Questions for the final

Notation: $|A|$ denotes the Lebesgue measure of a set A.
1. Show that measurable sets form a σ-algebra.
2. Give an example of a non-measurable set.
3. Show that $\limsup_n f_n$ is measurable if f_n are.
4. Let f_n be a sequence of measurable functions that converges to $f(x)$ a.e. on $[0,1]$. Show that for any $\varepsilon > 0$ there exists a compact set K_ε such that $|K_\varepsilon| < \varepsilon$ and f_n converges uniformly to f on $[0,1] \setminus K_\varepsilon$.
5. Let f_n be a sequence of measurable functions that converges to $f(x)$ a.e. on a measurable set E of a finite Lebesgue measure: $|E| < +\infty$. Assume that there exists $M > 0$ so that $|f_n(x)| \le M$ for all $n \ge 1$ and $x \in E$. Show that

$$\int_E f_n(x)dx \to \int_E f(x)dx.$$

6. Let f_n be a sequence of non-negative measurable functions which converges point-wise to a function f on a measurable set E, then

$$\int_E f d\mu \le \liminf_{n\to+\infty} \int_E f_n d\mu.$$

7. Let the functions f_n be measurable and defined on a measurable set E. Assume that $f_n(x) \to f(x)$ almost everywhere on E, and there exists a function $g(x)$ such that

$$\int_E g(x)d\mu < +\infty,$$

and $|f_n(x)| \le g(x)$ a.e. on E. Then we have

$$\int_E f d\mu = \lim_{n\to+\infty} \int_E f_n d\mu. \tag{1}$$

8. Let $E \subset \mathbb{R}$ with $m^*(E) < +\infty$ and let $\mathcal{J}$ be a fine cover of the set E by non-trivial closed intervals. Then for any $\varepsilon > 0$ there exists a finite subcollection of pairwise disjoint intervals $\{I_1, \ldots, I_N\}$ in $\mathcal{J}$ such that

$$m^*\left(E \setminus (\bigcup_{j=1}^N I_j)\right) < \varepsilon.$$

9. Let f be an increasing function on an interval $[a,b]$. Then $f'(x)$ exists almost everywhere on $[a,b]$ with respect to the Lebesgue measure and is a measurable function.
10. Give the definition of a function of a bounded variation. Show that a BV function is a difference of two monotone functions.
11. Show that if $f \in L^1[a,b]$ is integrable and

$$\int_a^x f(s)ds = 0 \tag{2}$$

for all $x \in [a,b]$ then $f(t) = 0$ a.e. on $[a,b]$.
12. Show that a function $F(x)$ defined on an interval $[a,b]$ is an indefinite integral of an integrable function, that is, it has the form

$$F(x) = F(a) + \int_a^x f(y)dy,$$

with $f \in L^1[a,b]$ if and only if F is absolutely continuous.
13. Let $f(x)$ be a continuous function on a compact set $K \subset \mathbb{R}^n$. Show that there exists a continuous function $\bar{f}(x)$ defined on all of $\mathbb{R}^n$ such that $f(x) = \bar{f}(x)$ for all $x \in K$.
14. Show that if a sequence f_n converges to a function f in $L^1[0,1]$, then there exists a subsequence f_{n_k} that converges to f a.e. on $[0,1]$.
15. Let μ be a Radon measure. Show that for each $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\limsup_{y \to x} \mu(\bar{B}(y,r)) \leq \mu(\bar{B}(x,r)).$$

16. Let μ and ν be Radon measures on $\mathbb{R}^n$. Then (i) there exist measures $\nu_{ac} \ll \mu$ and $\nu_s \perp \mu$ so that $\nu = \nu_{ac} + \nu_s$, and (ii) $D_\mu \nu(x) = D_\mu \nu_{ac}(x)$ and $D_\mu \nu_s = 0$, both for μ-a.e. x so that for each Borel set A we have

$$\nu(A) = \int_A (D_\mu \nu) d\mu + \nu_s(A).$$

17. Let μ be a Radon measure and assume that $f \in L^1_{loc}(\mathbb{R}^n, d\mu)$, then

$$\lim_{r \to 0} \frac{1}{|\bar{B}(x,r)|} \int_{\bar{B}(x,r)} f d\mu = f(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

18. Let ν be a signed measure, and E be a measurable set, $0 < \nu(E) < +\infty$. Show that there exists a positive set $A \subseteq E$ with $\nu(A) > 0$.
19. Show that if $f \in L^1_{per}([0,1])$ then its Fourier coefficients

$$f_k = \int_0^1 f(x) e^{-2\pi i k x} dx,$$

satisfy $\hat{f}_k \to 0$ as $k \to +\infty$.
20. Let $f \in L^1(\mathbb{T})$ satisfy the following condition at a point x: there exists $\delta > 0$ so that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < +\infty. \qquad (3)$$

Show that then the partial sums of the Fourier series satisfy $\lim_{N \to \infty} S_N f(x) = f(x)$.
21. Let f be a periodic function that is in all $L^p[0,1]$, $1 \leq p \leq +\infty$, and let

$$f_k = \int_0^1 f(x) e^{-2\pi i k x} dx,$$

be its Fourier coefficients. Define the partial sums of the Fourier series

$$S_N f(x) = \sum_{k=-N}^{N} f_k e^{2\pi i k x},$$

and their Cesaro averages

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x).$$

Show that $\sigma_N f(x)$ converges to $f(x)$ in all $L^p[0,1]$, $1 \le p < +\infty$.

22. Let $f(x)$ and $g(x)$ be two real-valued Schwartz class functions defined on $\mathbb{R}^n$. Prove that

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}} \hat{f}(x)g(x)dx,$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int |\hat{f}(\xi)|^2 d\xi.$$

Prove the Fourier inversion formula:

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

23. The Hilbert transform for a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is defined via its Fourier transform as

$$\widehat{Hf}(\xi) = (-i\text{sgn}\xi)\hat{f}(\xi).$$

The Hilbert transform is a bounded operator for any L^p to itself with $1 < p < +\infty$. Prove this when p is an even integer.

24. Let X_k be a sequence of independent, identically distributed random variables, such that $\mathbb{E}(|X_k|^{2016}) < +\infty$, and set

$$s_N = \frac{X_1 + \ldots + X_N}{N}.$$

Show that for any function $f \in \mathcal{S}(\mathbb{R})$ we have

$$\mathbb{E}(f(s_N)) \to f(0), \quad \text{as } n \to +\infty.$$

25. Let X_k be a sequence of independent, identically distributed random variables, such that $\mathbb{E}(|X_k|^{2016}) < +\infty$, and set

$$S_N = \frac{X_1 + \ldots + X_N}{\sqrt{N}}.$$

Show that for any function $f \in \mathcal{S}(\mathbb{R})$ we have

$$\mathbb{E}(f(s_N)) \to \int \hat{f}(x) e^{-D|x|^2} dx, \quad \text{as } n \to +\infty,$$

with some $D > 0$. What is D?