Homework # 4.

1. Let $f(t) \in L^p(\Omega)$, where $\Omega \in \mathbb{R}^n$. We say that

$$\mu(t) = m\{x \in \Omega : \ |f(x)| > t\}$$

is the distribution function of $|f|$. Show that (this is a version of the Chebyshev inequality)

$$\mu(t) \leq \frac{1}{t^p} \|f\|^p_{L^p(\Omega)}$$

and

$$\|f\|^p_{L^p(\Omega)} = p \int_0^\infty t^{p-1} \mu(t) dt.$$

More generally, show that we have for any differentiable increasing function $\phi(s)$ such that $\phi(0) = 0$:

$$\int_\Omega \phi(|f(x)|) dx = \int_0^\infty \phi'(\lambda) \mu(\lambda) d\lambda.$$

2. Given two functions $f(x)$ and $g(x)$, $x \in \mathbb{R}^m$, we define their convolution as

$$f \star g(x) = \int_{\mathbb{R}^m} f(x-y) g(y) dy.$$

Show that if $f \in L^p(\mathbb{R}^m)$, $1 \leq p < \infty$, $\phi \geq 0$, $\phi \in L^1(\mathbb{R}^m)$ with $\int \phi = 1$ and $\phi_t = t^{-m}\phi(x/t)$, then

$$\lim_{t \to 0} \|\phi_t \star f - f\|_p = 0.$$

3. Let μ be a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(X; d\mu)$ and let

$$\alpha_n = \int_X |f|^n d\mu, \ n \in \mathbb{N}.$$

Prove that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

4. Let

$$\phi_0(t) = \begin{cases} 1, \ x \in [0,1] \\ -1, \ x \in [1,2) \end{cases},$$

extend it periodically to all of $\mathbb{R}$, and define $\phi_n(t) = \phi_0(2^n t)$, $n \in \mathbb{N}$. Assume that $\sum |c_n|^2 < \infty$ and show that the series

$$\sum_{n=1}^\infty c_n \phi_n(t)$$

converges for almost every t.

5. Let $B(x, \delta)$ be a ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$ and define

$$\tilde{M} f(x) = \sup_{\delta > 0} \frac{1}{\mu(B(y,\delta))} \int_{B(y,\delta)} |f(y)| dy.$$

with the supremum taken over all balls $B(y, \delta)$ such that $x \in B(y, \delta)$.
(i) Use the covering lemmas to show that there exists a constant $c > 0$

that depends only on dimension n so that (m is the n-dimensional Lebesgue measure)

$$m\{x:\ \tilde{M}f(x)>\alpha\})\le \frac{c}{\alpha}\int_{\mathbb{R}^n}|f(y)|dy.$$

Now, show that if $f\in L^p(\mathbb{R}^n)$, then $\tilde{M}f\in L^p(\mathbb{R}^n)$. Hint: introduce $f_1(x)=f(x)$ if $|f(x)|>\alpha/2$ and $f_1(x)=0$ if $|f(x)|\le\alpha/2$ and show that $\{\tilde{M}(f)>\alpha\}\subset\{\tilde{M}(f_1)>\alpha/2\}$ so that

$$m(\{x:\ \tilde{M}f(x)>\alpha\})\le \frac{c}{\alpha}\int_{|f|\ge\alpha/2}|f(y)|dy.$$

Then use the relation

$$\int(\tilde{M}f)^p dy = p\int_0^\infty \mu(\tilde{M}f>\alpha)\alpha^{p-1}d\alpha$$

to finish the proof. This also proves that $\tilde{M}f$ is finite a.e.

(ii) Show that if $f\in L^1$ and $f\ne 0$ identically then there exist $C,R>0$ so that $\tilde{M}f(x)\ge C|x|^{-n}$ (here n is the space dimension) for all x with $|x|\ge R$. Hence $m(\{x:\ \tilde{M}(x)>\alpha\})\ge C'/\alpha$.

6. Set

$$Mf(x)=\sup_{\delta>0}\frac{1}{\mu(B(x,\delta))}\int_{B(x,\delta)}|f(y)|dy.$$

with the supremum taken over all balls $B(x,\delta)$.

(i) Show that if $f\in L^1$ and $f\ne 0$ identically then there exist $C,R>0$ so that $\tilde{M}f(x)\ge C|x|^{-n}$ (here n is the space dimension) for all x with $|x|\ge R$. Hence $m(\{x:\ \tilde{M}(x)>\alpha\})\ge C'/\alpha$. Show also that $M(x)\le\tilde{M}(x)\le 2^nM(x)$.

(ii) Show that $|f(x)|\le Mf(x)$ at every Lebesgue point of f if $f\in L^1(\mathbb{R}^n)$.