Jacobian is not straightforward. We will instead use an approximation argument to establish this property.

Lemma 3.2.12 *Let $D \in \mathbb{R}^d$ and $b(t,x)$ satisfy the assumptions of Lemma 3.2.6. Assume, in addition, that $\nabla \cdot b = 0$ in the distributional sense. Then, the trajectory map $\Phi_t(x)$ defined by the vector field $b(t,x)$ according to* (3.2.1) *is measure preserving on D.*

Proof. From the proof of Lemma 3.2.6 and Lemma 3.2.11, we already know that $\Phi_t(x)$, is a Hölder continuous bijection on D. It suffices to check the preservation of measure for an arbitrary d-dimensional interval lying in D, at a positive distance from ∂D. Fix such interval I and an arbitrary time $T > 0$. We will use a smooth incompressible flow that approximates $b(t, x)$ in a neighborhood of $\Phi_t(I)$. It is constructed as follows. According to the estimate (3.2.26), there exists $\kappa > 0$ such that

$$\operatorname{dist}(\Phi_t(I), \partial D) \geq \kappa \text{ for all } 0 \leq t \leq T.$$

Take any $\delta < \kappa/2$, and set

$$I_\delta := \{x \in D \,|\operatorname{dist}(x, I) < \delta\}.$$

Further decreasing δ if necessary, we may ensure that

$$\operatorname{dist}(\Phi_t(I_\delta), \partial D) \geq \kappa/2 \text{ for all } 0 \leq t \leq T.$$

Let $\eta(x)$ be a standard mollifier:

$$\eta \in C_0^\infty(\mathbb{R}^d),\ \eta(x) = 0 \text{ if } |x| \geq 1, \text{ and } \int_{\mathbb{R}^d} \eta(x)\, dx = 1.$$

Take any $\varepsilon < \kappa/4$, and define

$$b_\varepsilon = \eta_\varepsilon * b,$$

with $\eta_\varepsilon(x) = \eta(x/\varepsilon)$. The flow $b_\varepsilon(t, x)$ is defined for all x such that $\operatorname{dist}(x, \partial D) < \varepsilon$. In addition, it is smooth, and it is easy to check that $b_\varepsilon(t, x)$ is divergence free. Let us denote the trajectory map corresponding to $b_\varepsilon(t, x)$ by $\Phi_t^\varepsilon(x)$. We have

$$\begin{aligned}|\Phi_t(x) - \Phi_t^\varepsilon(x)| &\leq \left|\int_0^t (b(s, \Phi_s(x)) - b(s, \Phi_s^\varepsilon(x)))\, ds\right| + \left|\int_0^t (b(s, \Phi_s^\varepsilon(x))) - b_\varepsilon(s, \Phi_s^\varepsilon(x)))\, ds\right| \\ &\leq C \int_0^t \phi(|\Phi_s(x) - \Phi_s^\varepsilon(x)|) + C\phi(\varepsilon)t. \end{aligned} \tag{3.2.36}$$

Here we used the log-Lipschitz bound on b to estimate both terms. We have assumed above that $\Phi_t^\varepsilon(x)$ does not come within distance ε to the boundary ∂D, and we now verify that this indeed does not happen if we choose ε to be small enough. One can see from (3.2.36) that

$$|\Phi_t(x) - \Phi_t^\varepsilon(x)| \leq g(t),$$

where $g(t)$ satisfies

$$g'(t) = C\phi(g(t)) + C\phi(\varepsilon),\ \ g(0) = 0.$$

Exercise 3.2.13 Let $h(t)$ be the solution of

$$h'(t) = C\phi(h(t)),\ \ h(0) = C\phi(\varepsilon)T.$$

Show that $g(t) \leq h(t)$, for $0 \leq t \leq T$.

We can find $h(t)$ explicitly (at least while $h(t) \le 1$):

$$h(t) = (C\phi(\varepsilon)T)^{\exp(-Ct)} \exp(1 - \exp(-Ct)).$$

Therefore, there exists $\beta = \beta(T) > 0$ such that

$$|\Phi_t(x) - \Phi_t^\varepsilon(x)| \le C\varepsilon^\beta \tag{3.2.37}$$

for all $0 \le t \le T$. We can then choose ε so that, in particular, we have

$$|\Phi_t(x) - \Phi_t^\varepsilon(x)| \le \kappa/4 \text{ for all } 0 \le t \le T,$$

and so $\Phi_t^\varepsilon(x)$ stays at least distance ε away from ∂D for all $x \in I_\delta$ during this time interval.

Next, take a cut-off function $f \in C_0^\infty(I_\delta)$ such that

$$0 \le f(x) \le 1, \ \|\nabla f(x)\|_{L^\infty} \le C\delta^{-1}, \text{ and } f(x) = 1 \text{ if } x \in I,$$

Observe that

$$|\Phi_t^{-1}(I)| = \int_D \chi_I(\Phi_t(x))\,dx \le \int_D f(\Phi_t(x))\,dx \le \int_D \chi_{I_\delta}(\Phi_t(x))\,dx = |\Phi_t^{-1}(I_\delta)|, \tag{3.2.38}$$

and

$$|I| = |(\Phi_t^\varepsilon)^{-1}(I)| \le \int_D f(\Phi_t^\varepsilon(x))\,dx \le |(\Phi_t^\varepsilon)^{-1}(I_\delta)| = |I_\delta|. \tag{3.2.39}$$

We used in (3.2.39) the fact that $(\Phi_t^\varepsilon)^{-1}$ is measure preserving since this map is generated by a smooth incompressible velocity field. On the other hand, for $0 \le t \le T$ we have

$$\left|\int_D f(\Phi_t(x))\,dx - \int_D f(\Phi_t^\varepsilon(x))\,dx\right| \le \|\nabla f\|_{L^\infty}|D| \sup_{x \in I_\delta, 0 \le t \le T} |\Phi_t(x) - \Phi_t^\varepsilon(x)| \le \frac{C(D)\varepsilon^\beta}{\delta}. \tag{3.2.40}$$

We used (3.2.37) in the last step. Taking δ to zero, and simultaneously taking $\varepsilon = \delta^{2/\beta}$ to zero (so that the right hand side of (3.2.40) goes to zero too), and using (3.2.38), (3.2.39) and (3.2.40), we conclude that

$$|\Phi_t^{-1}(I)| \le |I|,$$

for every interval $I \subset D$ at a positive distance from ∂D, and any $0 \le t \le T$. It follows that the same is true for any open set $\Omega \subset D : |\Phi_t^{-1}(\Omega)| \le |\Omega|$. An analogous argument using

$$\int_D f(\Phi_t^{-1}(x))\,dx, \text{ and } \int_D f((\Phi_t^\varepsilon)^{-1}(x))\,dx,$$

leads to the inequality $|\Phi_t(\Omega)| \le |\Omega|$. Since Φ_t, Φ_t^{-1} are continuous and bijective, these two inequalities together imply that these maps are measure preserving.

Convergence of the approximation scheme

Let us now investigate the convergence of the sequence $(\omega^n, u^n, \Phi_t^n)$. The estimate (3.2.35) implies that for every $T > 0$, we have

$$\Phi_t^n(x) \in C^{\alpha(T)}([0,T] \times \bar{D}),$$

for some $\alpha(T) > 0$. The Arzela-Ascoli theorem implies that we can find a subsequence n_j such that $\Phi_t^{n_j}(x)$ converges uniformly to $\Phi_t(x) \in C([0,T] \times D)$. Moreover, since (3.2.35) is uniform in n, the limit $\Phi_t(x)$ also satisfies (3.2.35), thus

$$\Phi_t(x) \in C^{\alpha(T)}([0,T] \times \bar{D}).$$

In addition, as all Φ_t^n are measure-preserving, so is $\Phi_t(x)$.

Exercise 3.2.14 Prove this statement. You can use an argument similar to the proof of Lemma 3.2.12, or try a more direct approach.

The lower bound in (3.2.35) which applies to Φ_t^n uniformly implies that $\Phi_t(x)$ is invertible. As Φ_t^{-1} satisfies the same estimate (3.2.35), it also belongs to $C^{\alpha(T)}([0,T] \times \bar{D})$. We may then define the corresponding vorticity

$$\omega(t,x) = \omega_0(\Phi_t^{-1}(x)),$$

and the fluid velocity

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)\,dy.$$

For the simplicity of notation, we relabel the subsequence n_j by n.

Lemma 3.2.15 *We have* $|u(t,x) - u_n(t,x)| \to 0$, *as* $n \to \infty$, *uniformly in* $\bar{D}$ *for all* $t \in [0,T]$.

Proof. Note that

$$|u(t,x) - u_n(t,x)| = \left| \int_D \left(K_D(x, \Phi_t(z)) - K_D(x, \Phi_t^n(z))\right) \omega_0(z)\,dz \right|.$$

Given $\varepsilon > 0$, choose N so that $|\Phi_t(x) - \Phi_t^n(x)| < \delta$, for all $n \geq N$ and for all $x \in \bar{D}$, $t \in [0,T]$, with $\delta > 0$ to be determined later. Then we have

$$|u(t,x) - u_n(t,x)| \leq \|\omega_0\|_{L^\infty} \int_D |K_D(x,z) - K_D(x,y(z))|\,dz. \tag{3.2.41}$$

Note that by Lemma 3.2.12 the map $y(z) = \Phi_t^n \circ \Phi_t^{-1}(z)$ is measure preserving, and

$$|y(z) - z| = |\Phi_t^n(\Phi_t^{-1}(z)) - \Phi_t(\Phi_t^{-1}(z))| < \delta,$$

for every z. As usual, we split the integral in (3.2.41) into two regions: in the first one we have

$$\int_{B_{3\delta}(x) \cap D} |K_D(x,z) - K_D(x,y(z))|\,dz \leq 2C \int_{B_{3\delta}(x)} \frac{dz}{|x-z|} \leq 2C\delta,$$

while in the second

$$\int_{B_{3\delta}^c(x)\cap D} |K_D(x,z) - K_D(x,y(z))|\,dz \le C\delta \int_{B_{3\delta}^c(x)\cap D} |\nabla K_D(x,p(z))|\,dz$$
$$\le C\delta \int_{B_\delta^c} \frac{dz}{|x-z|^2} \le C\delta \log \delta^{-1}. \qquad (3.2.42)$$

Here, $p(z)$ is a point on a curve of length $\le 2\delta$ that connects z and $y(z)$. If the interval connecting these points lies in $\bar{D}$ then this interval can be used as this curve. If not, one can use an argument similar to that in the proof of Lemma 3.2.3. Thus choosing δ sufficiently small we can make sure that the difference of the velocities does not exceed ε.

Exercise 3.2.16 Fill in all the details in the last step in the proof of the Lemma.

We are now ready to show that

$$\frac{d}{dt}\Phi_t(x) = u(t, \Phi_t(x)).$$

Indeed, we have

$$\Phi_t^n(x) = x + \int_0^t u^n(\Phi_s^n(x), s)\,ds,$$

and, taking $n \to \infty$, using Lemma 3.2.15 and the definition of $\Phi_t(x)$, we obtain

$$\Phi_t(x) = x + \int_0^t u(\Phi_s(x), s)\,ds.$$

Thus, the limit triple $(\omega(t,x), u(t,x), \Phi_t(x))$ satisfies the Euler equations in our generalized sense, completing the proof of the existence of solutions. □

3.3 Existence and uniqueness of the solutions

Let us now, finally, state the main result on the existence and uniqueness of the solutions of the two-dimensional Euler equations with $\omega_0 \in L^\infty$. The existence part of this theorem summarizes what has been proved above using the approximation scheme.

Theorem 3.3.1 *Given $T > 0$, there exists $\alpha(T) > 0$ so that for any $\omega_0 \in L^\infty(D)$ there is a unique triple $(\omega(t,x), u(t,x), \Phi_t(x))$, with the vorticity $\omega \in L^\infty([0,T], L^\infty(D))$, the fluid velocity $u(t,x)$ uniformly bounded and log-Lipschitz in x, and $\Phi_t \in C^{\alpha(T)}([0,T]\times\bar{D})$ a measure preserving, invertible mapping of $\bar{D}$, which satisfy*

$$\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x)), \;\; \Phi_0(x) = x, \qquad (3.3.1)$$
$$\omega(t,x) = \omega_0(\Phi_t^{-1}(x)),$$
$$u(t,x) = \int_D K_D(x,y)\omega(y,t)\,dy.$$

It is clear from the statement of the theorem that $\omega(t,x)$ converges to $\omega_0(x)$ as $t \to 0$ in the weak-$*$ sense in L^∞: for any test function $\eta \in L^1(D)$ we have

$$\int_D \omega(t,x)\eta(x)dx = \int_D \omega_0(\Phi_t^{-1}(x))\eta(x)dx = \int_D \omega_0(x)\eta(\Phi_t(x))dx \to \int_D \omega_0(x)\eta(x)dx, \quad (3.3.2)$$

as $t \to 0$. Indeed, as ω is uniformly bounded in $L^\infty(D)$, it suffices to check (3.3.2) for smooth functions η, for which we have

$$\int_D |\eta(\Phi_t(x)) - \eta(x)|dx \le \|\nabla\eta\|_{L^\infty} \int_D |\Phi_t(x) - x|dx \le C(D)\|\nabla\eta\|_{L^\infty}\|u\|_{L^\infty} t.$$

Proof of Theorem 3.3.1. As we have already established the existence and regularity of the solutions, it remains only to prove the uniqueness. Suppose that there are two solution triples $(\omega^1, u^1, \Phi_t^1)$ and $(\omega^2, u^2, \Phi_t^2)$ satisfying the properties described in Theorem 3.3.1, and set

$$\eta(t) = \frac{1}{|D|}\int_D |\Phi_t^1(x) - \Phi_t^2(x)|\, dx.$$

Let us write

$$|\Phi_t^1(x) - \Phi_t^2(x)| \le \int_0^t |u^1(s,\Phi_s^1(x)) - u^1(s,\Phi_s^2(x))|\, ds + \int_0^t |u^1(s,\Phi_s^2(x)) - u^2(s,\Phi_s^2(x))|\, ds. \quad (3.3.3)$$

By Corollary 3.2.5, the first integral in the right side of (3.3.3) can be bounded by

$$C\|\omega_0\|_{L^\infty}\int_0^t \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\, ds.$$

For the second integral in (3.3.3), consider the difference

$$\begin{aligned} u^1(s,\Phi_s^2(x)) - u^2(s,\Phi_s^2(x)) &= \int_D K_D(\Phi_s^2(x), y)\omega^1(s,y)\, dy - \int_D K_D(\Phi_s^2(x), y)\omega^2(s,y)\, dy \\ &= \int_D \big(K_D(\Phi_s^2(x), \Phi_s^1(y)) - K_D(\Phi_s^2(x), \Phi_s^2(y))\big)\, \omega_0(y)\, dy, \end{aligned}$$

where we used the vorticity evolution formula in (3.3.1). Averaging (3.3.3) in x, we now obtain

$$\begin{aligned} \eta(t) &\le \frac{C\|\omega_0\|_{L^\infty}}{|D|}\int_0^t ds \int_D \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\, dx \\ &+ \frac{C}{|D|}\int_0^t ds \int_D |\omega_0(y)| \int_D |K_D(x,\Phi_s^1(y)) - K_D(x,\Phi_s^2(y))|\, dxdy \\ &\le C(D)\|\omega_0\|_{L^\infty}\int_0^t ds \int_D \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\frac{dx}{|D|}. \end{aligned} \quad (3.3.4)$$

We used Lemma 3.2.3 in the last step. As the function ϕ is concave, we may use Jensen's inequality to exchange ϕ and averaging in the last expression in (3.3.4):

$$\eta(t) \le C(D)\|\omega_0\|_{L^\infty}\int_0^t \phi(\eta(s))\, ds.$$

In addition, we have $\eta(0) = 0$. An argument very similar to the proof of uniqueness in Lemma 3.2.6 (based on the log-Lipschitz property of the function ϕ) can be now used to prove that $\eta(t) = 0$ for all $t \geq 0$.

Exercise 3.3.2 Work out the details of this argument.

This completes the proof of the theorem.

Regularity of the solutions for regular initial data

So far, we have only assumed that $\omega_0 \in L^\infty$. Of course, the Yudovich construction applies also if the initial condition ω_0 possesses additional regularity. In that case, the solution $\omega(t, x)$ inherits this extra regularity. This is expressed by the following theorem.

Theorem 3.3.3 *Suppose that $\omega_0 \in C^k(\bar{D})$, $k \geq 1$. Then the solution described in Theorem 3.3.1, satisfies, in addition, the following regularity properties, for each $t > 0$ fixed:*

$$\omega(t) \in C^k(\bar{D}), \quad \Phi_t(x) \in C^{k,\alpha(t)}(\bar{D}), \text{ and } u \in C^{k,\beta}(\bar{D}),$$

for all $\beta < 1$. In addition, the kth order derivatives of u are log-Lipschitz.

The regularity of the flow $u(t, x)$ is similar in spirit to that in Theorem 3.3.1 – there, L^∞ initial data for vorticity led to log-Lipschitz $u(t, x)$. Here, C^k initial condition $\omega_0(x)$ leads to a flow $u(t, x)$ which has a log Lipschitz derivative of the order k. The first proof of a result similar to Theorem 3.3.3 goes back to the work of Wolibner and of Hölder in the early 1930s. We will provide a detailed argument for the case of $k = 1$, larger values of k will be left as an exercise for the reader. The following result is classical.

Theorem 3.3.4 *Suppose that D is a domain in $\mathbb{R}^d$ with smooth boundary, and let ψ be the solution of the Dirichlet problem*

$$-\Delta\psi = \omega,$$
$$\psi|_{\partial D} = 0.$$

If $\omega \in C^\alpha(\bar{D})$, $\alpha > 0$, then $\psi \in C^{2,\alpha}(\bar{D})$, and

$$\|\partial_i \psi\|_{C^{1,\alpha}} \leq C(\alpha, D)\|\omega\|_{C^\alpha}.$$

This result was originally proved by Kellogg in 1931. Schauder later established a similar bound for more general elliptic operators. Such estimates are commonly called the Schauder estimates, the reader may consult [60, 80] for the proof. We will use this estimate for the stream function

$$\psi(t, x) = (-\Delta_D)^{-1}\omega, \quad u(t, x) = \nabla^\perp \psi(t, x).$$

We have already proved that if $\omega_0 \in L^\infty(\bar{D})$, then $\Phi_t^{-1}(x) \in C^{\alpha(t)}(\bar{D})$ for all $t \geq 0$, with $\alpha(t) = e^{-Ct}$. Since

$$\omega(t, x) = \omega_0(\Phi_t^{-1}(x)),$$

if, in addition, we know that $\omega_0 \in C^1(\bar{D})$, we then automatically have $\omega(t, x) \in C^{\alpha(t)}(\bar{D})$, so that the vorticity is Hölder continuous. By Theorem 3.3.4, we deduce that the flow $u(t, x)$ has a

Hölder continuous derivative: $u(t,x) \in C^{1,\alpha(t)}(\bar{D})$. However, this a priori Hölder exponent $\alpha(t)$ decreases as t grows, while we are looking to prove that $u(t,x) \in C^{1,\beta}(\bar{D})$, for all $\beta \in (0,1)$, hence this a priori information is not sufficient.

A simple calculation starting with the trajectories equation leads to

$$\frac{d}{dt}|\Phi_t(x) - \Phi_t(y)|^2 \leq 2\|\nabla u(t,\cdot)\|_{L^\infty}|\Phi_t(x) - \Phi_t(y)|^2, \tag{3.3.5}$$

where we now know that the derivatives of u are bounded for all t, even though their size may grow with time. Integrating (3.3.5) in time and using the initial condition

$$|\Phi_0(x) - \Phi_0(y)| = |x - y|,$$

we obtain

$$\exp\Big\{-\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty}\,ds\Big\} \leq \frac{|\Phi_t(x) - \Phi_t(y)|}{|x-y|} \leq \exp\Big\{\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty}\,ds\Big\}. \tag{3.3.6}$$

This inequality will be useful for us later. For now, we observe that it implies that $\Phi_t(x)$ is Lipschitz for every $t \geq 0$. We would like to show that, in fact, $\Phi_t(x) \in C^{1,\alpha(t)}(\bar{D})$ for all $t \geq 0$. For this purpose we need a couple of technical lemmas. In what follows, we adopt the summation convention: we sum over repeated indexes.

Lemma 3.3.5 *There exists a set $S \subseteq D$ of full measure so that for all $x \in S$ we have*

$$\partial_j \Phi_t^k(x) = \delta_{jk} + \int_0^t \partial_l u^k(s,\Phi_s(x))\partial_j\Phi_s^l(x)\,ds, \tag{3.3.7}$$

for all $t \geq 0$.

Proof. By the Rademacher theorem (see, e.g. [61]), it follows from (3.3.6) that $\Phi_t(x)$ is differentiable in x a.e. in $\bar{D}$, for each t fixed. Next, note that by the Fubini theorem, it follows that for a.e. x, $\Phi_t(x)$ is differentiable in x for a.e. t. We let S be the set of such x.

Let now $x \in S$, set

$$y = x + e_j\Delta x,$$

where e_j is a unit vector in jth direction, and consider the finite differences

$$\frac{\Phi_t^k(y) - \Phi_t^k(x)}{\Delta x} = \delta_{jk} + \int_0^t \frac{u^k(s,\Phi_s(y)) - u^k(s,\Phi_s(x))}{\Delta x}\,ds. \tag{3.3.8}$$

We may write, explicitly listing the coordinates

$$\begin{aligned}\frac{u^k(s,\Phi_s(y)) - u^k(s,\Phi_s(x))}{\Delta x} &= \frac{u^k(s,\Phi_s^1(y),\Phi_s^2(y)) - u^k(s,\Phi_s^1(x),\Phi_s^2(y))}{\Phi_s^1(y) - \Phi_s^1(x)}\frac{\Phi_s^1(y) - \Phi_s^1(x)}{\Delta x}\\ &+\frac{u^k(s,\Phi_s^1(x),\Phi_s^2(y)) - u^k(s,\Phi_s^1(x),\Phi_s^2(x))}{\Phi_s^2(y) - \Phi_s^2(x)}\frac{\Phi_s^2(y) - \Phi_s^2(x)}{\Delta x}.\end{aligned}$$

Since $u \in C^{1,\alpha}(\bar{D})$, it is not difficult to show, using the mean value theorem, that the first factors in the two products in the right side converge, as $\Delta x \to 0$, uniformly in x, to $\partial_l u^k(s, \Phi_s(x))$, $l = 1, 2$ respectively. On the other hand, the ratios

$$\frac{\Phi_s^l(y) - \Phi_s^l(x)}{\Delta x}$$

are controlled in L^∞ by the Lipschitz estimate (3.3.6). Moreover, for $x \in S$, the ratio converges to $\partial_j \Phi_s^l(x)$ for a.e. $s \in [0, t]$. By the Lebesgue dominated convergence theorem, we have the convergence of the integral in (3.3.8) to the integral in (3.3.7). □

Now, for $x, y \in S$ we find from (3.3.7) that

$$\partial_t \partial_j \Phi_t^k(x) = \partial_l u^k(t, \Phi_t(x)) \partial_j \Phi_t^l(x)$$

for all t, and similarly for y. Without loss of generality, we may confine our considerations to x, y such that $|x - y| \le 1$. Consider the expression

$$\begin{aligned}\partial_t(\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)) &= (\partial_l u^k(t, \Phi_t(x)) - \partial_l u^k(t, \Phi_t(y)))\partial_j \Phi_t^l(x) \\ &\quad + \partial_l u^k(t, \Phi_t(y))(\partial_j \Phi_t^l(x) - \partial_j \Phi_t^l(y)).\end{aligned}$$

It follows that

$$\partial_t |\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)| \le \|\Phi_t\|_{Lip} \|\nabla u\|_{C^{\alpha(t)}} |\Phi_t(x) - \Phi_t(y)|^{\alpha(t)} + \|\nabla u\|_{L^\infty} |\partial_j \Phi_t^l(x) - \partial_j \Phi_t^l(y)|,$$

where we denote by $\|\Phi_t\|_{Lip}$ the Lipschitz bound we have on $\Phi_t(x)$ in x for a given t. Let us denote

$$F(t) = \sum_{k,j} |\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)|.$$

Then we get

$$\dot{F}(t) \le \|\nabla u(\cdot, t)\|_{L^\infty} F(t) + |x - y|^{\alpha(t)} \|\Phi_t\|_{Lip}^2 \|\nabla u\|_{C^{\alpha(t)}}.$$

This inequality holds for every $t > 0$ with the corresponding value of $\alpha(t)$. Fix an arbitrary time interval $[0, T]$. By applying the Gronwall inequality, we conclude that for all $x, y \in S$ and all $t \in [0, T]$ we have

$$|\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)| \le C(\|\omega_0\|_{C^1}, T)|x - y|^{\alpha(T)}. \tag{3.3.9}$$

Note that the dependence of the constant in (3.3.9) on T can be pretty complex – it is controlled by the size of norms that we showed to be finite for every time but never traced their growth. We will obtain a more clear cut, quantitative bound on the possible growth later.

Now we need one more elementary lemma.

Lemma 3.3.6 *Suppose that $f : \bar{D} \subset \mathbb{R}^d \mapsto \mathbb{R}$ is Lipschitz. Suppose there exists a set of full measure S such that $\nabla f(x)$ exists for $x \in S$, and moreover for every $x, y \in S$ we have*

$$|\nabla f(x) - \nabla f(y)| \le C|x - y|^\gamma \tag{3.3.10}$$

for some fixed constant C and $0 < \gamma < 1$. Then $f \in C^{1,\gamma}(\bar{D})$.

Proof. Since S is full measure, we can extend ∇f by continuity to a function $g = (g_1, \dots, g_d)$ defined on all $\bar{D}$. Namely, we set $g(x) = \nabla f(x)$ if $x \in S$. If $x \notin S$, then we take any sequence $x_n \in S \to x$, and define $g(x) = \lim_{n\to\infty} \nabla f(x_n)$. Note that the sequence $\nabla f(x_n)$ is Cauchy due to (3.3.10), so the limit is well-defined. It is also straightforward to check that the definition is unambiguous (different sequences in S lead to the same limit), and that the resulting function $g \in C^\gamma(\bar{D})$. It remains to show that in fact f is everywhere differentiable and $\nabla f(x) \equiv g(x)$.

Without loss of generality, let us consider $\partial_1 f$. Let $x = (x_1, \widetilde{x}) \in D$, where $\widetilde{x} = (x_2, \dots, x_d)$; the case $x \in \partial D$ is similar. Given x_1, let us denote the set of $\widetilde{x}$ such that $(x_1, \widetilde{x}) \in D$ by F. Suppose first that $\widetilde{x}$ is such that $\nabla f(y_1, \widetilde{x})$ exists for a.e. y_1 such that $(y_1, \widetilde{x}) \in D$. We know that a.e. $\widetilde{x} \in F$ is like that, and we denote this set by G. We also know that if $\widetilde{x} \in G$, then $\nabla f(y_1, \widetilde{x}) = g(y_1, \widetilde{x})$ for those a.e. y_1 where it exists. Then for every $(y_1, \widetilde{x}) \in D$ and sufficiently close to $(x_1, \widetilde{x})$, we have

$$f(y_1, \widetilde{x}) = f(x_1, \widetilde{x}) + \int_{x_1}^{y_1} \partial_1 f(z_1, \widetilde{x})\, dz_1 = f(x_1, \widetilde{x}) + \int_{x_1}^{y_1} g_1(z_1, \widetilde{x})\, dz_1.$$

But this implies that $\partial_1 f(x_1, \widetilde{x})$ exists and is equal to $g(x_1, \widetilde{x})$. Assume now that $\widetilde{x}$ belongs to the exceptional measure zero set $F \setminus G$ where $\nabla f(y_1, \widetilde{x})$ fails to exist for a set of y_1 of positive measure. But then we can find $\widetilde{x}_n \in G$ such that $\widetilde{x}_n \to \widetilde{x}$ as $n \to \infty$. For each $\widetilde{x}_n$, we have

$$f(y_1, \widetilde{x}_n) = f(x_1, \widetilde{x}_n) + \int_{x_1}^{y_1} g_1(z_1, \widetilde{x}_n)\, dz_1$$

for all y_1 close enough to x_1. Passing to the limit in this equality, we find

$$f(y_1, \widetilde{x}) = f(x_1, \widetilde{x}) + \int_{x_1}^{y_1} g_1(z_1, \widetilde{x})\, dz_1.$$

This implies that $\partial_1 f(x_1, \widetilde{x})$ exists and is equal to $g_1(x_1, \widetilde{x})$ in this case, too. □

Exercise 3.3.7 Work out the details of the above argument in the case of $(x_1, \widetilde{x}) \in \partial D$.

We conclude that the following lemma holds.

Lemma 3.3.8 *For every $t \geq 0$, the function $\partial_j \Phi_t^k(x)$ belongs to $C^{\alpha(t)}(\bar{D})$ and (3.3.7) holds for all x, t.*

Now, the proof of Theorem 3.3.3 in the case $k = 1$ is straightforward.
Proof. Indeed, since $\Phi_t(x)$ is measure preserving, we have

$$\det \nabla \Phi_t = 1,$$

and then the derivatives of the inverse map $\Phi_t^{-1}(x)$ satisfy the bounds analogous to those of Φ_t (this can also be derived by solving the backwards characteristic equation). Then, Lemma 3.3.8 implies immediately that

$$\omega(t, x) = \omega_0(\Phi_t^{-1}(x))$$

is $C^1(\bar{D})$ for all times. □

Exercise 3.3.9 Carry out the analogous computations for $k > 1$, proving Theorem 3.3.3 in this case.

3.4 Examples of stationary solutions of the 2D Euler equations

Here, we discuss some basic examples of the stationary 2D Euler flows. We will construct them on the two-dimensional torus $\mathbb{T}^2 = [-\pi,\pi]\times[-\pi,\pi]$ rather than in a bounded domain, as the periodic boundary conditions are particularly convenient for explicit computations. In the periodic case, the Biot-Savart law has the form

$$u = \nabla^\perp(-\Delta)^{-1}\omega,$$

where $(-\Delta)$ is the Lapalacian on $\mathbb{T}^2$. The inverse of the Laplacian is easiest to define through the Fourier transform:

$$(-\Delta)^{-1}f(x) = \sum_{k\in\mathbb{Z}^2} e^{ikx}|k|^{-2}\hat{f}(k),$$

where

$$\hat{f}(k) = \int_{\mathbb{T}^2} e^{-ikx}f(x)\,\frac{dx}{(2\pi)^2}.$$

Note that the inverse Laplacian is only defined for functions which have mean zero. We will assume in this section that ω_0 satisfies this condition:

$$\int_{\mathbb{T}^2}\omega_0(x)dx = 0.$$

As the solution $\Phi_t(x)$ of the 2D Euler equations is measure-preserving, we see that then

$$\int_{\mathbb{T}^2}\omega(t,x)dx = \int_{\mathbb{T}^2}\omega_0(\Phi_t^{-1}(x))dx = \int_{\mathbb{T}^2}\omega_0(x)dx = 0,$$

so that the mean-zero condition on the vorticity holds for all times.

A stationary solution of the 2D Euler equations satisfies

$$(u\cdot\nabla)\omega = 0,$$

or

$$(\nabla^\perp(-\Delta)^{-1}\omega)\cdot\nabla\omega = 0, \tag{3.4.1}$$

at every x. Let us denote by

$$\psi = (-\Delta)^{-1}\omega$$

the stream function of the flow, so that $u = \nabla^\perp\psi$, and (3.4.1) is

$$\nabla^\perp\psi\cdot\nabla\omega = 0.$$

As

$$\omega = -\Delta\psi,$$

the flow is clearly stationary if the stream function satisfies

$$-\Delta\psi = f(\psi),$$

for some smooth function f. The simplest examples of stream functions of stationary flows are the eigenfunctions of the periodic Laplacian, with $f(\psi) = \lambda\psi$.

A shear flow

Consider a flow with the stream function

$$\psi(x_1, x_2) = \omega(x_1, x_2) = \cos x_2,$$

which is an eigenfunction of the Laplacian on the torus. The corresponding flow, called a shear flow, is unidirectional:

$$u(x_1, x_2) = (-\sin x_2, 0). \tag{3.4.2}$$

The trajectories of such shear flow are horizontal straight lines:

$$X_1(t) = x_1 - t\sin x_2, \quad X_2(t) = x_2. \tag{3.4.3}$$

Exercise 3.4.1 Describe how a vertical interval $\gamma_0 = \{(0, x_2), \ -\pi \le x_2 \le \pi\}$ is evolved by the shear flow.

A cellular flow

A cellular flow has the stream function

$$\psi(x_1, x_2) = \frac{1}{2}\omega(x_1, x_2) = -\sin x_1 \sin x_2,$$

which is also an eigenfunction of the Laplacian on the torus, and the corresponding flow is

$$u(x_1, x_2) = (-\cos x_2 \sin x_1, \cos x_1 \sin x_2).$$

This flow has four vortices in the four quadrants of the torus $[-\pi, \pi] \times [-\pi, \pi]$, and, in particular, a hyperbolic point at the origin. More generally, we informally refer to a flow as cellular if it is smooth, and its streamlines have the same geometry: four vortices separated by separatrices which are straight lines.

The "singular cross"

An important example of a Yudovich solution of the 2D Euler equations is the "singular cross" flow, considered by Bahouri and Chemin [5]. It corresponds to the vorticity ω_0 which equals to (-1) in the first and third quadrants of the torus $(-\pi, \pi] \times (-\pi, \pi]$, and to $(+1)$ in the other two quadrants:

$$\begin{aligned} &\omega_0(x_1, x_2) = -1 \text{ for } \{0 \le x_1, x_2 \le \pi\} \text{ and } \{-\pi \le x_1, x_2 \le 0\}, \\ &\omega_0(x_1, x_2) = 1 \text{ for } \{0 \le x_1 \le \pi, \ -\pi \le x_2 \le 0\}, \text{ and } \{-\pi \le x_1 \le 0, \ 0 \le x_2 \le \pi\}. \end{aligned} \tag{3.4.4}$$

As for a cellular flow, the singular cross has four vortices, one in each quadrant of the torus, and a hyperbolic point at the origin. We will next verify that the above ω_0 is a stationary Yudovich solution of the Euler equations. To do this, we will consider the vorticity equation with the initial condition ω_0 and show that the solution equals to ω_0 for all $t \ge 0$. We will see that for this initial condition, the fluid velocity u is just log-Lipschitz, and the flow map $\Phi_t(x)$ is indeed only Hölder continuous with the exponent that is exponentially decaying in time.

The first step is an important conservation of symmetry. Note that ω_0 has the symmetries

$$\omega_0(x_1, x_2) = -\omega_0(-x_1, x_2) = -\omega_0(x_1, -x_2) \tag{3.4.5}$$

on the torus $(-\pi, \pi] \times (-\pi, \pi]$.

Lemma 3.4.2 *If the initial condition $\omega_0 \in L^\infty$, and satisfies the symmetries (3.4.5), then the Yudovich solution of the 2D Euler equations satisfies the same symmetries for all $t \geq 0$:*

$$\omega(t, x_1, x_2) = -\omega(t, -x_1, x_2) = -\omega(t, x_1, -x_2). \tag{3.4.6}$$

This result, and the proof below, also apply when the 2D Euler equations are set in a domain D that is symmetric with respect to the coordinate axes.

Proof. Clearly, it is sufficient to prove the conservation of the odd symmetry with respect to the $\{x_1 = 0\}$ axis, since the choice of the coordinates does not affect the properties of the solutions. Suppose that the triple $(\omega(t, x_1, x_2), u(t, x_1, x_2), \Phi_t(t, x_1, x_2))$ is a Yudovich solution of the 2D Euler equations, and set

$$\omega^{(1)}(t, x_1, x_2) = -\omega(t, -x_1, x_2).$$

The stream functions $\psi(t, x_1, x_2)$ and $\psi^{(1)}(t, x_1, x_2)$ satisfy

$$-\Delta\psi(t, x_1, x_2) = \omega(t, x_1, x_2),$$

and

$$-\Delta\psi^{(1)}(t, x_1, x_2) = \omega^{(1)}(t, x_1, x_2) = -\omega(t, -x_1, x_2).$$

It is easy to see that these stream functions are related via

$$\psi^{(1)}(t, x_1, x_2) = -\psi(t, -x_1, x_2). \tag{3.4.7}$$

Hence, the corresponding flows

$$u(t, x_1, x_2) = \nabla^\perp\psi(t, x_1, x_2),$$

and

$$u^{(1)}(t, x_1, x_2) = \nabla^\perp\psi^{(1)}(t, x_1, x_2),$$

are related by

$$\begin{aligned} u^{(1)}(t, x_1, x_2) &= (\partial_2\psi^{(1)}(t, x_1, x_2), -\partial_1\psi^{(1)}(t, x_1, x_2)) = (-\partial_2\psi(t, -x_1, x_2), -\partial_1\psi(t, -x_1, x_2)) \\ &= (-u_1(t, -x_1, x_2), u_2(t, -x_1, x_2)). \end{aligned} \tag{3.4.8}$$

The flow map corresponding to $u^{(1)}$ is the solution of

$$\frac{d\Phi^{(1)}_{t,1}}{dt} = -u_1(t, -\Phi^{(1)}_{t,1}, \Phi^{(1)}_{t,2}), \quad \frac{d\Phi^{(1)}_{t,2}}{dt} = u_2(t, -\Phi^{(1)}_{t,1}, \Phi^{(1)}_{t,2}), \quad \Phi^{(1)}_t(0, x_1, x_2) = (x_1, x_2),$$

and is given by

$$\Phi^{(1)}_t(t, x_1, x_2) = (-\Phi^1_t(-x_1, x_2), \Phi^2_t(-x_1, x_2)).$$

Note that $\Phi^{(1)}_t$ is bijective on D and measure preserving since $u^{(1)}$ is incompressible and log-Lipschitz. Next, note that

$$\begin{aligned} &\omega^{(1)}\left(t, \Phi^{(1)}_{t,1}(x), \Phi^{(1)}_{t,2}(x)\right) = \omega^{(1)}\left(t, -\Phi^1_t(-x_1, x_2), \Phi^2_t(-x_1, x_2)\right) \\ &= -\omega\left(t, \Phi^1_t(-x_1, x_2), \Phi^2_t(-x_1, x_2)\right) = -\omega_0(-x_1, x_2) = \omega^{(1)}(0, x_1, x_2) \equiv \omega^{(1)}_0(x_1, x_2). \end{aligned}$$

Therefore, $\omega^{(1)}$ satisfies

$$\omega^{(1)}(t,x) = \omega_0^{(1)}\left((\Phi_t^{(1)})^{-1}(x)\right).$$

Hence, the triple

$$(\omega^{(1)}(t,x_1,x_2)), u^{(1)}(t,x_1,x_2), \Phi_t^{(1)}(t,x_1,x_2))$$

is also a Yudovich solution of the 2D Euler equations with the initial condition

$$\omega_0^{(1)}(x_1,x_2) = -\omega_0(-x_1,x_2).$$

If ω_0 is odd with respect to x_1, the initial data for (ω,u,Φ_t) and $(\omega^{(1)},u^{(1)},\Phi_t^{(1)})$ coincide. Hence by uniqueness of solutions these two solutions must coincide, and the therefore the symmetry is preserved in time. □

Now we can verify that the singular cross solution is a stationary solution.

Lemma 3.4.3 *The solution of the 2D Euler equations with the "singular cross" initial condition ω_0 is stationary, that is, $\omega(t,x) \equiv \omega_0(x)$ for all $t \geq 0$.*

Proof. Since ω_0 is odd with respect to both x_1 and x_2, by Lemma 3.4.2 the solution $\omega(t,x)$ has the same property. According to (3.4.8), u_1 is odd with respect to x_1, hence

$$u_1(t,0,x_2) = 0 \text{ for all } t \geq 0.$$

Similarly, we have

$$u_2(t,x_1,0) = 0 \text{ for all } t \geq 0.$$

An identical argument shows that

$$u_1(t,\pi,x_2) = u_2(t,x_1,\pi) = 0.$$

This and log-Lirschitz property of u shows that the particle trajectories never cross the lines $x_1 = 0,\pi$ and $x_2 = 0,\pi$. Thus, the whole trajectory $\Phi_t(x)$ which starts inside one of the four quadrants of the torus, will stay inside that quadrant. As

$$\omega(t,x) = \omega_0(\Phi_t^{-1}(x)),$$

and ω_0 is constant in each of the four quadrants, this shows that

$$\omega(t,x) \equiv \omega_0,$$

for all $t \geq 0$. □

The periodic Biot-Savart law

The following more explicit form of the periodic Biot-Savart law will be useful for us in showing that the flow generated by the singular cross is only log-Lipschitz, as well as in other examples in this chapter.

Proposition 3.4.4 *Let $\omega \in L^\infty(\mathbb{T}^2)$ be a mean zero function. Then the vector field*

$$u = \nabla^\perp(-\Delta)^{-1}\omega \tag{3.4.9}$$

is given by

$$u(x) = -\frac{1}{2\pi}\lim_{\gamma\to 0}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega(y)e^{-\gamma|y|^2}\,dy, \tag{3.4.10}$$

where ω has been extended periodically to all $\mathbb{R}^2$.

As will be clear from the proof, the formula holds for a broader class of ω than L^∞. However we will not pursue full generality here since $\omega \in L^\infty$ is all we need.

Proof. Relation (3.4.9) can be re-written as

$$u(x) = \sum_{k\in\mathbb{Z}^2} e^{ikx}\frac{ik^\perp}{|k|^2}\hat{\omega}(k)$$

with $k^\perp = (k_2, -k_1)$. To connect this expression to (3.4.10), observe first that for a smooth ω we have

$$\sum_{k\in\mathbb{Z}^2} e^{ikx}\frac{ik^\perp}{|k|^2}\hat{\omega}(k) = \lim_{\gamma\to 0}\int_{\mathbb{R}^2} e^{ipx}\frac{ip^\perp}{|p|^2}\int_{\mathbb{R}^2} e^{-ipy-\gamma|y|^2}\omega(y)\,\frac{dydp}{(2\pi)^2}, \tag{3.4.11}$$

where the function $\omega(y)$ is extended periodically to the whole plane.

Exercise 3.4.5 Check the above identity by substituting the Fourier series for $\omega(y)$ on the right hand side and integrating in y to obtain the Gaussian approximation of identity.

On the other hand, recall that the inverse Laplacian in the whole plane is given by

$$(-\Delta)^{-1}f(x) = \int_{\mathbb{R}^2} e^{ikx}\frac{1}{|k|^2}\int_{\mathbb{R}^2} e^{-iky}f(y)\,\frac{dydk}{(2\pi)^2} = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|f(y)\,dy \tag{3.4.12}$$

if the function f is sufficiently regular and rapidly decaying (see e.g. [60]). After an integration by parts, the expression in the right side of (3.4.11), with the help of (3.4.12), can be written as

$$\int_{\mathbb{R}^2} e^{ipx}\frac{1}{|p|^2}\int_{\mathbb{R}^2} e^{-ipy}\nabla^\perp\left(\omega(y)e^{-\gamma|y|^2}\right)\frac{dydp}{(2\pi)^2} = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\nabla^\perp\left(\omega(y)e^{-\gamma|y|^2}\right).$$

Integrating by parts, we obtain (3.4.10).

To extend (3.4.10) to ω which are just bounded, let us first show that the limit in (3.4.10) exists in this case. There is not enough decay in the integrand on the right hand side for absolute convergence when $\gamma = 0$, but periodicity and the mean zero property of ω turn out to be sufficient.

Take $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \phi(x) \le 1$, $\phi(x) = 1$ if $x \in B_{10}$, and $\phi(x) = 0$ if $x \in B_{20}^c$. Write the right hand side of (3.4.10) as

$$\lim_{\gamma\to 0}\left(\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\phi(x-y)\omega(y)e^{-\gamma|y|^2}\,dy + \int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}(1-\phi(x-y))\omega(y)e^{-\gamma|y|^2}\,dy\right). \tag{3.4.13}$$

Then in the first summand we can pass to the limit without any problem. In the second summand, write $\omega(y) = -\Delta\psi$; the function ψ satisfying this equality exists due to periodicity and the mean zero property of ω and can be found explicitly on the Fourier side. Note that by Sobolev imbedding, $\|\psi\|_{L^\infty} \leq C\|\omega\|_{L^\infty}$. In the second summand in (3.4.13), we will integrate by parts two times, taking Δ off ψ. If any of the derivatives fall on ϕ, we get finite integration region and can pass to the limit as well. If both derivatives fall on $\frac{(x-y)^\perp}{|x-y|^2}$, we get enough decay for absolute convergence when $\gamma = 0$, and so can pass to the limit by dominated convergence theorem. If both derivatives fall on $e^{-\gamma|y|^2}$, the resulting term is bounded by

$$C\gamma\|\psi\|_{L^\infty}\int_{\mathbb{R}^2\setminus B_{10}}\frac{1}{|y|}(1+\gamma|y|^2)e^{-\gamma|y|^2}\,dy \leq C\gamma^{1/2}\|\omega\|_{L^\infty}\int_{\mathbb{R}^2}\left(1+\frac{1}{|z|}\right)e^{-|z|^2}\,dz,$$

where we applied scaling $z = \gamma^{1/2}y$.

Exercise 3.4.6 Consider the remaining case where one derivative falls on $\frac{(x-y)^\perp}{|x-y|^2}$, while the other on $e^{-\gamma|y|^2}$.

Now that we know that the limit in (3.4.10) exists for $\omega \in L^\infty$, take $\omega_n \in C^\infty$ that converges to ω in all L^p, $p < \infty$: a standard approximation of identity mollification suffices here. Then ψ_n converge to ψ in $W^{2,p}$, and by the Sobolev imbedding $\|\psi - \psi_n\|_{L^\infty} \to 0$. Following through the arguments above, we can check that the limit in the right hand of (3.4.10) for ω_n converges to that for ω for every x. On the other hand, we proved that for smooth ω_n, (3.4.10) holds, and it is straightforward to check from (3.4.9) and Sobolev imbedding that u_n converges to u uniformly. Together, these three observations complete the proof of (3.4.10) for $\omega \in L^\infty$.

Expression (3.4.10) shows that the 2D Euler evolution on a torus can be equivalently viewed as the evolution on the plane with periodic initial data if we understand the Biot-Savart law in the sense of the principal value integral (3.4.10).

Log-Lipschitzianity of the singular cross

Next, we show that the flow of the singular cross is only log-Lipschitz, and not Lipschitz.

Proposition 3.4.7 *Consider the singular cross solution described above. Then, for small positive x_1, we have*

$$u_1(x_1,0) = \frac{4}{\pi}x_1\log x_1 + O(x_1). \tag{3.4.14}$$

The estimate (3.4.14) corresponds to u_1 being just log-Lipchitz near the origin. Hence the estimates on the fluid velocity in the Yudovich theory are qualitatively sharp.

Proof. Let us use the Biot-Savart law (3.4.10)

$$u_1(x_1,0) = \frac{1}{2\pi}\lim_{\gamma\to 0}\int_{\mathbb{R}^2}\frac{y_2}{(x_1-y_1)^2+y_2^2}\omega_0(y)e^{-\gamma|y|^2}\,dy, \tag{3.4.15}$$

with $\omega_0(y)$ given by (3.4.4). Let us denote $S = [-1,1]\times[-1,1]$, and represent $u_1(x_1,0)$ as a sum of two components:

$$u_1(x_1,0) = u_1^S(x_1,0) + u_1^F(x_1,0).$$

Here, $u_1^S(x_1,0)$ is the contribution from the integration over S in (3.4.15) (the "near field"), while $u_1^F(x_1,0)$ comes from the integration over the complement of S (the "far field"). We first claim the result of the following exercise.

Exercise 3.4.8 Verify that, for $0 \le x_1 \le 1/2$, we have

$$\left|\lim_{\gamma\to 0}\int_{\mathbb{R}^2\setminus S}\frac{y_2}{(x_1-y_1)^2+y_2^2}\omega_0(y)e^{-\gamma|y|^2}\,dy\right| \le Cx_1,$$

and thus

$$|u_1^F(x_1,0)| \le Cx_1,$$

One way to perform this computation is to use the odd symmetries of ω_0, and its mean-zero property, which leads to an extra cancellation and effectively a faster decay in the kernel. Alternatively, one can adapt an argument similar to the one in the proof of Proposition 3.4.4.

In the term $u_1^S(x_1,0)$, we can freely pass to the limit $\gamma \to 0$ and use the symmetry to simplify the expression:

$$\begin{aligned}\pi u_1^S(x_1,0) &= \frac{1}{2}\int_S \frac{y_2}{(x_1-y_1)^2+y_2^2}\omega_0(y)\,dy = \int_0^1 dy_2\int_{-1}^1 \frac{y_2}{(x_1-y_1)^2+y_2^2}\omega_0(y_1,y_2)dy_1\\ &= -2x_1\int_0^1\int_0^1 \frac{y_1y_2}{((x_1-y_1)^2+y_2^2)((x_1+y_1)^2+y_2^2)}\,dy_1dy_2. \qquad (3.4.16)\end{aligned}$$

In the last step, we used that $\omega_0(y_1,y_2)=-1$ on $[0,1]\times[0,1]$. Let us consider the contributions from different regions of integration in (3.4.16). The integral over the region $[0,1]\times[0,2x_1]$ can be estimated as

$$\begin{aligned}&\int_0^1\int_0^{2x_1}\frac{y_1y_2}{((x_1-y_1)^2+y_2^2)((x_1+y_1)^2+y_2^2)}\,dy_1dy_2\\ &\le C\int_0^1 dz_2\int_0^{x_1}dz_1\frac{x_1z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)} \le C\int_0^{x_1}\frac{x_1}{x_1^2+z_2^2}\,dz_2 \le C.\end{aligned}$$

The region $[0,2x_1]\times[2x_1,1]$ contributes

$$\begin{aligned}&\int_0^{2x_1}dy_1\int_{2x_1}^1 dy_2\frac{y_1y_2}{((x_1-y_1)^2+y_2^2)((x_1+y_1)^2+y_2^2)}\,dy_1dy_2\\ &\le C\int_0^{x_1}dz_1\int_{x_1}^1 dz_2\frac{x_1z_2}{(z_1^2+z_2^2)^2} \le C\int_0^{x_1}\frac{x_1dz_1}{z_1^2+x_1^2} \le C.\end{aligned}$$

We need to be slightly more careful in the region $[2x_1,1]\times[2x_1,1]$. Here, the first observation is

Exercise 3.4.9 Show that

$$\left|\int_{2x_1}^1\int_{2x_1}^1\frac{y_1y_2}{((x_1-y_1)^2+y_2^2)((x_1+y_1)^2+y_2^2)}\,dy_1dy_2 - \int_{x_1}^1\int_{x_1}^1\frac{y_1y_2}{(y_1^2+y_2^2)^2}dy_1dy_2\right| \le C.$$

The second step is to note that

$$\begin{aligned}\int_{x_1}^1\int_{x_1}^1\frac{y_1y_2}{(y_1^2+y_2^2)^2}dy_1dy_2 &= \int_{x_1}^1 y_1\left(\frac{1}{y_1^2+x_1^2}-\frac{1}{y_1^2+1}\right)dy_1\\ &= \int_{x_1^2}^1\frac{dz_1}{z_1+x_1^2}+O(1) = -2\log x_1+O(1).\end{aligned}$$

Collecting all the estimates, we arrive at (3.4.14).

The Hölder regularity of the singular cross flow map

A characteristic curve starting at a point $(x_1^0, 0)$, with $x_1^0 \in (0, \pi)$ is just the interval

$$\Phi_t((x_1^0, 0)) \equiv (x_1(t), 0),$$

moving towards the origin. If x_1^0 is sufficiently small, the component $x_1(t)$ will satisfy

$$\dot{x}_1(t) \le x_1(t) \log x_1(t),$$

and so

$$\frac{d}{dt}(\log x_1(t)) \le \log x_1(t),$$

so that

$$\log x_1(t) \le e^t \log x_1^0,$$

and

$$x_1(t) \le x_1(0)^{\exp(t)}. \tag{3.4.17}$$

This estimate has an interesting consequence for the Hölder regularity of the flow map. Since the origin is a stationary point of the flow, the inverse flow map $\Phi_t^{-1}(x)$ can be Hölder continuous only with a decaying in time exponent (at most e^{-t}). In fact, the exponent is a little weaker than that since our estimate on the characteristic convergence to zero is not sharp. Of course, the direct flow map $\Phi_t(x)$ also has a similar property; to establish it one needs to look at characteristic lines moving along the vertical separatrix.

Exercise 3.4.10 Verify the latter claim by a direct calculation. You do not have to redo the proof of Proposition 3.4.7, you can use symmetry to conclude the analogous asymptotic behavior for $u_2(0, x_2)$, with a different sign.

This observation shows that the Hölder bounds on the flow map in Yudovich theory are also qualitatively sharp.

3.5 An upper bound on the growth of the gradient of vorticity

We return now to the time-dependent two-dimensional Euler equations (3.1.21) in a smooth bounded domain D:

$$\begin{aligned} &\partial_t \omega + (u \cdot \nabla)\omega = 0, \\ &u = \nabla^\perp (-\Delta_D)^{-1} \omega, \\ &\omega(0, x) = \omega_0(x), \end{aligned} \tag{3.5.1}$$

with $\nabla^\perp = (\partial_2, -\partial_1)$. Recall that the boundary condition

$$u \cdot n = 0, \text{ on } \partial D, \tag{3.5.2}$$

holds automatically – see the remark below (3.1.23). We now consider a regular initial vorticity ω_0, and ask how fast the higher derivatives of the solution may grow. This issue is related to the small scale creation in fluids, a phenomenon that is ubiquitous in applications in physics and engineering. We witness this process in observing thin filaments in turbulent flows, in the structure of hurricanes and in boiling water in our kitchen. The main result in this section addresses such upper bound on the growth of the small scales in solutions. A similar bound is implicit already in the work of Wolibner [137] and Hölder [85], and has been stated explicitly by Yudovich.

Theorem 3.5.1 *Assume that $\omega_0 \in C^1(\bar{D})$. Then the gradient of the solution $\omega(t,x)$ satisfies the following bound*

$$\|\nabla\omega(\cdot,t)\|_{L^\infty} \leq (\|\nabla\omega_0\|_{L^\infty}+1)^{Ct\exp\|\omega_0\|_{L^\infty}} \tag{3.5.3}$$

for all $t \geq 0$.

This upper bound grows at a double exponential rate in time which is extremely fast. We will later see that, actually, this bound is sharp, at least in domains with boundaries.

The gradient growth for passive scalars

The 2D Euler vorticity is an active scalar: it satisfies an advection equation

$$\omega_t + (u\cdot\nabla)\omega = 0, \tag{3.5.4}$$

with a flow $u(t,x)$ which is related to the vorticity itself via the Biot-Savart law. Before we look at the vorticity gradient growth, and go to the proof of Theorem 3.5.1, let us see what happens for passive scalars: these are solutions of the advection equation

$$\varphi_t + (u\cdot\nabla)\varphi = 0, \quad \varphi(0,x) = \varphi_0(x), \tag{3.5.5}$$

with a prescribed flow $u(t,x)$ which does not depend on the solution $\varphi(t,x)$ of (3.5.5). The equation has exactly the same form as the 2D Euler vorticity equation, except u and φ are not coupled: u is given and the initial data φ_0 is arbitrary, and doesn't have to be the vorticity of $u(0,x)$. In order to see what can happen, we consider some explicit examples, coming from the stationary solutions of the 2D Euler equations that we have discussed above.

A shear flow

First, let us look at a passive scalar advected by the shear flow

$$u(x_1,x_2) = (-\sin x_2, 0). \tag{3.5.6}$$

Recall that the trajectories of this flow are

$$\Phi_t^1(x) = x_1 - t\sin x_2, \quad \Phi_t^2(x) = x_2. \tag{3.5.7}$$

The passive scalar $\varphi(t,x)$ is a solution of

$$\partial_t\varphi + (u\cdot\nabla)\varphi = 0, \;\; \varphi(x,0) = \varphi_0(x). \tag{3.5.8}$$

For the shear flow (3.5.6), we can solve for $\varphi(x,t)$ explicitly:

$$\varphi(t,\Phi_t^1(x),\Phi_t^2(x)) = \varphi_0(x_1,x_2).$$

Taking into account expressions (3.5.7) for $\Phi_t^1(x),\Phi_t^2(x)$, we get

$$\varphi(t,x_1,x_2) = \varphi_0(x_1 + t\sin x_2, x_2).$$

Therefore, $\nabla\varphi(t,x)$ grows only linearly in t in a shear flow.

A cellular flow

Next, consider a cellular flow

$$u(x_1,x_2) = (-\cos x_2 \sin x_1, \cos x_1 \sin x_2).$$

This flow is similar to the singular cross flow: it has four vortices in the four quadrants of the torus $[-\pi,\pi]\times[-\pi,\pi]$, and, in particular, a hyperbolic point near the origin. However, unlike the singular cross flow, the cellular flow is smooth. Note that near the origin the flow looks like $u \sim (x_1,-x_2)$, so that e_1 is the contracting direction and e_2 is the unstadle direction. A trajectory starting at a point $(x_1,0)$ on the x_1-axis is a straight line with $\Phi_t^1(x)$ a solution of

$$\dot{\Phi}_t^1(x) = -\sin\Phi_t^1(x).$$

Thus, if x_1 is small, then

$$\Phi_t^1(x) \sim x_1 e^{-t}.$$

Then for a solution $\varphi(x_1,x_2,t)$ of the passive scalar equation (3.5.8) with such u, we have

$$\varphi(t,x_1,0) \sim \varphi_0(x_1 e^t, 0). \tag{3.5.9}$$

Therefore, $\nabla\varphi(t,x)$ grows exponentially in time for a cellular flow.

The exponential in time growth of $\nabla\varphi(t,x)$ is actually the fastest one can have for a passive scalar advected by a smooth flow.

Exercise 3.5.2 Prove that if the flow $u = u(x)$ in (3.5.8) is smooth and time-independent, and the initial data $\varphi_0 \in C^\infty(\mathbb{T}^2)$, then

$$\|\varphi(x,t)\|_{H^s} \le Ce^{Ct},$$

for all times $t > 0$, where C depends only on u, s and φ_0. To show this, differentiate the passive scalar equation, and consider the resulting equation for $\varphi_j(t,x) = \partial\varphi/\partial x_j$.

A passive scalar advected by a singular cross

Looking back at a passive scalar advected by a cellular flow, we see that the limitation on the growth of $\nabla\varphi(t,x)$ comes from (3.5.9): a cellular flow trajectory starting at a point $(0,x_2)$ on the vertical line (with a small $x_2 > 0$) will approach the fixed point $(0,0)$ exponentially fast in time but not faster. The reason for this is the Lipschitz regularity of the flow. As we have previously seen, for a singular cross, these points get close at a rate which is doubly exponential in time. To articulate this further, consider a passive scalar advected by the singular cross flow:

$$\partial_t \varphi + (u \cdot \nabla)\varphi = 0, \quad \varphi(0, x) = \varphi_0(x).$$

Choose $\varphi_0(x)$ to be a smooth function such that $\varphi_0(0) = 0$ and $\varphi_0(\delta) = 1$ for a small number $\delta > 0$ such that $u_1(x_1, 0) \leq x_1 \log x_1$ for all $0 \leq x_1 \leq \delta$. Then, we have

$$\varphi(\Phi_t((\delta, 0)), t) = \varphi_0(\delta) = 1,$$

and $\varphi(0, t) = 0$ since the origin is a stagnation point. On the other hand, due to our estimates on the singular cross flow, we know that

$$\Phi_t(\delta, 0) \leq \delta^{\exp(t)}.$$

By the mean value theorem, we conclude that

$$\|\nabla \varphi(\cdot, t)\|_{L^\infty} \geq \delta^{-\exp(t)},$$

thus resulting in double exponential growth in the gradient of passively advected scalar. Thus, the singular cross flow can lead to a double exponential growth in the gradient of the passive scalar. The question we will soon look into is whether such scenario is also relevant for the 2D Euler equations with smooth initial data, where the vorticity is advected by the flow but is not a passive scalar.

The Kato inequality

A key step in the proof of Theorem 3.5.1 is the following inequality due to Kato.

Proposition 3.5.3 (Kato) *Let D be a smooth compact domain, $\omega \in C^\alpha(D)$, $\alpha > 0$, and u be given by the usual Biot-Savart law*

$$u = \nabla^\perp (-\Delta_D)^{-1} \omega.$$

Then

$$\|\nabla u\|_{L^\infty} \leq C(\alpha, D)\|\omega\|_{L^\infty} \left(1 + \log\left(1 + \frac{\|\omega\|_{C^\alpha}}{\|\omega\|_{L^\infty}}\right)\right). \tag{3.5.10}$$

The operators $\partial_{jk}(-\Delta)^{-1}$ are called the (iterated) Riesz transforms. The Calderon-Zygmund theory proves that the Riesz transforms are bounded on all L^p, $1 < p < \infty$ (see, e.g. [130]). The derivatives of the fluid velocity u are exactly the Riesz transforms of the vorticity. However, we need to control the L^∞ norm of ∇u since this is what appears in (3.3.6). The L^∞ bound on the Riesz transform of a function ω in terms of just $\|\omega\|_{L^\infty}$ is not true, and we need a little extra – a logarithm – of a higher order norm of ω to control the L^∞ norm of ∇u.

The proposition also has applications to the three dimensional case, where it leads to a well known conditional regularity statement for the solutions of 3D Euler equation, the Beale-Kato-Majda criterion [12]. In three dimensions, there is no control on $\|\omega\|_{L^\infty}$ anymore. However, using the bound (3.5.10), one can show that the finiteness of the integral

$$\int_0^T \|\omega\|_{L^\infty}\, dt,$$

implies the regularity of the solution on $[0,T]$. Thus $\|\omega\|_{L^\infty}$ "controls" the possible blow up in 3D case: solutions cannot develop a singularity without

$$\int_0^T \|\omega\|_{L^\infty}\,dt$$

also becoming infinite.

Before proving Proposition 3.5.3, we need the following lemma.

Lemma 3.5.4 *Let D be a smooth compact domain, $\omega \in C^\alpha(\bar{D})$ for some $\alpha > 0$, and $u(x)$ be given by the Biot-Savart law*

$$u(x) = \int_D K_D(x,y)\omega(y)\,dy,$$

then

$$\frac{\partial u_i(x)}{\partial x_j} = P.V.\int_D \frac{\partial K_{D,i}(x,y)}{\partial x_j}\omega(y)\,dy + \frac{(-1)^j}{2}\omega(x)(1-\delta_{ij}). \tag{3.5.11}$$

Proof. This is a fairly standard computation so we outline the main steps. Since $\omega \in C^\alpha(\bar{D})$ implies $u \in C^{1,\alpha}(\bar{D})$, it suffices to prove (3.5.11) for every point $x \in D$; the equality will extend to ∂D by continuity. Proposition 3.2.1 implies that $K_D(x,y)$ can be written as

$$K_D(x,y) = \frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2} + \nabla_x^\perp h(x,y). \tag{3.5.12}$$

The function $h(x,y)$ is smooth for $x \in D$, so the contribution of the second term on the ride hand side of (3.5.12) to $\nabla u(x)$ is simply

$$\int_D \nabla_x[\nabla_x^\perp h(x,y)]\omega(y)\,dy.$$

Set $\rho := \mathrm{dist}(x, \partial D) > 0$. Then, the first term in (3.5.12) can be split as

$$u^{(1)}(x) = \frac{1}{2\pi}\int_D \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\eta_\rho(x-y)\,dy + \frac{1}{2\pi}\int_D \frac{(x-y)^\perp}{|x-y|^2}\omega(y)(1-\eta_\rho(x-y))\,dy. \tag{3.5.13}$$

Here, $\eta(y) \equiv \eta(|y|)$ is a smooth cut-off function, so that $\eta(y) = 1$ if $|y| \le 1/2$ and $\eta(y) = 0$ if $|y| \ge 1$, and $\eta_\rho(y) = \rho^{-d}\eta(y/\rho)$. Note that the second term in (3.5.13) can be differentiated in a straightforward fashion, since the integrand is regular in x because of the cut-off. The first term is a convolution

$$u^{(11)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\eta_\rho(x-y)\,dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{y^\perp\eta_\rho(y)}{|y|^2}\omega(x-y)\,dy. \tag{3.5.14}$$

Let us assume for the moment that $\omega \in C^1(\bar{D})$. As the kernel

$$\widetilde{K}(y) = \frac{y^\perp}{|y|^2}\chi_\rho(y)$$

has an integrable singularity at $y=0$, we have

$$\frac{\partial u_i^{(11)}(x)}{\partial x_j}=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(y^\perp)_i}{|y|^2}\frac{\partial \omega(x-y)}{\partial x_j}\eta_\rho(y)\,dy=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(y^\perp)_i}{|y|^2}\frac{\partial \omega(x-y)}{\partial y_j}\eta_\rho(y)\,dy. \tag{3.5.15}$$

One can not immediately integrate by parts in (3.5.15) – that would create a non-integrable singularity in y. Thus, we write

$$\begin{aligned}\frac{\partial u_i^{(11)}(x)}{\partial x_j}&=-\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{(y^\perp)_i}{|y|^2}\frac{\partial \omega(x-y)}{\partial y_j}\eta_\rho(y)\,dy \qquad (3.5.16)\\ &=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\Big[\int_{|y|\geq\varepsilon}\frac{\partial}{\partial y_j}\Big(\frac{(y^\perp)_i}{|y|^2}\eta_\rho(y)\Big)\omega(x-y)\,dy+\int_{|y|=\varepsilon}\frac{(y^\perp)_i}{|y|^2}\frac{y_j}{|y|}\omega(x-y)\eta_\rho(y)\,dy\Big].\end{aligned}$$

The first term in (3.5.16), combined with the corresponding derivatives of the second terms on the right hand sides of (3.5.13) and (3.5.12), contributes the term involving derivatives of $K_D(x,y)$ in (3.5.11). The second term in (3.5.16) gives, in the limit $\varepsilon\to 0$:

$$\frac{1}{2\pi}\omega(x)\int_{|y|=1}(y^\perp)_i y_j dy=\frac{(-1)^j}{2}(1-\delta_{ij})\omega(x),$$

which is the second term in (3.5.11). The result holds for $\omega\in C^\alpha$ by approximation with C^1 functions.

Exercise 3.5.5 Carry out all the details of the computation above.

The proof of Proposition 3.5.3

Let us set

$$\delta=\min\left(c,\Big(\frac{\|\omega_0\|_{L^\infty}}{\|\omega(x,t)\|_{C^\alpha}}\Big)^{1/\alpha}\right).$$

Here, $c>0$ is some fixed constant that depends on D, chosen so that the set of points $x\in D$ with $\mathrm{dist}(x,\partial D)\geq 2\delta$ is not empty. Consider first any interior point x such that

$$\mathrm{dist}(x,\partial D)\geq 2\delta.$$

Let us look at the representation (3.5.11). The part of the integral over the complement of the ball centered at x with radius δ can be estimated as

$$\Big|\int_{B^c_\delta(x)}\nabla K_D(x,y)\omega(y)\,dy\Big|\leq C\|\omega_0\|_{L^\infty}\int_{B^c_\delta(x)}|x-y|^{-2}\,dy\leq C\|\omega_0\|_{L^\infty}(1+\log\delta^{-1}). \tag{3.5.17}$$

Here, we used a bound (3.2.9) from the Proposition 3.2.1.

Next, recall, once again, that the Dirichlet Green's function is given by

$$G_D(z,y)=\frac{1}{2\pi}\log|z-y|+h(z,y), \tag{3.5.18}$$

where h is harmonic in D in z for each fixed $y \in D$ and has boundary the value

$$h(z,y) = -\frac{1}{2\pi}\log|z-y|, \quad z \in \partial D.$$

Any second order partial derivative at $z = x$ of the first term in the right side of (3.5.18) is of the form

$$r^{-2}\Omega(\phi),$$

where r, ϕ are the polar coordinates centered at x, and $\Omega(\phi)$ is mean zero. For this part, we can write, using the mean-zero property of $\Omega(\phi)$:

$$\left|P.V.\int_{B_\delta(x)}[\partial^2_{x_ix_j}\log|x-y|]\omega(y)\,dy\right| = \left|\int_{B_\delta(x)}[\partial^2_{x_ix_j}\log|x-y|](\omega(y)-\omega(x))\,dy\right|$$
$$\leq C\|\omega(x,t)\|_{C^\alpha}\int_0^\delta r^{-1+\alpha}\,dr \leq C(\alpha)\delta^\alpha\|\omega(x,t)\|_{C^\alpha} \leq C(\alpha)\|\omega_0\|_{L^\infty} \quad (3.5.19)$$

by our choice of δ.

As for the function h in (3.5.18), note that our assumptions on x, the boundary values for h, and the maximum principle together guarantee that we have

$$|h(z,y)| \leq C\log\delta^{-1},$$

for all $y \in B_\delta(x)$, $z \in D$. Standard estimates for the harmonic functions (see e.g. [60]) give, for each fixed $y \in B_\delta(x)$,

$$|\partial^2_{x_ix_j}h(x,y)| \leq C\delta^{-4}\|h(z,y)\|_{L^1(B_\delta(x),dz)} \leq C\delta^{-2}\log\delta^{-1}.$$

This gives

$$\left|\int_{B_\delta(x)}\partial^2_{x_ix_j}h(x,y)\omega(y,t)\,dy\right| \leq C\|\omega_0\|_{L^\infty}\log\delta^{-1}. \quad (3.5.20)$$

Together, (3.5.20), (3.5.19) and (3.5.17) prove the Proposition at interior points.

Now if x' is such that

$$\mathrm{dist}(x',\partial D) < 2\delta,$$

find a point x such that $\mathrm{dist}(x,\partial D) \geq 2\delta$ and $|x'-x| \leq C(D)\delta$. By the Schauder estimate (see Theorem 3.3.4) we have

$$|\nabla u(x') - \nabla u(x)| \leq C(\alpha,D)\delta^\alpha\|\omega\|_{C^\alpha}. \quad (3.5.21)$$

At x, the interior bounds apply, which together with (3.5.21) gives the desired bound at any $x' \in D$. $\square$

The proof of Theorem 3.5.1

Given Proposition 3.5.3, the proof of the estimate (3.5.3) and so of Theorem 3.5.1 is straightforward. Let us come back to the two sided bound (3.3.6) and use the Kato estimate (3.5.10). We obtain

$$f(t)^{-1} \leq \frac{|\Phi_t(x)-\Phi_t(y)|}{|x-y|} \leq f(t), \quad (3.5.22)$$

where

$$f(t) = \exp\left(C\|\omega_0\|_{L^\infty} \int_0^t \left(1 + \log\left(1 + \frac{\|\nabla\omega(x,s)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)\right) ds \right).$$

Of course, since the bound (3.5.22) is two-sided, it also holds for Φ_t^{-1}. On the other hand, we have

$$\|\nabla\omega(x,t)\|_{L^\infty} = \text{sup}_{x,y} \frac{|\omega_0(\Phi_t^{-1}(x)) - \omega_0(\Phi_t^{-1}(y))|}{|x-y|} \leq \|\nabla\omega_0\|_{L^\infty} \text{sup}_{x,y} \frac{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}{|x-y|}. \tag{3.5.23}$$

Combining (3.5.23) and (3.5.22), we obtain

$$\|\nabla\omega(x,t)\|_{L^\infty} \leq \|\nabla\omega_0\|_{L^\infty} \exp\left(C\|\omega_0\|_{L^\infty} \int_0^t \left(1 + \log\left(1 + \frac{\|\nabla\omega(x,s)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)\right) ds \right),$$

or

$$\log\|\nabla\omega(x,t)\|_{L^\infty} \leq \log\|\nabla\omega_0\|_{L^\infty} + C\|\omega_0\|_{L^\infty} \int_0^t \left(1 + \log\left(1 + \frac{\|\nabla\omega(x,s)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)\right) ds.$$

Let $A = ||\omega_0||_{L^\infty}$, $B = ||\nabla\omega_0||_{L^\infty}$ and consider the solution $y = y(t)$ of

$$\frac{\dot{y}}{y} = CA\left(1 + \log(1+y)\right), \qquad y(0) = \frac{B}{A} = y_0 \,. \tag{3.5.24}$$

By Gronwall's lemma it is enough to bound $y(t)$. The solution of (3.5.24) is given by

$$\int_{y_0}^{y(t)} \frac{dy}{y\left(1 + \log(1+y)\right)} = CAt\,, \tag{3.5.25}$$

hence

$$\begin{aligned} &\log\left(1 + \log(1 + y(t))\right) - \log\left(1 + \log(1+y_0)\right) \\ &\quad + \int_{y_0}^{y(t)} dy \left[\frac{1}{y(1+\log(1+y))} - \frac{1}{(1+y)(1+\log(1+y))}\right] = CAt\,. \end{aligned}$$

The integrand in the last expression is positive and hence

$$1 + \log(1 + y(t)) \leq (1 + \log(1+y_0))\exp(CAt)\,. \tag{3.5.26}$$

This implies the double exponential upper bound we seek. □

We note that although we gave a proof of Theorem 3.5.1 in the case of the smooth bounded domain, it can be proved in a similar way for the case of the torus. This is due to the fact that the Green's function of the Laplacian on the torus satisfies the same estimates as in Proposition 3.2.1.

Exercise 3.5.6 Verify this claim.

The question of how sharp the double exponential bound is has been open for a long time. This is what we will discuss in the rest of this chapter.

3.6 The Denisov example

Theorem 3.5.1 gives only an upper bound on the growth of the gradient of the vorticity of the solutions of the 2D Euler equations, but not a way to construct solutions for which the gradient actually does grow. The first works constructing examples of flows with growth in the derivatives of vorticity are due to Yudovich [140, 141]. He considered growth on the boundary of the domain, and his construction required the boundary to have a flat piece. His bounds on the growth are not explicit, but it is shown that

$$\limsup_{t\to\infty} \|\nabla\omega(\cdot,t)\|_{L^\infty} = \infty.$$

Generally, the small scale generation at the boundary fits well with the physical intuition. It is known that boundaries generate interesting phenomena in fluid motion, and in particular influence onset of turbulence (see e.g. [77]). In later works [95, 109], it was shown that the small scale generation at the boundary is in some sense generic.

The basic idea behind many examples of the vorticity gradient growth is simple: find a stable stationary flow u_0 and perturb it a little. The background vorticity ω_0 satisfies

$$u_0 \cdot \nabla\omega_0 = 0.$$

The full vorticity of the perturbed flow $u(t,x)$ satisfies

$$\omega_t + u \cdot \nabla\omega = 0.$$

Writing $u = u_0 + v$, and $\omega = \omega_0 + \eta$, we obtain

$$\eta_t + u_0 \cdot \nabla\eta + v \cdot \nabla\omega_0 + v \cdot \nabla\eta = 0.$$

The point is that the background stable flow u_0 should be chosen so that if φ is a passive scalar advected by u_0:

$$\varphi_t + u_0 \cdot \nabla\varphi = 0,$$

then $\nabla\varphi(t,x)$ would grow rapidly for large t. Then the hope is that since u_0 is stable, the perturbation $v(t,x)$ would remain small, and the true nonlinear perturbation η would be similar to φ, and also have a large gradient. The plan is not easy to implement, however, since the problem is strongly nonlinear and nonlocal. No matter how small the perturbation is, it will interact with the background flow, and this interaction is difficult to control for large times.

Nadirashvili [113] used this strategy to construct examples of flows with a linear growth in the vorticity gradient in the bulk, when the domain is an annulus. He called such solutions "wandering", since, at least in a relatively strong norm, they travel to infinity as time passes. The argument is based on constructing a stable background flow that can stretch a small perturbation, creating small scales.

We will follow a similar philosophy and consider the Denisov example. This example provides the best known rate of growth for the gradient of vorticity in the bulk of the fluid, away from the boundary, when starting with smooth initial data. Incredibly, the rate of growth that we can rigorously get is just superlinear, leaving a huge gap with the double exponential upper bound. Further on, we will see an example showing that growth on the boundary (as opposed to in the bulk) can indeed happen at a double exponential rate, but such examples are not known for the bulk.

The Denisov example is set on the torus $\mathbb{T}^2$, and uses a cellular flow as the stable background flow. One could try instead to smooth out the singular cross flow, and arrange for a small perturbation of it to play the role of a passive scalar behind the singular behavior. If one could somehow arrange for the solution to approach, in some sense, in the long time limit, the singular cross solution, then one could provide an example of the double exponential in time growth. This idea was exploited by Denisov in [55] to design a finite time double exponential growth example. However, one would face serious difficulties in trying to extend this approach to infinite time. First, to keep the background scenario stable, one needs symmetry – and the odd symmetry bans a nonzero perturbation right where the velocity is most capable of producing double exponential growth for all times, on the $x_2 = 0$ separatrix. Second, it is not clear how to make a smooth solution approach the "cross" in some suitable sense. Third, the perturbation will not be passive, and, for large times, will be difficult to decouple from the equation. In the Denisov example, the nonlinearity is something we will fight: the growth of the vorticity gradient is driven by a linear mechanism. To build an example with double exponential growth, the nonlinearity would have to become our friend. We will consider such example in the next section. The growth of the vorticity gradient in that example will be double exponential, and it will happen at the boundary of the domain. We will see that the latter is crucial for the double exponential growth.

The superlinear growth in the Denisov example

We now start the construction of an example where the gradient of the vorticity of a solution of the 2D Euler equations grows faster than linearly in time. More precisely, we will prove the following theorem.

Theorem 3.6.1 *There exists $\omega_0 \in C^\infty(\mathbb{T}^2)$ such that for the corresponding solution $\omega(t,x)$ of 2D Euler equations, we have*

$$\frac{1}{T^2}\int_0^T \|\nabla\omega(t,\cdot)\|_{L^\infty}\,dt \overset{T\to\infty}{\longrightarrow} +\infty. \tag{3.6.1}$$

This shows a faster than linear growth on average, and, in particular, on a subsequence of times tending to infinity.

The background flow

Our basic background flow will be really similar to that the cellular flow example above. In fact, it will be the same flow, but arranged slightly differently. We will start with the background vorticity

$$\omega^*(x_1,x_2) = -\cos x_1 - \cos x_2 = -2\cos\left(\frac{x_1+x_2}{2}\right)\cos\left(\frac{x_1-x_2}{2}\right).$$

The background stream function is

$$\psi^*(x_1,x_2) = \omega^*(x_1,x_2),$$

and the background flow is

$$u^*(x_1,x_2) = (\sin x_2, -\sin x_1).$$

The torus $(-\pi,\pi]^2$ contains two stagnation points of the flow, $(0,0)$ and (π,π). The four lines $x_2 = \pm x_1 \pm \pi$ are separatrices of the flow, and the points $(\pi,0)$ and $(0,\pi)$ where the separatrices intersect are hyperbolic points.

Exercise 3.6.2 Draw the streamlines of this flow.

Consider the hyperbolic point $D \equiv (\pi,0)$. The change of coordinates

$$\xi = (x_1+x_2-\pi)/2,\ \ \eta = (x_2-x_1+\pi)/2$$

transforms the stream function into

$$\psi(x_1,x_2) = 2\sin\xi\sin\eta,$$

and the characteristic equations near the point D become

$$\dot{\xi} = \sin\xi\cos\eta,\ \ \ \dot{\eta} = -\sin\eta\cos\xi$$

(in what follows in this section we will replace the Φ notation for the trajectories with more compact $x_1(t)$, $x_2(t)$, $\xi(t)$, and $\eta(t)$ notation).

Adding a perturbation: symmetry and stability

Let us now consider the 2D Euler equations

$$\begin{aligned} &\partial_t\omega + (u\cdot\nabla)\omega = 0, \\ &u = (\partial_2(-\Delta)^{-1}\omega, -\partial_1(-\Delta)^{-1}\omega), \\ &\omega(0,x) = \omega_0(x),\ \ x\in\mathbb{T}^2. \end{aligned} \tag{3.6.2}$$

We set

$$\omega(t,x) = \omega^*(x) + \varphi(t,x),$$

and

$$u(t,x) = u^*(x) + v(t,x).$$

We will take the initial condition $\varphi(0,x)$ as a small perturbation of $\omega^*(x)$. We will need the stability and symmetry lemmas in the analysis, asserting that the solution will remain close to $\omega^*(x)$ in L^2 sense, and, in addition, will keep some symmetries of the initial condition.

Let P_1 be the orthogonal projection onto the unit sphere in $\mathbb{Z}^2$ on the Fourier side, and P_2 be the projection onto the orthogonal complement of functions supported on the unit sphere on Fourier side. Simply put, if

$$f(x) = \sum_{k\in\mathbb{Z}^2} \hat{f}(k)e^{ikx},$$

then

$$P_1 f(x) = \sum_{|k|=1} \hat{f}(k)e^{ikx},$$

and

$$P_2 f(x) = \sum_{|k|\neq 1} \hat{f}(k) e^{ikx}.$$

The following lemma limis the interaction of $P_1\omega$ and $P_2\omega$ generated by the Euler evolution.

Lemma 3.6.3 *[Stability Lemma] Let $\omega(t,\cdot)$ be $C^1(\mathbb{T}^2)$ solution of the 2D Euler equation. Suppose that the initial data $\omega_0(x)$ is mean zero and satisfies*

$$\|P_2\omega_0\|_{L^2} \leq \varepsilon,$$

for some $\varepsilon > 0$. Then, we have

$$\|P_2\omega(t,\cdot)\|_{L^2} \leq \sqrt{2}\varepsilon \text{ for all } t > 0.$$

Proof. Recall that the mean zero property is conserved by the 2D Euler evolution. There are two additional quantities conserved by Euler evolution:

$$\int_{\mathbb{T}^2} |\omega(t,x)|^2 \, dx = \int_{\mathbb{T}^2} |\omega_0(x)|^2 x, \tag{3.6.3}$$

and

$$\int_{\mathbb{T}^2} \omega(t,x)\psi(t,x) \, dx = \int_{\mathbb{T}^2} |u(t,x)|^2 \, dx = \int_{\mathbb{T}^2} |u_0|^2 \, dx. \tag{3.6.4}$$

Here, $\psi = (-\Delta_D)^{-1}\omega$ is the stream-function of u.

Exercise 3.6.4 Verify (3.6.3) and (3.6.4) directly from (3.6.2).

With (3.6.3) and (3.6.4) in hand, observe that then

$$\sum_{|k|>1} \left(1 - \frac{1}{|k|^2}\right) |\hat{\omega}(k,t)|^2 = \int_{\mathbb{T}^2} |\omega_0(x)|^2 x - \int_{\mathbb{T}^2} |u_0|^2 \, dx$$

also does not depend on time. At $t = 0$, by assumption, this expression does not exceed ε^2. The same is then true for all times. But since

$$1 - |k|^{-2} \geq 1/2 \text{ if } |k| > 1,$$

it follows that

$$\|P_2\omega(\cdot,t)\|_{L^2}^2 \leq 2\varepsilon^2,$$

finishing the proof. □

The Fourier transform of ω^* is supported on the unit sphere in $\mathbb{Z}^2$, with

$$\hat{\omega^*}(1,0) = \hat{\omega^*}(-1,0) = \hat{\omega^*}(0,1) = \hat{\omega^*}(0,-1) = -1/2.$$

We also work with real valued solutions $\omega(x,t)$, so

$$\hat{\omega}(k,t) = \overline{\hat{\omega}}(-k,t).$$

Yet, the Stability Lemma alone is not enough to conclude L^2 stability of ω^* to small perturbations, as energy might shift between different modes with $|k| = 1$.

We will also need to ensure that the perturbed flow respects certain symmetries.

Lemma 3.6.5 *[Symmetries Lemma] Consider the 2D Euler vorticity equation on $\mathbb{T}^2$ with a smooth initial condition $\omega_0(x)$.*
(1) If ω_0 is even: $\omega_0(x) = \omega_0(-x)$, then the solution $\omega(t, x)$ remains even for all $t > 0$.
(2) If ω_0 is invariant under the rotation by $\pi/2$: $\omega_0(x_1, x_2) = \omega_0(-x_2, x_1)$, then the solution $\omega(t, x)$ remains invariant under rotation by $\pi/2$.

Proof. The proof uses uniqueness of the smooth solutions to (3.6.2). Similarly to what we did in Lemma 3.4.2, we show that if $\omega(t, x_1, x_2)$ is a solution, then so are $\omega(t, -x_1, -x_2)$ and $\omega(t, -x_2, x_1)$. Given either of these two symmetry assumptions on the initial data, this would imply, by uniqueness, that the solution must possess the same symmetry for all $t \geq 0$. First, assume that $\omega_0(x)$ is even and set $\omega^{(1)}(t, x) = \omega(t, -x_1, -x_2)$. Let ψ and $\psi^{(1)}$ be the corresponding stream functions

$$-\Delta\psi = \omega, \quad -\Delta\psi^{(1)} = \omega^{(1)}, \quad x \in \mathbb{T}^2.$$

Then $\psi^{(1)}(t, x_1, x_2) = \psi(t, -x_1, -x_2)$, and the corresponding flows are

$$u(t, x) = (\partial_2\psi(t, x), -\partial_1\psi(t, x)),$$

and

$$u^{(1)}(t, x) = (\partial_2\psi^{(1)}(t, x), -\partial_1\psi^{(1)}(t, x)) = -u(t, -x).$$

Therefore, $\omega^{(1)}$ satisfies

$$\omega_t^{(1)}(t, x) + (u^{(1)} \cdot \nabla)\omega^{(1)}(t, x) = \omega_t(t, -x) + u(t, -x) \cdot \nabla\omega(t, -x) = 0,$$

and is also a solution of the 2D Euler equations in the vorticity formulation.

Furthermore, if $\omega_0(x_1, x_2) = \omega_0(-x_2, x_1)$, we consider $\omega^{(2)}(t, x_1, x_2) = \omega(t, -x_2, x_1)$. The corresponding stream function $\psi^{(2)}$, the solution of

$$-\Delta\psi^{(2)} = \omega^{(1)}, \quad x \in \mathbb{T}^2,$$

satisfies $\psi^{(2)}(t, x_1, x_2) = \psi(t, -x_2, x_1)$, and the flow $u^{(2)}(t, x)$ is

$$\begin{aligned} u^{(2)}(t, x) &= (\partial_2\psi^{(2)}(t, x), -\partial_1\psi^{(2)}(t, x)) = (-\partial_1\psi(t, -x_2, x_1), -\partial_2\psi(t, -x_2, x_1)) \\ &= (u_2(t, -x_2, x_1), -u_1(t, -x_2, x_1)). \end{aligned}$$

Therefore, $\omega^{(2)}$ satisfies

$$\omega_t^{(2)}(t, x) + (u^{(2)} \cdot \nabla)\omega^{(2)}(t, x) = \big[\omega_t + u_2\frac{\partial\omega}{\partial x_2} + u_1\frac{\partial\omega}{\partial x_1}\big](t, -x_2, x_1) = 0,$$

and is a solution of the 2D Euler vorticity equation as well. □

Exercise 3.6.6 Does the 2D Euler vorticity equation preserve the $\omega(x_1, x_2) = \omega(-x_1, x_2)$ symmetry? How about $\omega(x_1, x_2) = \omega(x_2, x_1)$ and $\omega(x_1, x_2) = -\omega(-x_1, -x_2)$?

The following Corollary follows from Lemma 3.6.5 and Lemma 3.6.3.

Corollary 3.6.7 *[Stability under symmetry] Suppose that $\omega_0(x)$ is even and symmetric under rotation by $\pi/2$. and, in addition,*

$$\|\omega_0(x) - \omega^*(x)\|_{L^2} \le \varepsilon.$$

Then, we have

$$\|\omega(t,x) - \omega^*(x)\|_{L^2} \le C_0\varepsilon \tag{3.6.5}$$

for all times $t > 0$.

Proof. Lemma 3.6.3 implies that

$$\|P_2\omega(t)\|_{L^2} \le \sqrt{2}\varepsilon.$$

Furthermore, it follows from Lemma 3.6.5 that $\omega(t,x)$ is even and invariant under the rotation by $\pi/2$. Hence, its Fourier coefficients satisfy

$$\hat{\omega}(t,1,0) = \hat{\omega}(t,0,1) = \hat{\omega}(t,-1,0) = \hat{\omega}(t,0,-1),$$

and are real valued. Therefore,

$$\omega(t,x) = c(t)\omega^*(x) + P_2\omega(x,t),$$

and, as $\|\omega(t)\|_{L^2} = \|\omega_0\|_{L^2}$ and the solution is continuous in time, it is easy to see that

$$|c(t) - 1| \le C_1\varepsilon,$$

from which (3.6.5) follows. □

Constructing the initial perturbation

Let us now specify the initial data that will lead to the growth of the vorticity gradient. While the construction is carried out on the torus, it is more convenient to think of functions defined on $\mathbb{R}^2$ which are $2\pi-$periodic in both x_1 and x_2. Let U_δ be a disc of radius $\sqrt{\delta}$ centered at the origin $(0,0)$. We will use the coordinates

$$\xi = \frac{x_1 + x_2 - \pi}{2}, \;\; \eta = \frac{x_2 - x_1 + \pi}{2},$$

in which the saddle point D corresponds to $\xi = \eta = 0$, and the characteristics have the form

$$\dot{\xi} = \sin\xi\cos\eta, \;\; \dot{\eta} = -\sin\eta\cos\xi.$$

The direction ξ is expanding at D and the direction η is contracting. Consider the rectangle

$$P_\delta = \{|\xi| < 0.1, \;\; |\eta| < \delta\},$$

and denote its image under the rotation by $\pi/2$ around the origin in the original (x_1, x_2) coordinate system by P'_δ. We will define ω_0 on $\mathbb{T}^2$ as follows: first,

$$\omega_0(x) = \omega^*(x) \text{ outside } P_\delta,\ P'_\delta, \text{ and } U_\delta.$$

Next, we set, in the (ξ, η) coordinate system,

$$\omega_0 = f(\xi, \eta) \text{ in } P_\delta.$$

The function $f \in C_0^\infty(P_\delta)$ is even in (ξ, η), and satisfies $-1 \leq f \leq 4$. The level set $f(\xi, \eta) = 4$ is

$$\{f(\xi, \eta) = 4\} = \{\eta = 0, \ \ |\xi| \leq 0.08\},$$

and the level set $f(\xi, \eta) = 3$ is the ellipse

$$\{f(\xi, \eta) = 3\} = \{(\xi/0.09)^2 + (2\eta/\delta)^2 = 1\}.$$

Moreover, we assume that

$$4 \geq f(\xi, \eta) > 3 \ \text{ if } \ (\xi/0.09)^2 + (2\eta/\delta)^2 < 1.$$

In the rectangle P'_δ, we define $\omega_0(x)$ so that it is invariant under rotation by $\pi/2$ with respect to the origin.

Next, inside the disk U_δ, we set

$$\omega_0 = \omega^* + \phi_\delta,$$

where $\phi_\delta \in C_0^\infty(U_\delta)$ is designed so that

$$\int_{\mathbb{T}^2} \omega_0(x)\, dx = 0,$$

and ω_0 obeys the symmetry conditions: it is even and symmetric with respect to the rotation by $\pi/2$.

A priori bound on the vorticity and the velocity of the flow

Observe that by this construction, ω_0 is smooth. Since we chose f to be even in the (ξ, η) coordinates, and ω^* is even with respect to the hyperbolic point D as well, we know that ω_0 is even with respect to the point D :

$$\omega_0(x_1 - \pi, x_2) = \omega_0(-x_1 - \pi, -x_2).$$

By Lemma 3.6.5, the solution $\omega(t, x)$ inherits this property. It follows from the proof of the same lemma (see the expression for $u^{(1)}$ in that proof), that

$$u(t, D) = \nabla^\perp(-\Delta)^{-1}\omega(D, t) = 0,$$

for all times $t \geq 0$, so the point D is fixed by the flow. Note that D is not necessarily hyperbolic anymore, as our definition of $f(\xi, \eta)$ may destroy the hyperbolicity near D. However, the flow still possesses the hyperbolic structure outside a small region near D, and we will use this to prove the growth of $\nabla\omega$.

Let us recall the notation

$$\omega(t, x) = \omega^*(x) + \varphi(x, t),$$

and

$$u(t, x) = u^*(x) + v(t, x).$$

By Corollary 3.6.7 and the definition of ω_0, we know that

$$\|\varphi(t)\|_{L^2} \leq C\delta^{1/2},$$

for all $t \geq 0$. Due to the L^∞ maximum principle for $\omega(t,x)$, we also have

$$\|\varphi(t)\|_{L^\infty} \leq C.$$

Interpolating, we get

$$\|\varphi(t)\|_{L^p} \leq C(p)\delta^{1/p},$$

for every $p \geq 2$.

Lemma 3.6.8 *We have*

$$\|v(t)\|_{L^\infty} \leq C(p)\delta^{1/p},$$

for every $p > 2$.

Proof. Recall that by (3.4.10),

$$v_{1,2}(t,x) = \frac{1}{2\pi}\lim_{\gamma\to 0}\int_{\mathbb{R}^2}\frac{\pm y_{2,1}}{|y|^2}\varphi(t,x-y)e^{-\gamma|x-y|^2}\,dy,$$

where φ is extended periodically to all $\mathbb{R}^2$. We split the integral into two parts, over the unit ball B_1 and its complement. Then by the Hölder inequality,

$$\left|\int_{B_1}|y|^{-1}|\varphi(t,x-y)|\,dy\right| \leq C\|\varphi(t)\|_{L^p}(2-q)^{-1/q} \leq C(p)\delta^{1/p},$$

where $p^{-1}+q^{-1}=1$, $p>2$. For the rest of the estimate, note that φ is mean zero since ω_0, and ω^* are, and set

$$\varphi = \Delta\psi.$$

Integrating by parts, we obtain

$$\left|\int_{B_1^c}\frac{y_{1,2}}{|y|^2}\Delta\psi(x-y)e^{-\gamma|x-y|^2}\,dy\right| \leq C\int_{\partial B_1}\left(\left|\frac{\partial\psi}{\partial n}\right|+|\psi|\right)d\sigma + \int_{B_1^c}\psi(x-y)\Delta\left(\frac{y_{1,2}}{|y|^2}e^{-\gamma|x-y|^2}\right)dy.$$

Using the Sobolev embedding theorem, trace theorem, and the bounds we have for φ, we can complete the proof of the lemma similarly to the proof of Proposition 3.4.4. □

Exercise 3.6.9 Carry out the calculations carefully to complete the proof.

Trajectories near the saddle point

We choose δ so that $\|v(t,\cdot)\|_{L^\infty} \leq 0.001$ for all t. Let us zoom into the point D. The characteristic curves near D are

$$\dot{x}_1(t) = \sin x_2 - v_1(x,t), \quad \dot{x}_2(t) = -\sin x_1 - v_2(x,t).$$

In the ξ, η coordinates this becomes

$$\dot{\xi} = \cos\eta \sin\xi - (v_1 + v_2)/2, \quad \dot{\eta} = -\sin\eta\cos\xi + (v_1 - v_2)/2.$$

We will write this in a shortcut notation

$$\dot{\xi} = \sin\xi\cos\eta + \mu_1, \quad \dot{\eta} = -\sin\eta\cos\xi + \mu_2, \tag{3.6.6}$$

where $\|\mu_{1,2}\|_{L^\infty} \leq 0.001$. We will need the following lemma on the behavior of the trajectories.

Lemma 3.6.10 *Let $\xi(t), \eta(t)$ be the solution of the Cauchy problem* (3.6.6) *with the initial condition $\xi(t_0) = \xi_0$, $\eta(t_0) = \eta_0$. If $|\xi_0| \leq 0.03$ and $|\eta_0| \leq 0.1$, then $|\eta(t_0 + 1)| \leq 0.1$. Furthermore, if $0.03 \geq |\xi_0| \geq 0.02$ and $|\eta_0| \leq 0.1$, then $|\xi(t_0 + 1)| > 0.03$. More generally, if $0.03 \geq |\xi_0| \geq (3 - \tau)/100$, $0 \leq \tau \leq 1$, and $|\eta_0| \leq 0.1$, then $|\xi(t_0 + \tau)| > 0.03$.*

Proof. Observe that $|\xi'| < |\xi| + 0.001$, $|\xi_0| \leq 0.03$ imply that

$$|\xi(t)| \leq |\xi_0| e + 0.001 \int_0^1 e^t \, dt < 0.04e$$

for $t \in (t_0, t_0 + 1)$. Now, at $\eta = 0.1$ we have

$$\dot{\eta} \leq -\sin 0.1 \cos 0.2 + 0.001 < 0,$$

for all times in $(t_0, t_0 + 1)$ and so the trajectory cannot move past or arrive from the inside at this value of η. The case of $\eta = -0.1$ is similar. Thus, for $t \in (t_0, t_0 + 1)$ we have $|\eta(t)| \leq 0.1$. For the second statement of the lemma notice that for $0.03 \geq \xi_0 \geq 0.02$, due to bounds we showed, we have

$$\dot{\xi}(t) \geq 0.9\xi - 0.001$$

in the time interval $(t_0, t_0 + 1)$, and thus

$$\xi(t) \geq 0.02e^{0.9} - 0.001(e - 1) > 0.03.$$

For the last statement, following the same estimates, we have to check that

$$(3 - \tau)e^{0.9\tau} - 0.1(e^\tau - 1) \geq (3 - \tau)(1 + 0.9\tau) - 0.3\tau \geq 3 + 0.5\tau \geq 3,$$

which is correct. □

Proof of Theorem 3.6.1

We denote by R_s the rectangle $|\eta| < 0.1$, $|\xi| < 0.01s$, and by $E(t_1, t_2)$ the Euler flow map from time t_1 to time t_2. The map $E(t_1, t_2)$ is a smooth area preserving diffeomorphism given by the solutions to the characteristic equations (3.6.6). It has a fixed point D and is centrally symmetric with respect to D. Consider the set

$$S_0 = R_3 \cap \{x : \ \omega_0(x) \geq 3\}.$$

This set is bounded by the intervals lying on the lines $\xi = \pm 0.03$ and by parts of the ellipse where $\omega_0(x) = 3$. We split this set as

$$S_0 = S_0^1 \cup S_0^2,$$

where $S_0^1 = S_0 \cap R_2$, and S_0^2 is the rest of S_0. Consider the image $F_0 = E(0,1)S_0$. By Lemma 3.6.10, this set is contained in $|\eta| < 0.1$. Denote then $S_1 = F_0 \cap R_3$, and keep only the simply connected component of this set containing the point D. The set S_1 is bounded by the intervals lying on the lines $\xi = \pm 0.03$ and by parts of the level set $\omega(t = 1, x) = 3$. By virtue of the same Lemma, the set $F_0^2 = E(0,1)S_0^2$ gets transported out of R_3, and so

$$|S_1| \leq |S_0| - |S_0^2|.$$

Note that S_1 contains a part of the level set $\omega(t = 1, x) = 4$. Moreover, since F_0 is contained in $|\eta| < 0.1$ and the ends of $\omega_0 = 4$ curve get transported out of R_3, the part of the level set $\omega = 4$ lying in S_1 contains a curve passing through the point D and connecting two points $P_1^{\pm}$ lying on $\xi = \pm 0.03$. Now let us split $S_1 = S_1^1 \cup S_1^2$, where $S_1^1 = S_1 \cap R_2$, and S_1^2 is the rest of S_1. We now iterate time in unit steps, obtaining a sequence of sets

$$S_{n+1} = E(n, n+1)S_n \cap R_3.$$

All properties of the set S_1 described above continue to hold for S_n. In particular, we have

$$|S_{n+1}| \leq |S_n| - |S_n^2|,$$

which implies that

$$\sum_n |S_n^2| < \infty.$$

On the other hand, for each fixed ξ, $0.02 < |\xi| < 0.03$, a section of the set S_n^2 at level ξ must contain an interval $[\eta_1, \eta_2]$ such that $\omega(\eta_1, \xi) = 3$ and $\omega(\eta_2, \xi) = 4$, with

$$|\eta_1 - \eta_2| \geq \frac{1}{\|\nabla\omega(t = n)\|_{L^\infty}}.$$

This implies

$$|S_n^2| \geq 0.01 \|\nabla\omega(t = n)\|_{L^\infty}^{-1}.$$

We need one more elementary lemma.

Lemma 3.6.11 *Let $a_j > 0$ be such that $\sum_{j=1}^{\infty} a_j < \infty$. Then*

$$\frac{1}{N^2} \sum_{j=1}^{N} a_j^{-1} \stackrel{N\to\infty}{\longrightarrow} \infty.$$

Proof. Observe that

$$\min_{x_i > 0, x_1 + \dots + x_n = \sigma} \sum_{i=1}^{n} x_i^{-1} = n^2 \sigma^{-1},$$

and the minimum is achieved when $x_i = \sigma/n$ for each i.

Exercise 3.6.12 Prove this claim.

Now, set

$$\tau_N = \sum_{j=N/2}^{N} a_j \overset{N\to\infty}{\longrightarrow} 0.$$

Finally, note that

$$\frac{1}{N^2}\sum_{j=1}^{N} a_j^{-1} \geq \frac{1}{N^2}\sum_{j=N/2}^{N} a_j^{-1} \geq N^{-2}\frac{N^2}{4}\frac{1}{\tau_N} \to \infty$$

as $N \to \infty$. □

An application of Lemma 3.6.11 then gives

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=0}^{\infty}\|\nabla\omega(\cdot,n)\|_{L^\infty} = +\infty.$$

This is a discrete version of (3.6.1). One can obtain the continuous version by using the last statement of Lemma 3.6.10.

Exercise 3.6.13 Prove (3.6.1), and thus finish the proof of the Theorem. You will need to use the last statement of Lemma 3.6.10, taking τ small (and passing to the limit $\tau \to 0$). Otherwise, the argument above will require only a few adjustments.

3.7 The double exponential growth in a bounded domain

In this section, we will describe an example of a solution of the 2D Euler equations in a bounded domain (a disk) that has the vorticity gradient that grows at a double exponential rate in time [94]. As we will see, the gradient growth will happen exactly at the boundary. Essentially, the boundary will play a role of a separatrix in a singular cross type flow. The main difference here is that while the voriticity has to vanish on a separatrix if we want to keep the symmetry of the cross (and this depletion makes achieving the double exponential growth of the vorticity gradient in the bulk difficult), it does not have to be zero on the boundary – this will allow us to bring the points where the vorticity differs by a $O(1)$ quantity at distance that goes to zero at a double exponential in time rate.

Theorem 3.7.1 *Consider the two-dimensional Euler equations on a unit disk D. There exist smooth initial data ω_0 with $\|\nabla\omega_0\|_{L^\infty}/\|\omega_0\|_{L^\infty} > 1$ such that the corresponding solution $\omega(x,t)$ satisfies*

$$\frac{\|\nabla\omega(x,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \geq \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{c\exp(c\|\omega_0\|_{L^\infty}t)} \tag{3.7.1}$$

for some $c > 0$ and for all $t \geq 0$.

The initial data and the basic mechanism

It will be convenient for us to take the system of coordinates centered at the lowest point of the disk, so that the center of the unit disk D is at the point $(0, 1)$. Here is a rough idea of the construction. The initial vorticity will be equal to -1 in the right half of the disk, and to $+1$ in the left half of the disk, except for a narrow strip around the vertical axis, where the two values would be smoothly interpolated. Recall that the flow on the boundary is always tangential to the boundary because of the boundary conditions. Thus, if we can show that the flow on the boundary would always point toward the bottom point $(0, 0)$, it would bring closer and closer the boundary points on the left half-circle where initially $\omega_0 = +1$ and those on the right half-circle where $\omega_0 = -1$. As these regions approach $(0, 0)$ they will generate log-Lipschitz fluid velocity asymptotic behavior similar to that in Proposition 3.4.7 describing the Bahouri-Chemin example. This nonlinear enhancement is the crucial part of the mechanism, and will lead to the double exponential growth in the vorticity gradient. The difficulty is in the details: understanding and controlling all the nonlinear effects, and, in fact, using them to get the double exponential boost.

Our initial data $\omega_0(x)$ will be odd with respect to the vertical axis:

$$\omega_0(x_1, x_2) = -\omega_0(-x_1, x_2).$$

We have checked the preservation of this symmetry on the torus. It can be verified similarly for any domain with the vertical symmetry axis, in particular, for the disk. As we have described above, we will take smooth initial data $\omega_0(x)$ so that $\omega_0(x) \le 0$ for $x_1 > 0$, which, by symmetry, means that $\omega_0(x) \ge 0$ for $x_1 < 0$. This configuration makes the origin a hyperbolic fixed point of the flow.

Exercise 3.7.2 Show that under the above assumptions u_1 vanishes on the vertical axis, thus $\omega(t, x) \le 0$ for $x_1 > 0$ for all $t \ge 0$.

A flow estimate near the bottom

We will now derive an estimate on the flow that will allow us to capture the hyperbolic nature of the flow near the origin. The Dirichlet Green's function for the disk is given explicitly by (see e.g. [60])

$$G_D(x, y) = \frac{1}{2\pi}(\log|x - y| - \log|x - \bar{y}| - \log|y - e_2|),$$

where, with our choice of the coordinates,

$$\bar{y} = e_2 + (y - e_2)/|y - e_2|^2, \quad e_2 = (0, 1),$$

is the reflection of the point y with respect to the unit disk D.

Exercise 3.7.3 Show that for any $y \in D^+$ we have

$$|\bar{y}| = \frac{|y|}{|y - e_2|}, \tag{3.7.2}$$

and

$$\frac{\bar{y}_1}{|\bar{y}|^2} = \frac{y_1}{|y|^2}, \quad \frac{\bar{y}_2}{|\bar{y}|^2} = 1 - \frac{y_2}{|y|^2}. \tag{3.7.3}$$

Given the symmetry of ω, we have

$$u(t,x) = -\nabla^\perp \int_D G_D(x,y)\omega(t,y)\,dy = -\frac{1}{2\pi}\nabla^\perp \int_{D^+} \log\left(\frac{|x-y||\widetilde{x}-\bar{y}|}{|x-\bar{y}||\widetilde{x}-y|}\right)\omega(t,y)\,dy, \quad (3.7.4)$$

where D^+ is the right half disk where $x_1 \geq 0$, and $\widetilde{x} = (-x_1, x_2)$. The following Lemma will be crucial for the proof of Theorem 3.7.1. For each point $(x_1, x_2) \in D^+$, let us introduce the region

$$Q(x_1,x_2) = \{(y_1,y_2) \in D^+ : \quad x_1 \leq y_1, \quad x_2 \leq y_2\},$$

and set

$$\Omega(t,x_1,x_2) = \frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(t,y)\,dy_1 dy_2. \quad (3.7.5)$$

Lemma 3.7.4 *Take any γ, such that $0 < \gamma < \pi/2$, and let D_1^γ the intersection of D^+ with a sector $0 \leq \phi \leq \pi/2 - \gamma$, where ϕ is the usual angular variable. Then there exists $\delta > 0$ so that for all $x \in D_1^\gamma$ such that $|x| \leq \delta$ we have*

$$u_1(t,x_1,x_2) = x_1\Omega(t,x_1,x_2) + x_1 B_1(x_1,x_2,t), \quad (3.7.6)$$

where $|B_1(x_1,x_2,t)| \leq C(\gamma)\|\omega_0\|_{L^\infty}$.

Similarly, if we denote by D_2^γ the intersection of D^+ and the sector $\gamma \leq \phi \leq \pi/2$, then for all $x \in D_2^\gamma$ such that $|x| \leq \delta$ we have

$$u_2(t,x_1,x_2) = -x_2\Omega(t,x_1,x_2) + x_2 B_2(t,x_1,x_2), \quad (3.7.7)$$

where $|B_2(t,x_1,x_2)| \leq C(\gamma)\|\omega_0\|_{L^\infty}$.

Exercise 3.7.5 This lemma holds more generally than in the disk. Perhaps, the simplest proof is when D is a square. The computation for that case is quite similar to the Bahouri-Chemin example. Carry out the proof of the Lemma for this case.

The exclusion of a small sector does not appear to be a technical artifact. The vorticity can be arranged (momentarily) in a way that the hyperbolic picture provided by the Lemma is violated outside of D_1^γ, for example the direction of u_1 may be reversed near the vertical axis, where u_1 is small.

The terms involving $\Omega(t,x_1,x_2)$, as will become clear soon, can be thought of as the main terms in these estimates in a certain regime. Indeed, while the remainder in (3.7.6), (3.7.7) satisfies the Lipschitz estimates, the nonlocal term $\Omega(x_1,x_2,t)$ can grow as a logarithm if the support of the vorticity approaches the origin. This growth through the nonlinear feedback can lead to the double exponential growth in the gradient of solution. Essentially, Lemma 3.7.4 makes it possible to ensure in certain regimes that the flow near the origin is hyperbolic, so that the fluid trajectories are hyperbolas to the leading order. The speed of motion along the trajectories is controlled by $\Omega(t,x_1,x_2)$ in (3.7.6), (3.7.7), and this factor is the same for both u_1 and u_2.

We also note a monotonicity property imbedded in the form of $\Omega(t,x_1,x_2)$: the size of the expression in (3.7.5) tends to increase as x approaches the origin since the region of integration grows. This will be important in the construction of our example.

Proof. Let us prove (3.7.6), the proof of (3.7.7) is similar. Fix a small $\gamma > 0$, and take a point $x = (x_1, x_2) \in D_1^\gamma$, $|x| \le \delta$, where δ is to be determined soon. By the definition of D_1^γ, we have $x_2 \le x_1/\tan\gamma$. Hence, if we set

$$r = 10(1 + \cot\gamma)x_1,$$

then $x \in B_r(0)$. Let us assume that δ is small enough so that $r < 0.1$ whenever $|x| \le \delta$. The contribution to u_1 from the integration over the disk $B_r(0)$ in the Biot-Savart law (3.7.4) does not exceed

$$C\|\omega_0\|_{L^\infty} \int_{D^+ \cap B_r(0)} \frac{1}{|x-y|}\, dy \le C(\gamma)\|\omega\|_{L^\infty} x_1.$$

Exercise 3.7.6 Verify this directly from (3.7.4).

For the integration over $D^+ \setminus B_r(0)$, we note that for y in this region we have $|y| \ge 10|x|$. Let us rewrite the four logarithms in (3.7.4) as

$$\begin{aligned} \pi G_D(x,y) = &\frac{1}{4}\log\left(1 - \frac{2x\cdot y}{|y|^2} + \frac{|x|^2}{|y|^2}\right) - \frac{1}{4}\log\left(1 - \frac{2x\cdot \bar{y}}{|\bar{y}|^2} + \frac{|x|^2}{|\bar{y}|^2}\right) \\ &- \frac{1}{4}\log\left(1 - \frac{2\widetilde{x}\cdot y}{|y|^2} + \frac{|x|^2}{|y|^2}\right) + \frac{1}{4}\log\left(1 - \frac{2\widetilde{x}\cdot \bar{y}}{|\bar{y}|^2} + \frac{|x|^2}{|\bar{y}|^2}\right). \end{aligned} \tag{3.7.8}$$

For small s, we have

$$\log(1+s) = s - \frac{s^2}{2} + O(s^3).$$

With the help of this approximation, a direct computation starting with (3.7.8) leads to

$$\pi G_D(x,y) = -\frac{x_1 y_1}{|y|^2} + \frac{x_1 \bar{y}_1}{|\bar{y}|^2} - \frac{2x_1x_2y_1y_2}{|y|^4} + \frac{2x_1x_2\bar{y}_1\bar{y}_2}{|\bar{y}|^4} + O\left(\frac{|x|^3}{|y|^3}\right). \tag{3.7.9}$$

We used above that $|\bar{y}| \ge |y|$ for $y \in D^+$, as follows from (3.7.2). Using (3.7.3), we simplify (3.7.9) to

$$-\pi G_D(x,y) = \frac{4x_1x_2y_1y_2}{|y|^4} - \frac{2x_1x_2y_1}{|y|^2} + O\left(\frac{|x|^3}{|y|^3}\right). \tag{3.7.10}$$

Exercise 3.7.7 Verify that the expression (3.7.10) can be differentiated with respect to x_2, yielding

$$-\pi\frac{\partial G_D(x,y)}{\partial x_2} = \frac{4x_1y_1y_2}{|y|^4} - \frac{2x_1y_1}{|y|^2} + O\left(\frac{|x|^2}{|y|^3}\right). \tag{3.7.11}$$

Observe that the contribution of the error term in (3.7.11) to the integral in the Biot-Savart law over $D^+ \setminus B_r$ is controlled:

$$\int_{D^+\setminus B_r} \frac{|x|^2}{|y|^3}\, dy \le C|x|^2 \int_r^1 \frac{1}{s^2}\, ds \le Cr^{-1}|x|^2 \le C(\gamma)x_1.$$

For the corresponding integral of the second term in the right side of (3.7.11) we have a bound

$$\int_{D^+\setminus B_r} \frac{y_1}{|y|^2}\, dy \le C\int_r^1 ds \le C.$$

Therefore, the last two terms in (3.7.11) give regular contributions to u_1, and are contributing to the second term in the right side of (3.7.6).

Thus, to prove (3.7.6), it remains only to reconcile the regions of integration in the main term in (3.7.11), namely to show that

$$\int_{D^+\setminus B_r} \frac{y_1y_2}{|y|^4}\omega(t,y)\,dy = O(1) + \int_{Q(x_1,x_2)} \frac{y_1y_2}{|y|^4}\omega(t,y)\,dy.$$

To this end, note first that

$$\int_{B_r\cap Q(x_1,x_2)} \frac{y_1y_2}{|y|^4}\,dy \leq \int_{x_1}^{Cx_1} dy_1 \int_0^{Cx_1} dy_2 \frac{y_1y_2}{|y|^4}$$
$$\leq C\int_{x_1}^{Cx_1} y_1 \int_0^{C^2x_1^2} \frac{1}{(y+y_1^2)^2}\,dydy_1 = C\int_{x_1}^{Cx_1}\frac{dy_1}{y_1} \leq C.$$

Finally, the set $D^+\setminus(Q(x_1,x_2)\cup B_r)$ consists of two strips, one along the x_1 axis, and another along the x_2 axis. The contribution of the integral over the strip along the x_2 axis does not exceed

$$\int_0^{x_1} dy_1 \int_{x_1}^1 dy_2 \frac{y_1y_2}{|y|^4} \leq \int_0^{x_1}\frac{y_1}{y_1^2+x_1^2}\,dy_1 \leq C.$$

The integral over the strip along the x_1 axis does not exceed

$$\int_0^{x_2} dy_2 \int_{x_1}^1 dy_1 \frac{y_1y_2}{|y|^4}.$$

Since $x_2 \leq C(\gamma)x_1$, the latter integral can be bounded by a constant via a similar computation. This completes the proof of the lemma. □

An exponential in time growth of the vorticity gradient

Before proving Theorem 3.7.1, we make a simpler observation: with the aid of Lemma 3.7.4 it is fairly straightforward to find examples with an exponential in time growth of the vorticity gradient. Indeed, take smooth initial data $\omega_0(x)$ which is equal to one everywhere in D^+ except on a thin strip of width δ near the vertical axis $x_1 = 0$, where $0 < \omega_0(x) < 1$. Recall that ω_0 must vanish on the vertical axis by our symmetry assumptions. Due to the incompressibility of the flow, the distribution function of $\omega(t,x)$ is the same for all times. In particular, the measure of the complement of the set where $\omega(t,x) = 1$ does not exceed 2δ. In this case for every $|x| < \delta$, $x \in D^+$, we can derive the following estimate for the integral appearing in the representation (3.7.6):

$$\int_{Q(x_1,x_2)} \frac{y_1y_2}{|y|^4}\omega(t,y)\,dy_1dy_2 \geq \int_{2\delta}^1\int_{\pi/6}^{\pi/3} \omega(t,r,\phi)\frac{\sin 2\phi}{2r}\,d\phi dr \geq \frac{\sqrt{3}}{4}\int_{2\delta}^1\int_{\pi/6}^{\pi/3}\frac{\omega(t,r,\phi)}{r}\,d\phi dr.$$

The value of the integral on the right hand side is minimized when the area where $\omega(r,\phi)$ is less than one is concentrated at small values of the radial variable. As this area does not exceed 2δ, we obtain

$$\frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1y_2}{|y|^4}\omega(t,y)\,dy_1dy_2 \geq c_1\int_{c_2\sqrt{\delta}}^1\int_{\pi/6}^{\pi/3}\frac{1}{r}\,d\phi dr \geq C_1\log\delta^{-1}, \tag{3.7.12}$$

where c_1, c_2 and C_1 are positive universal constants.

Using the estimate (3.7.12) in (3.7.6), we get that for all for $|x| \le \delta$, $x \in D^+$ that lie on the disk boundary, we have

$$u_1(t,x) \le -x_1(C_1 \log \delta^{-1} - C_2),$$

where $C_{1,2}$ are universal constants. We can thus choose $\delta > 0$ sufficiently small so that

$$u_1(t,x) \le -x_1, \text{ for all times if } |x| < \delta. \tag{3.7.13}$$

As we have discussed, due to the boundary condition on u, the trajectories which start at the boundary stay on the boundary for all times. Taking such trajectory starting at a point $x_0 \in \partial D$ with the first component satisfying $0 < x_{0,1} \le \delta$, we get from (3.7.13)

$$\Phi^1_{t,1}(x_0) \le x_{0,1} e^{-t}.$$

Since $\omega(t,x) = \omega(\Phi_t^{-1}(x))$, we see that $\|\nabla\omega(x,t)\|_{L^\infty}$ grows exponentially in time if we choose ω_0 which does not vanish identically at the boundary near the origin, that is, if

$$\omega_0(\delta, 1 - \sqrt{1-\delta^2}) > 0.$$

Proof of Theorem 3.7.1

To construct examples with the double exponential growth of $\nabla\omega(t,x)$, we have to work a little harder. For the sake of simplicity, we will build our example with ω_0 such that $\|\omega_0\|_{L^\infty} = 1$.

We first fix a small $\gamma > 0$. We will take the smooth initial data as in the example of the exponential growth of the vorticity gradient, with $\omega_0(x) = -1$ for $x \in D^+$ apart from a narrow strip of width at most $\delta > 0$ (with δ small enough so that the estimates (3.7.6), (3.7.7) apply) near the vertical axis where $0 \ge \omega_0(x) \ge -1$. This ensures that the lower bound (3.7.12) still holds.We will require one more feature in ω_0 that we will describe later. In addition, choose δ so that

$$C_1 \log \delta^{-1} > 100C(\gamma).$$

Here, $C(\gamma)$ is the constant in the bound for the error terms B_1, B_2 appearing in (3.7.6), (3.7.7).

Given $0 < x_1' < x_1'' < 1$, we set

$$\mathcal{O}(x_1', x_1'') = \left\{(x_1,x_2) \in D^+\,,\ x_1' < x_1 < x_1'',\ x_2 < x_1\right\}\,. \tag{3.7.14}$$

We also define, for $0 < x_1 < 1$,

$$\underline{u}_1(t,x_1) = \min_{(x_1,x_2)\in D^+\,,\, x_2 \le x_1} u_1(t,x_1,x_2) \tag{3.7.15}$$

and

$$\overline{u}_1(t,x_1) \quad = \max_{(x_1,x_2)\in D^+\,,\, x_2 \le x_1} u_1(t,x_1,x_2)\,. \tag{3.7.16}$$

As the initial condition ω_0 is smooth, so is $\omega(t,x)$, as well as $u(t,x)$, thus these functions are locally Lipschitz in x_1 on $[0,1)$, with the Lipschitz constants being locally bounded in time. Hence, we can uniquely define $a(t)$ by

$$\dot{a} = \overline{u}_1(a,t)\,, \quad a(0) = \varepsilon^{10}\,, \tag{3.7.17}$$

and $b(t)$ by

$$\dot{b} = \underline{u}_1(b,t)\,, \quad b(0) = \varepsilon\,, \tag{3.7.18}$$

where $0 < \varepsilon < \delta$ is sufficiently small. Its exact value is to be determined later. We set

$$\mathcal{O}_t = \mathcal{O}(a(t), b(t))\,. \tag{3.7.19}$$

At this stage, we have not yet ruled out that $\mathcal{O}_t$ perhaps might become empty for some $t > 0$, that is, $b(t)$ would catch up with $a(t)$. Note that $b(t)$ starts to the right of $a(t)$ but it moves to the left faster than $a(t)$ if both $\underline{u}_1$ and $\overline{u}_1$ are negative, as expected. However, it is clear from the definitions that $\mathcal{O}_t$ will be non-empty at least on some non-trivial interval of time. Our estimates below show that in fact $\mathcal{O}_t$ will be non-empty for all $t > 0$.

Now we will make one more assumption on ω_0. We ask that ω_0 includes a "bullet": $\omega_0 = -1$ in the small trapezoid $\mathcal{O}_0$, with a smooth but narrow cutoff into D^+, so that $\|\nabla\omega_0\|_{L^\infty} \lesssim \varepsilon^{-10}$. This leaves some ambiguity in the definition of $\omega_0(x)$ in the strip of width δ next to the x_2 axis apart from $\mathcal{O}_0$. We will see that it does not really matter how we define ω_0 there, as long as we satisfy all the conditions stated so far. Using the estimates (3.7.6), (3.7.7), the estimate (3.7.12) and our choice of δ ensuring that $C_1 \log\delta^{-1} \gg C(\gamma)$, we see that both a and b are decreasing functions of time and that near the diagonal $x_1 = x_2$ in $\{|x| < \delta\}$ we have

$$\frac{x_1(\log\delta^{-1} - C)}{x_2(\log\delta^{-1} + C)} \le \frac{(-u_1)(x_1,x_2)}{u_2(x_1,x_2)} \le \frac{x_1(\log\delta^{-1} + C)}{x_2(\log\delta^{-1} - C)}. \tag{3.7.20}$$

This means that all particle trajectories on the diagonal, at all times are directed into the region $\phi > \pi/4$. We claim that then $\omega(t,x) = -1$ on $\mathcal{O}_t$. Indeed, it is clear that the "fluid particles" which at the time $t = 0$ are outside of $\mathcal{O}_0$ cannot enter $\mathcal{O}_{t'}$ through the diagonal $\{x_1 = x_2\}$ due to (3.7.20) at any time $0 \le t' \le t$. The definition of the dynamics of $a(t), b(t)$ means that neither can they enter the set $\mathcal{O}_{t'}$ through the vertical segments $\{(a(t'), x_2) \in D^+\,,\ x_2 < a(t')\}$ or $\{(b(t'), x_2) \in D^+\,,\ x_2 < b(t')\}$ at any time $0 \le t' \le t$. Finally, they obviously cannot enter through the boundary points of D. Hence the "fluid particles" in $\mathcal{O}_t$ must have been in $\mathcal{O}_0$ at the initial time and thus $\omega(t,x) = -1$ in $\mathcal{O}_t$.

Let us now estimate how rapidly the set $\mathcal{O}_t$ approaches the origin. As $\|\omega(t,x)\|_{L^\infty} \le 1$ by our choice of the initial data ω_0, by Lemma 3.7.4, we have

$$\underline{u}_1(b(t),t) \ge -b(t)\,\Omega(b(t), x_2(t)) - C\,b(t),$$

for some $0 \le x_2(t) \le b(t)$. A simple calculation shows that, for any $0 \le x_2 \le b(t)$ we have

$$\Omega(b(t), x_2) \le \Omega(b(t), b(t)) + C.$$

Indeed, since $x_2(t) \le b(t)$ we can write

$$\int_b^2 \int_{x_2(t)}^b \frac{y_1 y_2}{|y|^4}\,dy_2 dy_1 \le \int_b^2 \int_0^b \frac{y_1 y_2}{|y|^4}\,dy_2 dy_1 = \frac{1}{2}\int_b^2 y_1 \left(\frac{1}{y_1^2} - \frac{1}{y_1^2 + b^2}\right)\,dy_1$$
$$\le b^2 \int_b^2 y_1^{-3} dy_1 \le C. \tag{3.7.21}$$

Thus, we get

$$\underline{u}_1(b(t),t) \ge -b(t)\,\Omega(b(t), b(t)) - 2C\,b(t). \tag{3.7.22}$$

In the same vein, for suitable $\widetilde{x}_2(t)$ with $0 \le \widetilde{x}_2(t) \le a(t)$, we have

$$\overline{u}_1(a(t),t) \le -a(t)\,\Omega(a(t),\widetilde{x}_2(t)) + \widetilde{C}a(t) \le -a(t)\,\Omega(a(t),0) + 2Ca(t),$$

by an estimate similar to (3.7.21) above. Observe also that

$$\Omega(a(t),0) \ge \frac{4}{\pi}\int_{\mathcal{O}_t} \frac{y_1 y_2}{|y|^4}\omega(t,y)\,dy_1 dy_2 + \Omega(b(t),b(t)).$$

Since $\omega(t,y) = 1$ on $\mathcal{O}_t$, we have

$$\int_{\mathcal{O}_t} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy_1 dy_2 \ge \int_{\varepsilon}^{\pi/4}\int_{a(t)/\cos\phi}^{b(t)/\cos\phi} \frac{\sin 2\phi}{2r}\,dr d\phi > \frac{1}{8}(-\log a(t) + \log b(t)) - C,$$

thus

$$\overline{u}_1(a(t),t) \le -\,a(t)\left(\frac{1}{2\pi}(-\log a(t) + \log b(t)) + \Omega(b(t),b(t))\right) + 2Ca(t). \tag{3.7.23}$$

It follows from the estimates (3.7.22), (3.7.23) that $a(t)$ and $b(t)$ are monotone decreasing in time, and by the finiteness of $\|u\|_{L^\infty}$ these functions are Lipschitz in t. Hence we have sufficient regularity for the following calculations:

$$\frac{d}{dt}\log b(t) \ge -\Omega(b(t),b(t)) - 2C\,, \tag{3.7.24}$$
$$\frac{d}{dt}\log a(t) \le \frac{1}{2\pi}\left(\log a(t) - \log b(t)\right) - \Omega(b(t),b(t)) + 2C. \tag{3.7.25}$$

Subtracting (3.7.24) from (3.7.25), we obtain

$$\frac{d}{dt}\left(\log a(t) - \log b(t)\right) \le \frac{1}{2\pi}\left(\log a(t) - \log b(t)\right) + 4C. \tag{3.7.26}$$

From (3.7.26), the Gronwall lemma leads to

$$\log a(t) - \log b(t) \le \log\left(a(0)/b(0)\right)\exp(t/2\pi) + 4C\exp(t/2\pi) \le (9\log\varepsilon + 4C)\exp(t/2\pi). \tag{3.7.27}$$

We should choose our ε so that $-\log\varepsilon$ is larger than the constant $4C$ that appears in (3.7.27). In this case, we obtain from (3.7.27) that

$$\log a(t) \le 8\exp(t/2\pi)\log\varepsilon,$$

and so

$$a(t) \le \varepsilon^{8\exp(t/2\pi)}.$$

Note that by the definition of $a(t)$, the first coordinate of the characteristic that originates at the point on ∂D near the origin with $x_1 = \varepsilon^{10}$ does not exceed $a(t)$. To arrive at (3.7.1), it remains to note that we chose ω_0 so that $\|\nabla\omega_0\|_{L^\infty} \lesssim \varepsilon^{-10}$. □

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