

### Homework # 3.

The first two problems need not be turned in:

0.1. Given  $p > 1$  and  $q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$  prove the Hölder inequality:

$$\int |fg| \leq \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q}.$$

You may find this in every book but try to do that on your own. First show that for any numbers  $\alpha, \beta \geq 0$  and  $0 < s < 1$  we have  $\alpha^s \beta^{1-s} \leq s\alpha + (1-s)\beta$ . Then assume that  $\|f\|_p = \|g\|_q = 1$ , use this inequality with  $\alpha = |f(t)|^p$ ,  $\beta = |g(t)|^q$  and  $s = 1/p$ ,  $1-s = 1/q$  and integrate in  $t$ . Finally, show that the case  $\|f\|_p = \|g\|_q = 1$  is sufficient.

0.2. Show that the spaces  $L^p$  are complete: this is again in every book but do try to do this on your own. One way to prove this is to show first that a metric space is complete if every absolutely summable series is summable. To do that take a Cauchy sequence, choose a nice subsequence and make a telescoping series out of it. Make sure that the telescoping series is absolutely summable and show that this implies existence of the limit of the Cauchy sequence. Then apply this result to  $L^p$ : take an absolutely summable series  $f_n$  in  $L^p$  and consider the sequence of partial sums  $g_n = \sum_{k=1}^n |f_k|$ . It converges point-wise to a function  $g$ . Applying the Fatou lemma we obtain that  $g$  is in  $L^p$ . It remains only to verify that convergence of  $\sum_{k=1}^\infty |f_k|$  implies convergence of  $\sum_{k=1}^\infty f_k$ .

1. (i) Show that step functions and continuous functions are dense in  $L^p[0, 1]$ ,  $1 \leq p < \infty$ . Is this true for  $L^\infty$ ?

(ii) Show that if  $f$  is integrable on  $E$  then

$$\lim_{t \rightarrow 0} \int_E |f(x+t) - f(x)| dx = 0.$$

2. A family  $f_n$  is uniformly integrable on a set  $E$  if  $\mu(E) < \infty$ ,

$$\lim_{\alpha \rightarrow \infty} \sup_n \int_{|f_n| \geq \alpha} |f_n| dx = 0.$$

Show that if  $f_n \rightarrow f$  a.e. then the following are equivalent: (i)  $f_n$  are uniformly integrable, (ii)  $\int |f - f_n| \rightarrow 0$ , and (iii)  $\int |f_n| \rightarrow \int |f|$ .

3. (i) Show that if  $|f_n| \leq g \in L^1(\Omega)$  then  $f_n$  are uniformly integrable on  $\Omega$ . Does there exist a family of functions  $f_n$  that is uniformly integrable but there is no integrable function  $g$  so that  $|f_n| \leq g$ ?

(ii) Let  $f_k$  and  $g_k$  be  $\mu$ -measurable functions such that  $f_k \rightarrow f$   $\mu$ -a.e.,  $g_k \rightarrow g$   $\mu$ -a.e. and  $|f_k| \leq g_k$ . Suppose that  $\int g_k d\mu \rightarrow \int g d\mu$  and show that then  $\int f_k d\mu \rightarrow \int f d\mu$ .

4. A monotone function is called singular if  $f' = 0$  a.e.

(i) Show that any monotone increasing function is a sum of an absolutely continuous function and a singular function.

(ii) Does there exist a strictly increasing monotone function that is singular?

5. Construct an absolutely continuous strictly increasing function on  $[0, 1]$  such that  $g' = 0$  on a set of positive measure.

6. Let  $\mathcal{J}$  be a collection of non-degenerate closed intervals on  $\mathbb{R}$ , and let  $A$  be the set of centers of all intervals in  $\mathcal{J}$ . Assume that for every  $\varepsilon > 0$  and every center  $x \in A$  there exists an interval  $I \in \mathcal{J}$  centered at  $x$  with  $|I| < \varepsilon$ . Finally, assume that

$$D = \sup\{|I| : I \in \mathcal{J}\} < +\infty.$$

Show by hand that there exist two sub-collections  $\mathcal{F}_1$  and  $\mathcal{F}_2$  of intervals in  $\mathcal{J}$  so that each  $\mathcal{F}_{1,2}$  is a countable collection of pairwise disjoint intervals and the set  $A$  is covered by  $\mathcal{F}_1 \cup \mathcal{F}_2$ .

7\*. In dimension two the answer is that you need not two but 19 such sub-collections. Try to prove this.