Math 205A - Fall 2016
Homework # 1
Solutions

Problem 1: Show that f is well-defined, monotone, continuous and that it is constant on every interval contained in the complement of the Cantor set.

Let x have two ternary expansions $0.a_1a_2\ldots$ and $0, a_1'a_2'\ldots$, with a tail of 2s in the first, and let N, N' and b_n, b_n' be the respective quantities of the problem. Then for some m,

$$\begin{aligned} &a_n = a_n' \quad \text{for } n \le m,\\ &a_{m+1} = 0 \text{ or } 1; \quad a_{m+2} = a_{m+3} = \ldots = 2,\\ &a_{m+1}' = a_{m+1} + 1; \quad a_{m+2}' = a_{m+3}' = \ldots = 0. \end{aligned}$$

Note first that $b_n = b_n'$ for $n \le m$, so if if $N \le m$, $b_n = b_n'$ for all n. Otherwise:
If $a_{m+1} = 0$: $a_{m+1}' = 1$, $N = \infty$ and $N' = m+1$, then

$$\sum_1^N \frac{b_n}{2^n} = \sum_1^m \frac{b_n}{2^n} + \frac{0}{2^{m+1}} + \sum_{m+2}^\infty \frac{1}{2^n} = \sum_1^m \frac{b_n}{2^n} + \frac{1}{2^{m+1}}, \qquad \sum_{n=1}^{N'} \frac{b_n'}{2^n} = \sum_{n=1}^m \frac{b_n}{2^n} + \frac{1}{2^{m+1}}.$$

If $a_{m+1} = 1$: $a_{m+1}' = 2$, $N = m+1$ and $N' = \infty$, then

$$\sum_1^N \frac{b_n}{2^n} = \sum_1^m \frac{b_n}{2^n} + \frac{1}{2^{m+1}}, \qquad \sum_1^{N'} \frac{b_n'}{2^n} = \sum_1^m \frac{b_n}{2^n} + \frac{1}{2^{m+1}} + \sum_{m+2}^\infty \frac{0}{2^n} = \sum_1^m \frac{b_n}{2^n} + \frac{1}{2^{m+1}}.$$

In all cases the results coincide. Hence f is well-defined.

Now let $R_n(x)$ be 0 if the interval removed during the n$^{\text{th}}$ step of the construction of the Cantor set lies before x, and 1 otherwise. Then $f(x) = \sum R_n(x)2^{-n}$, and is monotone increasing in x, as so is each $R_n(x)$.

Given an interval I in C^c, each removed interval other then I lies either before, or after I, so all the corresponding $R_n(x)$'s coincide for $x \in I$. When I is removed, that R_n is 1 for all $x \in I$. Hence, $f(x)$ is constant on I.

Finally, f is onto, because given $\sum b_n 2^{-n}$ in $[0,1]$, it's preimage contains at least $\sum (2\cdot b_n)3^{-n}$. So $f : [0,1] \longrightarrow [0,1]$ is monotone and onto, hence continuous. $\square$

Problem 2: Show that $g(x) = f(x) + x$ is a homeomorphism of $[0,1]$ onto $[0,2]$, $m[g(C)] = 1$ and that there exists a measurable set A so that $g(A)$ is not measurable. Show that there is a measurable set that is not a Borel set.

g is continuous, strictly increasing (hence 1-1) and $g(0) = 0$, $g(1) = 2$, so it is onto. Thus g^{-1} exists. g^{-1} is also strictly increasing and onto, hence it is continuous.

C^c is a union of disjoint intervals I_j with $\sum |I_j| = m(C^c) = 1$, and f is constant on each of them. On an interval I_j, $g(x) = f(x_j) + x$, so $m(g(I)) = m(I_j + f(x_j)) = |I_j|$. g being a homeomorphism implies that $m(g(C^c)) = \sum m(g(I_j)) = \sum |I_j| = m(C^c) = 1$, hence $m(g(C)) = 1$.

Next we prove that every measurable subset $E \subseteq [0,2]$ with $m(E) > 0$ has a non-measurable subset: Let $P \in [0,1)$ be the non-measurable set constructed by picking exactly one element of each of the equivalence classes of the relation $x \sim y \iff x - y \in \mathbb{Q}$. Let also $P_q = P + q$ with addition mod 2, and $E_q = E \cap P_q$. Then $[0,2] = \cup_{q\in\mathbb{Q}\cap[0,2]} P_q$, hence $E = \cup_{q\in\mathbb{Q}\cap[0,2]} E_q$, with a disjoint union on the right (same proof as in class). We claim that E_q is non-measurable for some q: in deed if not, $E_q + r$'s are disjoint for $r \in \mathbb{Q}\cap[0,2]$, so $\sum_{r\in\mathbb{Q}\cap[0,2]} m(E_q + r) = m(\cup_{r\in\mathbb{Q}\cap[0,2]} E_q + r) \le m([0,2]) = 2$, so $m(E_q) = 0$ for all q. But then $m(E) = m(\sum_{q\in\mathbb{Q}\cap[0,2]} E_q) = 0$, a contradicting the positivity of $m(E)$. Hence, some $E_q \subset E$ is non-measurable.

Now let $E \subset g(C)$ be non-measurable, and let $A = g^{-1}(E)$. A is measurable since $A \subset C$ and $m(C) = 0$, but $g(A) = E$ is not. This A is also an example of a measurable but not Borel set, because g being a homeomorphism, implies that A is Borel iff E is Borel.

$\square$

Problem 3: f is *lower semicontinuous* at y if $f(y) \le \liminf_{x\to y} f(x)$. Show that:
(i) f is lower semicontinuous if and only if $\{x : f(x) > \lambda\}$ open for all λ.
(ii) If f, g lower semicontinuous at y, so are $f+g$ and $\max\{f, g\}$.
(iii) If f_n are lower semicontinuous, so its $f(x) = \sup_n f_n(x)$.
(iv) A function f is lower semicontinuos if and only if there is a monotone increasing sequence ϕ_n of continuous functions such that $f(x) = \lim \phi_n(x)$.

(i) ($\Rightarrow$) Let f be lsc, and fix $\lambda \in \mathbb{R}$ and $x_0 \in A_{f,\lambda} = \{f(x) > \lambda\}$. Then $f(x_0) > \lambda$ so $\epsilon = (f(x_0) - \lambda)/2 > 0$. By the definition of lim inf, exists $\delta > 0$ s.t. $\inf_{B_\delta(x_0)\setminus x_0} f(x) \ge \liminf_{x\to x_0} f(x) - \epsilon/2 \ge f(x_0) - \epsilon/2 > \lambda$. Hence, $f(x) > \lambda$ for $x \in B_\delta(x_0)$, ie $B_\delta(x_0) \subset A_{f,\lambda}$. Since λ and x_0 were arbitrary, $A_{f,\lambda}$ is open, $\forall\lambda$.

($\Leftarrow$) Let $A_{f,\lambda}$ be open for all λ, and x_0. Let $\epsilon > 0$. Since $A_{f,f(x_0)-\epsilon}$ is open and contains x_0, $f(x) \ge f(x_0) - \epsilon$ in some neighborhood of x_0, therefore $\liminf_{x\to x_0} \ge f(x_0)$. This is true for all ϵ, so $\liminf_{x\to x_0} f(x) \ge f(x_0)$.

(ii) Given $\epsilon > 0$, clearly $\inf_{B_\delta(y)\setminus y} (f(x) + g(x)) \ge \inf_{B_\delta(y)\setminus y} f(x) + \inf_{B_\delta(y)\setminus y} g(x)$, so taking limit as $\epsilon \to 0$ gives $\liminf_{x\to y}(f(x) + g(x)) \ge \liminf_{x\to y} f(x) + \liminf_{x\to y} g(x) \ge f(y) + g(y)$ since f and g are lsc at y.

Again, given $\epsilon > 0$, $\inf_{B_\delta(y)\setminus y} max\{f(x), g(x)\} \ge \inf_{B_\delta(y)\setminus y} f(x)$ since $max\{f, g\} \ge f$. Taking limit as $\epsilon \to 0$ gives $\liminf_{x\to y} \max\{f(x) + g(x)\} \ge \liminf_{x\to y} f(x) \ge f(y)$, since f is lsc at y. Analogously $\liminf_{x\to y} \max\{f(x) + g(x)\} \ge g(y)$, hence $\liminf_{x\to y} \max\{f(x) + g(x)\} \ge max\{f(y), g(y)\}$.

(iii) $A_{f,\lambda} = \cup_n A_{f_n,\lambda}$, since given x, if all $f_n(x) \le \lambda$, $f(x) \le \lambda$ too. $A_{f_n,\lambda}$'s are open, hence so is $A_{f,\lambda}$.

(iv) ($\Rightarrow$) Divide $\mathbb{R}$ in intervals $[\frac{k}{2^n}, \frac{k+1}{2^{n+1}})$ and let $f_n(x)$ be $infimum$ of f in $[\frac{k}{2^n} - \frac{1}{2^{n+2}}, \frac{k+1}{2^n} + \frac{1}{2^{n+2}})$ for all x in $[\frac{k}{2^n} + \frac{1}{2^{n+2}}, \frac{k+1}{2^n} - \frac{1}{2^{n+2}})$, with linear interpolation in between. Using compactness of the closure of the interval and lower semicontinuity, one can easily show that the $infimum$ is not $-\infty$.

Now let $\phi_n = \max\{f_1, \ldots, f_n\}$. ϕ_n's are continuous, since so are f_n's. They are clearly increasing in n. Finally, $f_n(y) \le f(y)$, so $\sup_n \phi_n(y) \le f(y)$. On the other hand, given ϵ, $\{x : f(x) > f(y) - \epsilon\}$ is open, so $f(x) > f(y) - \epsilon$ in a neighborhood of y. Then, for a large enough n, $f_n(y) \ge \{\inf f(x)$ over a small interval of $y\} \ge f(y) - \epsilon$, giving $\sup \phi_n(y) \ge f(y) - \epsilon$. This is true for all ϵ, therefore $\sup \phi_n(y) = f(y)$.

($\Leftarrow$) $f(x) = \sup \phi_n(x)$ since the sequence is increasing, and ϕ_n's are continuous, hence lsc, thus by (iii), f is lsc too. $\square$

Problem 4:
(i) The set of points of continuity of f is a $\mathcal{G}_\delta$.
(ii) $\mathbb{Q}$ is not a $\mathcal{G}_\delta$.

(i) Let C be the set of continuity of f. f is continuous at y iff $\liminf_{x\to y} f(x) = \limsup_{x\to y} f(x)$, so

$$C^c = \{y : \liminf_{x\to y} f(x) < \limsup_{x\to y} f(x)\} = \cup_{p<q\in\mathbb{Q}}\{\{\liminf_{x\to y} f(x) \le p\} \cap \{\limsup_{x\to y} f(x) \ge q\}\}.$$

Hence, it's enough to show that $\{\liminf_{x\to y} f(x) \le p\}$ and $\{\limsup_{x\to y} f(x) \ge q\}$ are closed (this implies C^c is F_σ). But one can easily show that $\{\liminf_{x\to y} f(x) > p\}$ and $\{\limsup_{x\to y} f(x) < q\}$ are open, proving therefore that C is G_δ.

(ii) Suppose $\mathbb{Q} = \cap O_n$, O_n's open. Let $q_1, q_2, \ldots$ be an enumeration of the rationals, and consider $O'_n = O_n\setminus\{q_n\}$. Now $\overline{O'}_n = \overline{O}_n = \mathbb{R}$, so Baire's category theorem implies $\cap O'_n$ is dense too, contradicting $\cap O'_n = \emptyset$. $\square$

Problem 5: Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$.

$\emptyset$ is countable, hence in $\mathcal{M}$. If $E \in \mathcal{M}$, it's either countable or co-countable, hence so is E^c. Finally, if $\{E_n\} \subseteq \mathcal{M}$, then either all of them are countable, hence so is their union; or at least one is co-countable, say E_m, so $(\cup E_n)^c \subseteq E^c_m$ is countable too. Therefore $\mathcal{M}$ is a σ-algebra.

Clearly μ is non-negative and $\mu(\emptyset) = 0$. Now if $E, E_n \in \mathcal{M}$ with $E \subset \cup E_n$, then either all of E_n are countable, in which case so is their union and therefore E, giving $\mu(E) \le \sum \mu(E_n)$ (all of them 0); or at least one E_n is co-countable. In this case $\sum \mu(E_n) \geq 1$, whereas $\mu(E)$ is either 0 or 1, so again $\mu(E) \leq \sum \mu(E_n)$. Hence, μ is a measure on $\mathcal{M}$. $\square$

Problem 6: Construct a Borel set E such that for every interval I, $0 < m(E \cap I) < m(I)$.

First note that neither E nor E^c can contain an open set, which suggests that E should be nowhere dense. The strategy is to find inside each bounded rational interval $I_1, I_2, \ldots$ two disjoint nowhere dense Borel sets of positive measure. We start by claiming that any interval I has a nowhere dense subset A of positive measure. In deed, let $0 < \epsilon < m(I)/2$, and $q_1, q_2, \ldots$ the rationals in I. Let $A = I \backslash (\cup B_{\epsilon 2^{-n}}(q_n))$. Then $(op\,(cl\,(A)))^c = cl\,((cl\,(A))^c) = cl\,(op\,(A^c)) = cl\,(op\,(I^c \cap (\cap B_{\epsilon 2^{-n}}(q_n)))) = cl\,(op\,(I^c) \cap (\cap B_{\epsilon 2^{-n}}(q_n))) \supseteq cl\,(\mathbb{Q}) = \mathbb{R}$, so A is nowhere dense, Borel, and $m(A) \geq m(I) - \sum m(B_{\epsilon 2^{-n}}(q_n)) \geq m(I)/2$.

Now define inductively S_n, E_n, F_n as follows:

$E_1, F_1 \subset I_1$ disjoint, Borel, nowhere dense, of positive measure (pick E_1, then $I_1 \backslash E_1$ will have some open interval, so pick F_1 there).

Define $S_n = (E_1 \cup \ldots E_{n-1}) \cup (F_1 \cup \ldots F_{n-1})$ for $n \geq 2$.

Given S_n, define $E_n, F_n \subset I_n \backslash S_n$ disjoint, Borel, nowhere dense, of positive measure, and this can be done because S_n is nowhere dense.

Let $E = \cup_n E_n$, $F = \cup_n F_n$, both disjoint Borel sets. If I is any non-empty interval, it contains some I_n, so

$$m(I \cap E) \geq m(I_n \cap E_n) = m(E_n) > 0,$$
$$m(I) \geq m(I \cap E) + m(I \cap F) \geq m(I \cap E) + m(I_n \cap F_n) > m(I \cap E).$$

$\square$