Homework # 2.

1. Let μ be a Borel measure on $[0,1]$ with $\mu([0,1]) = 1$. Show that there exists a compact set $K \subseteq [0,1]$ so that $\mu(K) = 1$ but $\mu(H) < 1$ for any proper compact subset H of K. K is called the support of μ. Show that every compact subset of $[0,1]$ is the support of some Borel measure.

2. Construct a function such that each set $\{f(x) = \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$ but the set $\{f(x) > 0\}$ is not measurable.

3. Construct a monotone function that is discontinuous on a dense set on $[0,1]$.

4. Let ϕ be a non-negative continuous function on $\mathbb{R}^n$ such that $\int \phi = 1$. Given $t > 0$ define $\phi_t(x) = t^{-n}\phi(x/t)$. Show that if $g \in C^\infty(\mathbb{R}^n)$ with compact support then

$$\phi_t(g) = \int_{\mathbb{R}^n} \phi_t(x) g(x) dx \to g(0).$$

Because of that ϕ_t is called an approximation of identity. How much can you weaken the regularity assumptions on ϕ and g?

5. Let E_k be a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(E_k) < +\infty.$$

Show that then almost all x lie in at most finitely many of the sets E_k.

6. Let

$$\psi(x) = \begin{cases} x, & 0 \le x \le 1/2, \\ 1 - x, & 1/2 \le x \le 1, \end{cases}$$

and extend $\psi(x)$ to a periodic function on all of $\mathbb{R}$. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{4^n} \psi\Big(4^n x\Big).$$

Show that $f(x)$ is continuous on $\mathbb{R}$ but is nowhere differentiable.