Math 205A - Fall 2016
Homework #3
Solutions

Problem 1: **(i)** Show that step functions and continuous functions are dense in $L^p[0,1], 1 \leq p < \infty$. Is this true for L^∞?
(ii) Show that if f is integrable in E, then $\lim\limits_{t\to 0} \int_E |f(x+t) - f(x)|dx = 0$.

(i) A step function is a function of the form $\sum_1^n c_i \chi_{A_i}$, where A_i's are finitely many disjoint intervals.
Let $\mathcal{S}$ and $\mathcal{C}$ be the vector spaces of step functions and of continuous functions in $[0,1]$ respectively. (Note that finite linear combinations of step functions are again step functions, and the same holds true for continuous functions.) Want to show that $L^p = \bar{\mathcal{S}}$ and $L^p = \bar{\mathcal{C}}$, where closure is taken w.r.t. the L^p norm.

$\underline{\bar{\mathcal{S}} = L^p:}$ Note that $\mathcal{S} \subset L^p$, so $\bar{\mathcal{S}} \subset L^p$. For the reverse inclusion $L^p \subset \bar{\mathcal{S}}$, we show first that characteristic functions lie in $\bar{\mathcal{S}}$, then extend the result to simple functions, and finally conclude the proof using their density in L^p.
In deed, $\chi_A \in \bar{\mathcal{S}}$ for $A \subset [0,1]$ measurable: given ϵ, $A \subset U$ for some open U, with $m(U - A) < \epsilon^p/2$. U is a countable union of disjoint intervals I_i. Pick finitely many of those to ensure $m(U - \cap_1^n I_i) \leq \epsilon^p/2$, so $m(A - \cup_1^n I_i) \leq \epsilon^p$. Then $||\chi_A - \sum_1^n \chi_{U_i}||_p = m(A - \cup_1^n U_i)^{1/p} \leq \epsilon$. Hence, χ_A can be approximated by step functions.
The same is true for simple functions, because if $f = \sum_1^\infty c_i \chi_{A_i}$ for disjoint A_i's and $c_i \neq 0$, we can pick finitely many of those to ensure $||f - \sum_1^n c_i \chi_{A_i}||_p < \epsilon/2$, and then approximate each χ_{A_i} by the step function s_i such that $||f - s_i||_p < 2^{-(i+1)} c_i^{-1} \epsilon$. $s = \sum_1^n s_i$ is again a step function, and $||f - s||_p < \epsilon$.
So simple functions are included in $\bar{\mathcal{S}}$, and since their closure in L^p is L^p itself, then $L^p \subset \bar{\mathcal{S}}$.

$\underline{\bar{\mathcal{C}} = L^p:}$ Exactly the same reasoning, up to an interpolation argument, works, because the step functions themselves can be approximated by continuous functions (in the L^p norm). Indeed, given ϵ and an interval $[a,b] \subset [0,1]$, the function f that equals 0 on $[a,b]^c$, 1 on some $[a',b'] \subset [a,b]$ with $m([a,b] \backslash [a',b']) < \epsilon^p$, and linear interpolation in between, satisfies $||\chi_{[a,b]} - f||_p < \epsilon$. So continuous functions are dense in the step functions, and hence, L^p.

Step functions are <u>not</u> dense in L^∞: let $f = \sum_{n=2}^\infty \chi_{[\frac{1}{2n+1}, \frac{1}{2n}]}$. $f \in L^\infty$, but $||f - s||_\infty \geq 1/2$ for any step function s. Continuous functions are <u>not</u> dense in L^∞, because $[0,1]$ is compact, and uniform (L^∞) limits of uniformly continuous functions are continuous.

(ii) Note that the problem makes sense only if f is defined on $E' := E + (-\delta, \delta) : \{x + t : x \in E, t \in (-\delta, \delta)\}$ and all the t's in consideration are $|t| < \delta$, so let's extend f by 0 outside E. f is integrable on E', hence in $L^1(E')$. Step functions are dense there, so given ϵ, pick s a step function on E', such that $||f - s||_{L^1(E')} < \epsilon$. Since $f = 0$ on $E' \backslash E$, we can take w.l.o.g. $s = 0$ there too. Because

$$\int_E |f(x+t) - f(x)| \leq \int_E |f(x+t) - s(x+t)| + \int_E |s(x+t) - s(x)| + \int_E |f(x) - s(x)| < 2\epsilon + \int_E |s(x+t) - s(x)|,$$

it's enough to prove the statement for the step functions $s = \sum_1^n c_i \chi_{[a_i,b_i]}$ in E ($f \in L^1$ implies $s \in L^1$, so all the intervals must be bounded). $|s(\cdot + t) - s(\cdot)|$ are dominated by $2\sum_1^n |c_i| \chi_{[a_i - \delta, b_i + \delta]}$, which is integrable, and they converge pointwise to 0 a.e.,so by dominated convergence $\int_E |s(x+t) - s(x)| \to 0$ as $t \to 0$. □

Problem 2: Show that if $\mu(E) < \infty$ and $f_n \to f$ a.e., then the followings are equivalent:
(i) f_n are uniformly integrable, (ii) $\int |f_n - f| \to 0$, (iii) $\int |f_n| \to \int |f|$.

We will assume in this problem that f is integrable, otherwise one can come up with counterexamples for the equivalence (for example $f_n = n\chi_{[0,1/n]} + 1/x \chi_{(1/n,1)} \to 1/x = f$ a.e., and $\int_{[0,1]} |f_n| \to \int_{[0,1]} |f| = \infty$, but f_n are not uniformly integrable).
(i) $\implies$ **(ii):** Let $\epsilon > 0$. From uniform integrability there exists an α_ϵ such that $\int_{\{|f_n| \geq \alpha_\epsilon\}} |f_n| < \epsilon/4$. Now pick $\delta < \frac{\epsilon}{4\alpha_\epsilon}$ such that $\mu(A) \leq \delta \implies \int_A |f| \leq \epsilon/4$. From Egorov's theorem, exists a measurable set $E_\epsilon \subset E$ such that

$\mu(E\backslash E_\epsilon) \le \min\{\epsilon/4, \delta\}$, and $f_n \to f$ uniformly in E_ϵ. This implies that exists some N_ϵ such that for all $n > N_\epsilon$, $\int_{E_\epsilon} |f_n - f| < \epsilon/4$, and therefore

$$\begin{aligned}\int_E |f_n - f| &\le \int_{E_\epsilon} |f_n - f| + \int_{E\backslash E_\epsilon} |f_n| + \int_{E\backslash E_\epsilon} |f| \\ &\le \tfrac{\epsilon}{4} + \int_{\{|f_n|\ge\alpha_\epsilon\}} |f_n| + \int_{\{|f_n|<\alpha_\epsilon\}\cap\{E\backslash E_\epsilon\}} |f_n| + \tfrac{\epsilon}{4} \\ &\le \tfrac{\epsilon}{2=4} + \alpha_\epsilon \mu(E\backslash E_\epsilon) + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} \le \tfrac{3\epsilon}{4} + \alpha_\epsilon \cdot \delta \le \epsilon.\end{aligned}$$

This proves that $\lim \int |f_n - f| \le \epsilon$ for all ϵ, hence (ii).
(ii) $\Longrightarrow$ (iii): $\limsup \int |f_n| \le \lim \int (|f_n - f| + |f|) = \int |f|$. On the other hand, $\int |f| \le \int |f_n - f| + \int |f_n|$, so $\int |f| \le \liminf \int |f_n - f| + \liminf \int |f_n| = \liminf \int |f_n|$. Hence, $\lim \int |f_n - f|$ exists and equals $\int |f|$.
(iii) $\Longrightarrow$ (i): Suppose not. Then there exists an $\epsilon > 0$, $\alpha_k \to \infty$ and a sequence of n_k's such that $\int_{|f_{n_k}|\ge\alpha_k} |f_{n_k}| \ge \epsilon$. On the other hand, $g_k = |f_{n_k}|\chi_{\{|f_{n_k}|\ge\alpha_k\}} \to 0$ a.e. ($\chi_{\{|f_{n_k}|\ge\alpha_k\}} \to 0$ because $\int |f_n| \le \int |f| + \delta$ after some N). $g_k \le |f_{n_k}|$ and $\int |f_{n_k}| \to \int |f|$, so by problem 3, $\int g_k \to 0$, contradicting $\int g_k = \int_{|f_{n_k}|\ge\alpha_k} |f_{n_k}| \ge \epsilon$ □

Problem 3: **(i)** Show that if $|f_n| \le g \in L^1(\Omega)$, then f_n are uniformly integrable in Ω. Does there exist a uniformly integrable family $\{f_n\}$ with no integrable g such that $|f_n| < g$?
(ii) Let f_k and g_k be μ-measurable, such that $f_k \to f$ μ-a.e., $g_k \to g$ μ-a.e., $|f_k| \le g_k$ and $\int g_k \to \int g$. Show that $\int f_k \to \int f$.

(i) $\int_{|f_n|\ge\alpha} |f_n| \le \int_{g\ge\alpha} g$ since $|f_n| \le g$, and $\lim_{\alpha\to\infty} \int_{g\ge\alpha} g = 0$ since $g \ge 0$ is integrable.
Yes. $f_n = \chi_{[n,n+1]}$ on $\mathbb{R}$ are uniformly integrable (consider $\alpha > 1$), and if $f_n \le g$, then $g \ge 1$ on $(0, \infty)$, which is not integrable.
(ii) Note that here we need to assume that g is integrable, otherwise the result is not true.
$|f_k| \le g_k$ implies $g_k - f_k \ge 0$ and $g_k + f_k \ge 0$. Apply Fatou's lemma to both:
$\int (g - f) \le \liminf \int (g_k - f_k) = \int g - \limsup \int f_k \implies \limsup \int f_k \le \int f$.
$\int (f + g) \le \liminf \int (f_k + g_k) = \liminf \int f + \int g \implies \liminf \int f_k \ge \int f$.
Combining these gives the result. □

Problem 4: **(i)** Show that any increasing function is a sum of an absolutely continuous and a singular function.
(ii) Does there exist a strictly increasing singular function?

(i) Let f be a monotone function. f' exists a.e., so let $g(x) = \int_0^x f'$, and $h = f - g$. Then g is absolutely continuous, and h is singular.
(ii) *Yes.* Consider the strictly increasing function $f(x) = \sum_{q_n\in\mathbb{Q}} 2^{-n}\chi_{(q_n,\infty)}$ from HW2, and let h be its singular part. h is increasing because $h(y) - h(x) = f(y) - f(x) - \int_x^y f' \ge 0$ for $y > x$. If h wasn't strictly increasing, then it would be constant on some interval $[x, y]$, hence continuous there, so $f = g + h$ would also be continuous on $[x, y]$, contradicting the discontinuity of f on a dense subset of $[0, 1]$. □

Problem 5: Construct an absolutely continuous strictly increasing function on $[0, 1]$ such that $g' = 0$ on a set of positive measure.

Consider the set E from HW1 with $0 < m(E \cap I) < |I|$ for all intervals I in [0.1], and let $f(x) = \int_0^x \chi_E$. f is an indefinite integral, hence *absolutely continuous*. It is *strictly increasing* since for $y > x$, $f(y) - f(x) = m(E \cap [x, y]) > 0$. This in turn implies that $f' = \chi_E$, so $f' = 0$ on E^c with $m(E^c) > 0$. □

Problem 6: Show that there exist two countable sub-collections $\mathcal{F}_1$, $\mathcal{F}_2$ of pairwise disjoint intervals, such that $\mathcal{F}_1 \cup \mathcal{F}_2$ covers A.

We'll first cover $A \cap (0,1)$, then extend the argument to the whole $\mathbb{R}$, so assume for now that $A \subset (0,1)$. The strategy is to initially cover A inductively by a countable collection of intervals that are *not necessarily disjoint*; afterwards we'll rearrange these intervals into 2 sub-collections, each of them disjoint.

Step 1. Constructing a countable cover for A.
Let $A_1 = A$, $\mathcal{G}_1 = \{I \in \mathcal{F} : I \subset (0,1) \text{ and center of } I \text{ is in } A_1\}$, $\alpha_1 = \sup\{|I| : I \in \mathcal{G}_1\} \leq 1$. If $A_1 = \emptyset$, there's nothing to prove. Otherwise $\alpha_1 \neq 0$ because of the non-degeneracy, so choose $I_1 \in \mathcal{G}_1$ centered at $x_1 \in A_1$ with $|I_1| > 3/4\alpha_1$.

Given $A_i, \mathcal{G}_i, I_i$ for $i = 1, \ldots, n-1$, define $A_n = A \backslash \cup_{i=1}^{n-1} I_i$, $\mathcal{G}_n = \{I \in \mathcal{F} : I \subset (0,1) \text{ and center of } I \text{ is in } A_n\}$, and $\alpha_n = sup\{|I| : I \in \mathcal{G}_n\}$. If $\alpha_n = 0$, then $A \subseteq \cup_{i=1}^{n-1} I_i$ (remember, the intervals are non-degenerate). Otherwise again pick $I_n \in \mathcal{G}_n$ centered at $x_n \in A_n$ with $|I_n| > 3/4\alpha_n$.

First, $\alpha_n \to 0$: In deed, if $\alpha_n = 0$ for some n, we're done. Otherwise $\alpha_{n+1} \leq \alpha_n$, so say $\alpha_n \downarrow \alpha \geq 0$. If $m > n$, then $x_m \notin I_n$, so $|x_m - x_n| \geq |I_n|/2 \geq 3/8\alpha_n \geq 3/8\alpha$. Therefore we have an infinite sequence x_n of elements in (0,1) with distance between any two $\geq 3/8\alpha$, which can only happen if $\alpha = 0$.

Now we claim that $A \subseteq \cup I_n$. If not, let $x \in A \backslash (\cup I_n)$, and $I \subset (0,1)$ any interval in $\mathcal{F}$ centered at x. Since $x \in A \backslash (\cup I_n)$, then for all n, $x \in A_n$, so $I \in \mathcal{G}_n$, therefore $|I| \leq \alpha_n$. But $\alpha_n \to 0$, hence $|I| = 0$, contradicting the non-degeneracy assumption.

Step 2. Getting rid of the 'redundant intervals'.

We'll now get a new sub-collection I'_n that has 'less' overlaps than the original one as follows: If $A \subset \cup_2^\infty I_n$, let $I'_1 = \emptyset$, otherwise $I'_1 = I_1$. In step n, if $A \subset (\cup_1^{n-1} I'_i) \cup ((\cup_{n+1}^\infty I_i))$, let $I'_n = \emptyset$, otherwise $I'_n = I_n$. Then $A \subset \cup I'_n$, because by construction every point in A is contained only in finitely many of the $I_n s$ (if $x \in I_k$ for the first time, then $dist(x, x_l) > 0$ for $l > k$, and $|I_n| \downarrow 0$), so we could not have removed all of them.

What we achieved this way is that at most two of the non-empty I'_n's overlap at any point, because if I_i, I_j, I_k all intersect, then one of them is included in the others, say I_i. But then $I'_i = \emptyset$, contradicting the non-emptiness.

Step 3. Obtaining $\mathcal{F}_1$ and $\mathcal{F}_2$.

There are many ways to do this, but one nice way is using graph theory: let each I'_n be a vertex of a (possibly infinite) graph, and connect two vertexes iff the corresponding intervals overlap. By the remark above, this can have no cycles, so it's a tree, and hence bipartite. This means that the vertices can be arranged into two sets $S_{1,2}$, each of them with no edges in between. Then put the intervals belonging to the set S_i into $\mathcal{F}_i$!

Step 4. Covering A (not only $A \cap (0,1)$).

For each $n \in \mathbb{Z}$, pick an interval J_n of radius $< 1/2$ if $n \in A$, otherwise do nothing. $\mathbb{R} \backslash (\cup J_n)$ is a disjoint union of open intervals, each $\subseteq (n, n+1)$ for some n, so pick the disjoint collections $\mathcal{F}_1^n, \mathcal{F}_2^n$, also disjoint from the J_n's. Then $\mathcal{F}_1 = \cup_n \mathcal{F}_1^n \cup \{J_n\}_n$ and $\mathcal{F}_2 = \cup_n \mathcal{F}_2^n$. □