Math 205A - Fall 2016
Midterm solutions

Problem 1:
(i) Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$. Show that
$$u(x) = (f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$$
is bounded and continuous.
(ii) Let $E \in \mathbb{R}^n$ be a bounded measurable set. Show that
$$E - E = \{x - y : x, y \in E\}$$
contains an open ball centered at the origin.

(i) $|u(x)| \le \displaystyle\int |f(y)| \sup |g|\,dy = ||g||_\infty \int |f(y)|\,dy = ||g||_\infty ||f||_1 < \infty$, hence u is *bounded.*
u is *continuous* since for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$,
$$|u(x+h) - u(x)| = \left|\int (f(y+h) - f(y))g(x-y)\,dy\right| \le ||g||_\infty \int |f(y+h) - f(y)|\,dy,$$
and the integral goes to 0 as $h \to \infty$ (similar to problem 4).

(ii) We'll prove a slightly stronger result: if A, B are bounded and measurable, with $m(A \cap B) > 0$, then $A - B$ contains an open ball centered at the origin. In deed, let $f = 1_A$ and $g = 1_{-B}$ above. Then $u(x) = m(A \cap (x + B))$, it's continuous at 0 and $u(0) = m(A \cap B) > 0$. This means that for $\delta > 0$ small, $x \in B_\delta(0) \implies u(x) > 0$ too. But $u(x) > 0 \implies a = x + b \in A \cap (x + B)$ for some $a \in A, b \in B$, ie $x = a - b \in A - B$. Hence, $B_\delta(0) \subset A - B$. □

Problem 2: Let $p \in (0, 1)$, $0 < a < p$, and suppose $A_1, \ldots, A_N$ are Lebesgue measurable subsets of $[0, 1]$ with average measure
$$\frac{1}{N}\sum_{i=1}^N \mu(A_i) \ge p.$$
Let
$$E = \{x \in [0, 1] : x \in A_i \text{ for at least } aN \text{ values of } i\}.$$
Show that $\mu(E) \ge (1 - p)/(1 - a)$.

Let $f(x) = \displaystyle\frac{1}{N}\sum_{i=1}^N 1_{A_i}(x)$. The number of i's for which $x \in A_i$ is $Nf(x)$, therefore $E = \{x : f(x) \ge a\}$, and
$$\mu(E) = \mu(\{x : f(x) \ge a\}) = \mu(\{x : 1 - f(x) \le 1 - a\})$$
$$\overset{Chebychev}{\le} \frac{1}{1-a}||1 - f||_1 = \frac{1}{1-a}(1 - \frac{1}{N}\sum_{i=1}^N \mu(A_i)) \le \frac{1-p}{1-a}.$$
□

Problem 3: Let $f, g \in L^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and assume that for all $\psi \in C^\infty(\mathbb{T})$, we have
$$\int_0^{2\pi} f(t)\psi'(t)\,dt = -\int_0^{2\pi} g(t)\psi(t)\,dt,$$
i.e. that g is a *weak derivative* of f. Show that f is absolutely continuous, and that $f' = g$ almost everywhere.

f is absolutely continuous if and only if it's an indefinite integral with an L^1 integrand, and that's what we'll try to show, i.e. that $f(t) = \int_0^t g(s)\,ds + a$ a.e. t, with a to be determined later. In deed, let $F(t) = f(t) - \int_0^t g(s)\,ds - a$. Then for any C^∞ function ψ we have

$$\begin{aligned}
\int_0^{2\pi} F(t)\psi'(t)\,dt &= \int_0^{2\pi} f(t)\psi'(t)\,dt - \int_0^{2\pi}\int_0^t g(s)\psi'(t)\,ds\,dt - \int_0^{2\pi} a\psi'(t)\,dt \\
&= \int_0^{2\pi} f(t)\psi'(t)\,dt - \underbrace{\int_0^{2\pi}\int_s^{2\pi} g(s)\psi'(t)\,dt\,ds}_{\text{Fubini}} - a\underbrace{(\psi(2\pi)-\psi(0))}_{=0 \text{ on } \mathbb{T}}\,dt \\
&= \int_0^{2\pi} f(t)\psi'(t)\,dt - \int_0^{2\pi} g(s)(\psi(2\pi)-\psi(s))\,ds \\
&= \underbrace{\int_0^{2\pi} f(t)\psi'(t)\,dt + \int_0^{2\pi} g(s)\psi(s)\,ds}_{I} - \psi(2\pi)\underbrace{\int_0^{2\pi} g(s)ds}_{II} = 0.
\end{aligned}$$

In the last step, $I = 0$ from the condition of the problem, and $II = 0$ by applying the same condition with $\psi \equiv 1$.

So $\int_0^{2\pi} F(t)\psi'(t)\,dt = 0$ for all C^∞ ψ's. In general, for $\phi \in C^\infty$ we have

$$\phi(t) = \psi'(t) + b, \quad \text{where} \quad b = \int_0^{2\pi} \phi(s)\,ds \quad \text{and} \quad \psi(t) = \int_0^t (\phi(s) - b)\,ds \in C^\infty(\mathbb{T}),$$

so

$$\int_0^{2\pi} F\phi = \int_0^{2\pi} F\psi' + b\int_0^{2\pi} F = b\int_0^{2\pi} F.$$

Now we can choose a in the definition of F so that $\int_0^{2\pi} F = 0$, assuring that $\int_0^{2\pi} F\phi = 0$ for all $\phi \in C^\infty(\mathbb{T})$.

The last step is to choose ϕ in C^∞ with compact support in $[0, 2\pi]$ and $\int \phi = 1$, let $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$ on $[0, 2\pi]$, extended periodically on $\mathbb{R}$. Then $F * \phi_t \xrightarrow{L^1} F$, for example by homework 4 (or you could use pointwise convergence from homework 1). But $F * \phi_t = 0$ by what we just proved, so $||F||_1 = 0$, implying $F = 0$ a.e.
So $F = 0$ a.e., hence $f(t) = \int_0^t g(s)\,ds + a$ a.e., and therefore it's absolutely continuous, with $f' = g$ a.e. □

Problem 4: Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{t\to 0}\int_{\mathbb{R}} |f(x+t) - f(x)|\,dx = 0.$$

Step functions are dense in L^1, so given ϵ, pick a step function s such that $||f - s||_{L^1} < \epsilon$. Because

$$\int |f(x+t) - f(x)| \le \int |f(x+t) - s(x+t)| + \int |s(x+t) - s(x)| + \int |f(x) - s(x)| < 2\epsilon + \int |s(x+t) - s(x)|,$$

it's enough to prove the statement for the step functions $s = \sum_1^n c_i\chi_{[a_i,b_i]}$ ($f \in L^1$ implies $s \in L^1$, so all the intervals must be bounded). $|s(\cdot+t) - s(\cdot)|$ are dominated by $2\sum_1^n |c_i|\chi_{[a_i,b_i]}$, which is integrable, and they converge pointwise to 0 a.e.,so by dominated convergence $\int |s(x+t) - s(x)| \to 0$ as $t \to 0$. □

Problem 5: Suppose $f_n(x) \ge 0$, f_n are Lebesgue measurable on $[0, 1]$ and $f_n(x) \to 0$ a.e. on $[0, 1]$. Assume that

$$\int_0^1 \phi(f_n(x))\,dx \le 1$$

for some continuous function ϕ such that $\lim_{t\to\infty} \phi(t)/t = \infty$. Show that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0.$$

Fix $\epsilon > 0$. By Egorov's theorem, there exists N_δ and a set B_δ with $|B_\delta| \le \delta$ such that for all $n \ge N_\delta$, $|f_n(x)| \le \delta$ in $[0,1]\backslash B_\delta$, with δ to be chosen later. Then for $n \ge N_\delta$,

$$\int_0^1 f_n \le \delta + \int_{B_\delta} f_n = \delta + \int_{\{f_n \le M\} \cap B_\delta} f_n + \int_{\{f_n > M\} \cap B_\delta} f_n \le \delta + M\delta + \int_{\{f_n > M\} \cap B_\delta} f_n,$$

for any M which will be chosen independently of δ at the end. Finally for the last integral note that given N, there exists M_N such that for all $t \ge M_N$, $\phi(t)/t \ge N$, i.e. $t \le \phi(t)/N$. Then if $M \ge M_N$ and $f(x) \ge M \ge M_N$, $\phi(f_n(x)) \le \phi(f_n(x))/N$, so

$$\int_{\{f_n > M\} \cap B_\delta} f_n \le \frac{1}{N} \int_0^1 \phi(f_n) \le \frac{1}{N}.$$

Hence,

$$\int_0^1 f_n \le \delta + M\delta + \frac{1}{N}.$$

Now we can pick the constants $\delta = \epsilon/3$, N such that $1/N \le \epsilon/3$, and $M = \max\{1/3, M_N\}$, to ensure that for all $n \ge N_\delta$, $\int_0^1 f_n \le \epsilon$. ☐

Problem 6: Let f be an absolutely continuous function on $[0,1]$. Show that if a set A is measurable, then $f(A)$ is measurable. Hint: start by showing that if E has Lebesgue measure 0, then $f(E)$ has Lebesgue measure 0 as well.

Let us prove first the assertion of the hint: if $m(E) = 0$, then $m(f(E)) = 0$. In deed, fixing ϵ, from the absolute continuity of f, there exists a δ such that $\sum |y_i - x_i| \le \delta$ implies $\sum |f(y_i) - f(x_i)| \le \epsilon$. Now Lebesgue measure is a Radon measure, so there exists an open set O_δ with $m(O_\delta \backslash E) \le \delta$. Write O_δ as $O_\delta = \bigcup (a_i, b_i)$ with $\sum |b_i - a_i| \le \delta$. f is continuous, so it achieves its maximum and minimum over $[a_i, b_i]$ at c_i and d_i respectively. This means that $f((a_i, b_i))$ is contained in the interval with endpoints $f(c_i)$ and $f(d_i)$, and $m(f((a_i, b_i))) \le |f(c_i) - f(d_i)|$. On the other hand, $|c_i - d_i| \le |b_i - a_i|$.

Combining the assertions above, $m(f(E)) \le m(O_\delta) \le \sum |f(c_i) - f(d_i)|$. But $\sum |c_i - d_i| \le \sum |a_i - b_i| \delta$, so $\sum |f(c_i) - f(d_i)| \le \epsilon$ from the absolute continuity of f. Hence, $m(f(E)) \le \epsilon$ for all ϵ, implying that $m(f(E)) = 0$.

Now A is measurable, so there exist compact sets K_i with $m(A \backslash K_i) \le 1/i$. Then $A = (\bigcup_i K_i) \bigcup E$, where $m(E) = 0$. The image of compact sets under continuous functions is compact, so $f(K_i)$ is compact, hence measurable for all i's. Also $f(E)$ is measurable since it has measure 0. Combining these we get that $f(A) = (\bigcup_i f(K_i)) \bigcup f(E)$ is measurable. ☐