Homework # 5

1. (i) Let a_k be the Fourier coefficients of the function $u(x) = 1/2 - x$, $0 < x < 1$, extended periodically:

$$a_k = \int_0^1 (\frac{1}{2} - x)e^{-2\pi ikx}dx.$$

Show that there exists a constant $M > 0$ so that for all $x \in [0,1]$ and all $N \in \mathbb{Z}$ the partial sums of the Fourier series satisfy

$$\left|\sum_{k=-N}^{N} a_k e^{2\pi ikx}\right| \leq M.$$

(ii) Let two integer sequences $N_k \geq 1$ and $m_k \geq 1$ be such that

$$N_{k+1} - m_{k+1} > N_k + m_k,$$

and define

$$f(x) = \sum_{k=1}^{\infty} e^{2\pi iN_k x} \frac{B_k(x)}{k^2},$$

with

$$B_k(x) = \sum_{j=-m_k}^{m_k} a_j e^{2\pi ijx}.$$

Show that the function $f(x)$ is continuous and find its Fourier coefficients in terms of a_k.
(iii) Show that the partial Fourier sums $S_{N_k} f(0)$ satisfy a lower bound

$$S_{N_k} f(0) \geq \frac{C \log m_k}{k^2}.$$

(iv) Find a choice of m_k and N_k so that the Fourier series of $f(x)$ at $x = 0$ diverges.

2. (i) A measurable function f is said to be in weak L^1 denoted by L^1_w if $m(\lambda) = \lambda \times |\{x : \ |f(x)| > \lambda\}|$ is a bounded function of $\lambda \geq 0$. Show that L^1_w contains L^1 but is larger than L^1.
(ii) Let $f(x) \in C(\mathbb{R})$, $f(x) > 0$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Show that the function $h_c(x) = \sup_n \{n^c f(nx)\}$ is (i) in $L^1(\mathbb{R})$ if $c \in (0,1)$, (ii) is in $L^1_w(\mathbb{R})$ but not in $L^1(\mathbb{R})$ if $c = 1$, (iii) not in L^1_w if $c > 1$.

4. (i) Let $g_n = \chi_{[-n,n]}(x)$, compute $h_n = g_n \star g_1$ and show that h_n is a Fourier transform of a multiple of the function

$$f_n(x) = \frac{\sin x \sin(nx)}{x^2}.$$

(ii) Use (i) to show that the Fourier transform maps L^1 into a proper subset of $C_0(\mathbb{R})$ and not onto $C_0(\mathbb{R})$.
(iii) Show that the image of the Fourier transform is dense in $C_0(\mathbb{R})$.

5. (i) Let $f \in C(\mathbb{S}^1)$ have a modulus of continuity $\omega(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$. Show that $|\hat{f}(n)| \leq C\omega(1/2n)$.
(ii) Assume that f is absolutely continuous, show that $\hat{f}(n) = o(1/n)$ as

$n \to +\infty$.

(iii) Show that $f \in L^1(\mathbb{S}^1)$ is equal to an analytic function a.e. on $\mathbb{S}^1$ if and only if there exist $c > 0$ and $A > 0$ so that $|\hat{f}(n)| \leq Ae^{-cn}$.