Math 205A - Fall 2016
Homework #2
Solutions

Problem 1: Show that there is a support of every Borel measure. Show that every compact set is the support of some Borel measure.

Let $\mathcal{F} = \{(p,q) : p,q \in \mathbb{Q}, \mu((p,q)) = 0\}$. If $\mathcal{F} = \emptyset$, then $K = [0,1]$ because each open interval has positive measure. Otherwise, we claim that $K = O^c$, where $O = (\cup_{\mathcal{F}}(p,q))^c$. Clearly K is compact and $\lambda(K) = 1 - \lambda(O) = 0$. If the compact set $H \subset K$ is a proper subset, $K \backslash H$ contains a rational interval (p,q), and $(p,q) \subset K = O^c$, so (p,q) is not in $\mathcal{F}$. This implies that $\lambda(p,q) > 0$, therefore $\lambda(H) \leq 1 - \lambda(p,q) < 1$.

If $K \subset [0,1]$ is compact & non-empty, it has a dense sequence $x_1, x_2, \ldots$, possibly finite. If it is a finite sequence $x_1, \ldots, x_N$, just let $x_n := x_N$ for $n > N$. Define μ by on the Borel σ-algebra via $\mu(S) := \sum 2^{-n}\delta_{x_n}(S)$. Then $\mu([0,1]) = 1$, and its support is K. □

Problem 2: Construct a function such that each set $\{f(x) = \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$, but the set $\{f(x) > 0\}$ is not measurable.

Let $P \in [0,1)$ be the non-measurable set constructed by picking exactly one element of each of the equivalence classes of the relation $x \sim y \iff x - y \in \mathbb{Q}$. Define f by

$$f(x) = \begin{cases} x & x \in P \\ -|x| & x \notin P. \end{cases}$$

$\{f(x) = \alpha\} \subseteq \{-\alpha, +\alpha\}$ is discrete, hence measurable, whereas $\{f(x) > 0\} = P$, which is not measurable. □

Problem 3: Construct a monotone function that is discontinuous on a dense set of $[0,1]$.

Let $q_1, q_2, \ldots$ be an enumeration of rationals, and let $f(x) := \sum 2^{-n}\chi_{(q_n,\infty)}$. f is monotone, as a sum of increasing functions. Furthermore, it's discontinuous in $\mathbb{Q}$: given n, for all $x > q_n$, $f(x) \geq 2^{-n} + f(q_n)$. □

Problem 4: **(i)** Show that $\phi_t(g) \to g(0)$.
(ii) How much can we weaken the regularity assumptions on ϕ and g?

Let $M = \sup|g(x)| < \infty$ (since g is C^∞ and with compact support), and let $\epsilon > 0$. Then there exists a $\delta > 0$ s.t. $|x| < \delta$ implies $|g(x) - g(0)| < \epsilon$. Now noting that by a change of variables $\int \phi_t(x)dx = 1$, we get:

$$\begin{aligned} |\phi_t(g) - g(0)| = & \left|\int \phi_t(x)(g(x) - g(0))dx\right| \leq \int\limits_{|x|<\delta} \phi_t(x)|g(x) - g(0)|dx + \int\limits_{|x|\geq\delta} \phi_t(x)|g(x) - g(0)|dx \\ \leq & \ \epsilon \int\limits_{|x|<\delta} \phi_t(x)dx + 2M \int\limits_{|x|\geq\delta} \phi_t(x)dx \leq \epsilon \int\limits_{\mathbb{R}^n} \phi_t(x)dx + 2M \underbrace{\int\limits_{|y|\geq\delta/t} \phi(y)dy}_{\text{change of variables } x=ty} \\ \leq & \ \epsilon + 2M \int\limits_{|y|\geq\delta/t} \phi(y)dy \end{aligned}$$

But $\int \phi = 1$ implies that $\int_{|y|\geq\delta/t} \phi \to 0$ as $t \to 0$. Hence, $\lim\limits_{t\to 0} |\phi_t(g) - g(0)| \leq \epsilon$. The result follows by sending $\epsilon \to 0$.

(ii) In the proof above, all we needed was that g is in $L^1 \bigcap L^\infty$ and continuous at 0, and that ϕ is in L^1, non-negative, with $\int \phi = 1$. $\square$

Problem 5: Show that if $\sum_{k=0}^{\infty} \mu(E_k) < \infty$, then almost every x lies in at most finitely many of the E_k's.

The event E that x lies in at most finitely many E_k's can be written as $E = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k^c$. Now

$$\mu(E^c) = \mu(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k) = \lim_{n \to \infty} \mu(\bigcup_{k \geq n} E_k) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0, \quad \text{since } \sum \mu(E_k) < \infty.$$

Hence, $\mu(E) = 1$, i.e. almost every x lies in at most finitely many E_k's. $\square$

Problem 6: Show that $f(x) = \sum 4^{-n} \psi(4^n x)$ is continuous, but nowhere differentiable.

The series is uniformly convergent by Weierstrass M-test, since $|4^{-n}\psi(4^n x)| \leq 4^{-n}$, and $\sum 4^{-n} \leq \infty$. Then f is continuous as the uniform limit of a sequence of continuous functions.

Nowhere differentiability: Fix $x \in \mathbb{R}$. We will show that f is not differentiable at x by finding a sequence $h_m \to 0$ such that $\frac{f(x+h_m)-f(x)}{h_m}$ diverges as $m \to \infty$. Let $h_m = \pm 4^{-m}$, with the sign such that $4^{m-1}x$ and $4^{m-1}(x + h_m)$ have no fraction of the form $k/2, k \in \mathbb{Z}$ between them. Then

$$\frac{f(x + h_m) - f(x)}{h_m} = \sum_{n=1}^{\infty} 4^{-n} \frac{\psi(4^n x + 4^n h_m) - \psi(4^n x)}{h_m} = \sum_{n=1}^{m-1} 4^{-n} \frac{\psi(4^n x + 4^n h_m) - \psi(4^n x)}{h_m},$$

because for $n \geq m$, $4^n h_m = \pm 4^{n-m} \in \mathbb{Z}$, so $\psi(4^n x + 4^n h_m) - \psi(x) = 0$ by periodicity. On the other hand, for $n < m$, $4^n x$ and $4^n(x + h_m)$ can not have a fraction of the form $k/2, k \in \mathbb{Z}$ in between either, otherwise so would $4^{m-1}x$ and $4^{m-1}(x + h_m)$. This means that they both lie on the same linear part of ψ, hence

$$\frac{\psi(4^n x + 4^n h_m) - \psi(4^n x)}{h_m} = \frac{c_{n,m} 4^n h_m}{h_m} = c_{n,m} 4^n,$$

where $c_{n,m} = \pm 1$, giving

$$\frac{f(x + h_m) - f(x)}{h_m} = \sum_{n=1}^{m-1} c_{n,m}.$$

But this is for all m an integer that changes parity whenever m does. Hence, it can not have a limit as $m \to \infty$. $\square$