Dispersive Navier-Stokes Systems for Gas Dynamics: Mathematical Theories

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We consider a dispersive Navier-Stokes (DNS) system in the form

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0 \,, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x (\rho\theta) &= \nabla_x \cdot \Sigma + \nabla_x \cdot \widetilde{\Sigma} \,, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) &= \nabla_x \cdot (\Sigma u + q) + \nabla_x \cdot (\widetilde{\Sigma} u + \tilde{q}) \,, \\
(\rho, u, \theta)(x, 0) &= (\rho^{in}, u^{in}, \theta^{in})(x) \,,
\end{aligned}
\tag{1}
$$

where $\rho(x,t)$ is the mass density, $u(x,t)$ is the bulk velocity, and $\theta(x,t)$ is the temperature at a position $x \in \mathbb{R}^d$ and time $t \geq 0$. We assume that $d \geq 2$. The total energy density is given by $\rho e = \frac{1}{2}\rho|u|^2 + \frac{d}{2}\rho\theta$.

The Navier-Stokes stress tensor Σ and heat flux $-q$ are given by

$$
\Sigma = \mu(\theta)\, D_x u \,, \qquad q = \kappa(\theta)\, \nabla_x \theta \,, \tag{2}
$$

where the strain-rate tensor is given by $D_x u = \nabla_x u + (\nabla_x u)^T - \frac{2}{d} I \nabla_x \cdot u$, while $\mu(\theta) \geq 0$ and $\kappa(\theta) \geq 0$ are the coefficients of shear viscosity and heat conductivity.

The dispersive corrections to the stress tensor $\widetilde{\Sigma}$ and the heat flux $\tilde{q}$ are given by

$$
\begin{aligned}
\widetilde{\Sigma} &= \tau_1(\rho,\theta)\left(\nabla_x^2\theta - \tfrac{1}{d}\Delta_x\theta I\right) \\
&\quad + \tau_2(\rho,\theta)\,\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{d}|\nabla_x\theta|^2 I\right) \\
&\quad + \tau_3(\rho,\theta)\left(\nabla_x\rho\otimes\nabla_x\theta + \nabla_x\theta\otimes\nabla_x\rho - \tfrac{2}{d}\nabla_x\rho\cdot\nabla_x\theta I\right)\,, \\
\tilde{q} &= \tau_4(\rho,\theta)\left(\Delta_x u + \tfrac{d-2}{d}\nabla_x\nabla_x\cdot u\right) \\
&\quad + \tau_5(\rho,\theta)\,D_x u\cdot\nabla_x\theta + \tau_6(\rho,\theta)\,D_x u\cdot\nabla_x\rho \\
&\quad + \tau_7(\rho,\theta)\left(\nabla_x u - (\nabla_x u)^T\right)\cdot\nabla_x\theta\,,
\end{aligned}
\tag{3}
$$

where $\tau_i(\rho,\theta)$ for $i = 1,\cdots,7$ are additional transport coefficients.

The previous lecture presented the kinetic origins of such systems. Here we present a local well-posedness result over the whole space.

The dispersive Navier-Stokes (DNS) system (1) has transport coefficients $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 7$ with forms that depend upon details of the underlying kinetic equation. In particular, the transport coefficients $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 6$ satisfy the relations

$$\tau_4 = \frac{\theta}{2}\tau_1\,, \qquad \frac{\tau_2}{\theta} + \frac{2\tau_5}{\theta^2} = \partial_\theta\left(\frac{\tau_4}{\theta^2}\right)\,, \qquad \theta\tau_3 + \tau_6 = 2\partial_\rho\tau_4\,. \tag{4}$$

These relations ensure that the DNS system (1) inherits an entropy structure from the underlying kinetic equation in which the mathematical entropy density η is essentially the Euler entropy given by

$$\eta = \rho \log\left(\frac{\rho}{\theta^{d/2}}\right)\,.$$

Direct calculation from system (1) shows that η satisfies

$$
\begin{aligned}
&\partial_t \eta + \nabla_x \cdot \left(\eta\, u + \frac{q}{\theta} + \frac{\tilde{q}}{\theta} \right) \\
&\qquad = - \left(\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2} \cdot \nabla_x \theta \right) - \left(\frac{\widetilde{\Sigma}}{\theta} : \nabla_x u + \frac{\tilde{q}}{\theta^2} \cdot \nabla_x \theta \right) .
\end{aligned}
\tag{5}
$$

It follows from the constitutive relations (2) that

$$
\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2} \cdot \nabla_x \theta = \frac{\mu}{2\theta} |D_x u|^2 + \frac{\kappa}{\theta^2} |\nabla_x \theta|^2 \geq 0 \,,
$$

while it follows from constitutive relations (3) and (4) that

$$
\widetilde{\Sigma} : \frac{\nabla_x u}{\theta} + \tilde{q} \cdot \frac{\nabla_x \theta}{\theta^2} = \nabla_x \cdot \left(\frac{\tau_1}{2\theta} D_x u \cdot \nabla_x \theta \right) .
$$

One thereby sees that the dispersion terms containing $\widetilde{\Sigma}$ and $\tilde{q}$ contribute only to the entropy flux in the entropy equation (5). DNS systems (1) derived from kinetic equations therefore formally dissipate the Euler entropy in the same way as the compressible Navier-Stokes system.

Local Well-Posedness

The above calculation indicates that the DNS system is formally well-posed over domains without boundary. More specifically, its linearization about any nonzero constant state is well-posed in $L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$. Our main theorem establishes the local well-posedness of the DNS system.

Because our theory is local in time, we will not need the entropy structure of the system, and so will not assume that (4) holds. We will however assume that $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho, \theta)$ for $i = 1, \cdots, 7$ are smooth functions of ρ and θ with $\mu(\theta)$, $\kappa(\theta)$, and $\tau_1(\rho, \theta)\tau_4(\rho, \theta)$ being strictly positive whenever ρ and θ are bounded away from zero.

In our proof of local well-posedness, dispersive regularization plays a crucial role. We use the fact that solutions of dispersive equations gain spatial differentiability provided the initial data satisfy certain asymptotic flatness conditions at infinity.

This type of smoothing was noticed by Kato when he showed that solutions of the KdV equation gain half a spatial derivative compared to its initial data. Better, Constantin and Saut showed that solutions of dispersive equations of order m gain $\frac{m-1}{2}$ derivatives locally for positive times.

Based on this smoothing, various well-posedness results have been established for semilinear or quasi-linear dispersive equations and systems with strict or uniform dispersive effects. However, these existing results do not apply directly to the DNS system because its dispersion is degenerate.

One degeneracy occurs because the mass equation has no dissipative or dispersive terms. Another degeneracy occurs because the dispersive terms $\nabla_x \cdot \widetilde{\Sigma}$ and $\nabla_x \cdot \tilde{q}$ have the form

$$\nabla_x \cdot \widetilde{\Sigma} = \tfrac{d-1}{d}\tau_1 \, \Delta_x \nabla_x \theta + \text{lower order terms}\,,$$
$$\nabla_x \cdot \tilde{q} = 2\tfrac{d-1}{d}\tau_4 \, \Delta_x \nabla_x \cdot u + \text{lower order terms}\,.$$

Hence, if u is decomposed into its divergence free part and its gradient part then only the gradient part is smoothed by dispersion.

These degeneracies suggest that we decompose the DNS system into a strictly dispersive subsystem and a nondispersive subsystem. We treat the strictly dispersive subsystem in the style of Kenig, Ponce, and Vega (2004). The coupling of these subsystems is treated using both dissipative and dispersive regularization.

Our main result implies the following.

Well-Posedness Theorem. In dimension $d \geq 2$, let s_1, $s \in \mathbb{R}_+$ such that $s_1 \geq d/2 + 6$ and $s = \max\{s_1 + 6, N + d/2 + 4\}$ where $N(d) \in \mathbb{N}$ is to be given. Let $\bar{\rho} > 0$ and $\bar{\theta} > 0$ be constants. Let ρ^{in}, u^{in}, and θ^{in} satisfy:

- $$\left\|\rho^{in} - \bar{\rho}\right\|_{H^{s+1}} + \left\|(u^{in}, \theta^{in} - \bar{\theta})\right\|_{H^s} + \sum_{1 \leq |\alpha| \leq s_1} \left(\left\|\langle x \rangle^2 \partial_x^\alpha \rho^{in}\right\|_{H^1} + \left\|\langle x \rangle^2 \partial_x^\alpha (u^{in}, \theta^{in})\right\|_{L^2} \right) \leq C^{in} < \infty \,, \tag{6}$$

  where $\alpha \in \mathbb{N}^d$ denote multi-indices with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and we define $\langle x \rangle^2 \equiv 1 + |x|^2$;

- there exists a constant $\alpha^{in} > 0$ such that for every $x \in \mathbb{R}^d$

$$\begin{aligned} \alpha^{in} \le \rho^{in}(x)\,, \qquad & \alpha^{in} \le \theta^{in}(x)\,, \\ \alpha^{in} \le \mu(\theta^{in})\,, \qquad & \alpha^{in} \le \kappa(\theta^{in})\,, \\ \alpha^{in} \le \tfrac{4(d-1)^2}{d^3} \cdot \frac{\tau_1(\rho^{in}(x),\theta^{in}(x))\tau_4(\rho^{in}(x),\theta^{in}(x))}{\rho^{in}(x)^2} & ; \end{aligned} \tag{7}$$

- the Hamiltonian defined by

$$h^{in}(x,\xi) = \tfrac{2(d-1)}{d^{3/2}} \left(\frac{\tau_1(\rho^{in}(x),\theta^{in}(x))\tau_4(\rho^{in}(x),\theta^{in}(x))}{\rho^{in}(x)^2} \right)^{\frac{1}{2}} |\xi|^3 \tag{8}$$

generates a flow that is non-trapping.

Then for some $T_0 > 0$ depending only on C^{in}, α^{in}, and d there exists unique functions ρ, u, and θ with

$$\begin{aligned} \rho - \bar{\rho} &\in C([0,T_0];H^s) \cap L^\infty([0,T_0];H^{s+1})\,, \\ (u, \theta - \bar{\theta}) &\in C([0,T_0];H^{s-1}) \cap L^\infty([0,T_0];H^s)\,, \end{aligned} \tag{9}$$

such that (ρ, u, θ) solves the DNS initial-value problem (1).

Here L^2 denotes the Lebesgue space $L^2(\mathbb{R}^d;\mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\|\cdot\|_{L^2}$ denotes its norm. Similarly, H^s denotes the Sobolev space $H^s(\mathbb{R}^d;\mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\|\cdot\|_{H^s}$ denotes its norm.

To prove the above theorem, we construct an approximating sequence of solutions by adding an artificial hyperviscosity term to the DNS system (1). An *a priori* estimate is established that is independent of the artificial hyperviscosity. Then using this a priori estimate and letting the artificial hyperviscosity term vanish, we show that the approximating sequence converges to a solution of the original system. Uniqueness is also shown by the a priori estimate.

In our proof we first establish an estimate for a linear system that we will later use to construct our approximating sequence of solutions to the DNS system (1) plus an artificial hyperviscosity. We then establish the *a priori* estimate for this regularized DNS system. Finally, we show the existence of the approximating sequence and the convergence of this sequence to the unique solution to the original DNS system.

Regularized DNS System

We first express the DNS system (1) as a evolution system for (ρ, u, θ) and then add fourth-order artificial hyperviscosity terms to obtain the regularized DNS system

$$
\begin{aligned}
\partial_t \rho &= -\epsilon\, \Delta_x^2 \rho - \rho \nabla_x \cdot u - u \cdot \nabla_x \rho \,, \\
\partial_t u &= -\epsilon\, \Delta_x^2 u + \frac{1}{\rho} \nabla_x \cdot \Sigma + \frac{1}{\rho} \nabla_x \cdot \widetilde{\Sigma} - \frac{1}{\rho} \nabla_x (\rho\theta) - u \cdot \nabla_x u \,, \\
\partial_t \theta &= -\epsilon\, \Delta_x^2 \theta + \tfrac{2}{d} \frac{1}{\rho} \nabla_x \cdot q + \tfrac{2}{d} \frac{1}{\rho} \nabla_x \cdot \tilde{q} + \tfrac{2}{d} \frac{\widetilde{\Sigma} : \nabla_x u}{\rho} + \tfrac{2}{d} \frac{\Sigma : \nabla_x u}{\rho} \\
&\qquad - \tfrac{2}{d} \theta \nabla_x \cdot u - u \cdot \nabla_x \theta \,, \\
&(\rho, u, \theta)(x, 0) = (\rho^{in}, u^{in}, \theta^{in})(x) \,,
\end{aligned}
\tag{10}
$$

where Σ and q are given by (2) while $\tilde{\Sigma}$ and $\tilde{q}$ are given by (3).

The structure of this system becomes more explicit if we use (3) to express $\widetilde{\Sigma}$ and $\tilde{q}$ in terms of the fluid variables (ρ, u, θ). One finds that

$$
\begin{aligned}
\nabla_x \cdot \widetilde{\Sigma} = & \tfrac{d-1}{d} \tau_1(\rho,\theta) \Delta_x \nabla_x \theta \\
& + A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2 \rho + A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2 \theta \\
& + B^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta) \cdot \nabla_x \rho + B^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta) \cdot \nabla_x \theta \,, \\
\nabla_x \cdot \tilde{q} = & \tfrac{2(d-1)}{d} \tau_4(\rho,\theta) \Delta_x \nabla_x \cdot u \\
& + A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta) \vdots \nabla_x^2 u + \tau_5(\rho,\theta) D_x u : \nabla_x^2 \theta \\
& + \tau_6(\rho,\theta) D_x u : \nabla_x^2 \rho + B^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x u \,,
\end{aligned}
$$

where $A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$, $A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$, and $A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$ are $d{\times}d{\times}d$ three-tensors that are linear in $(\nabla_x\rho,\nabla_x\theta)$, while $B^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$, $B^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$, and $B^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta)$ are $d \times d$ two-tensors that are quadratic in $(\nabla_x\rho,\nabla_x\theta)$. The forms of B^ρ and B^θ are not uniquely specified above, but their specific forms do not affect our subsequent arguments.

The regularized DNS system (10) thereby has the form

$$
\begin{aligned}
\partial_t \rho &= -\epsilon\, \Delta_x^2 \rho - \rho \nabla_x \cdot u - u \cdot \nabla_x \rho \,, \\
\partial_t u &= -\epsilon\, \Delta_x^2 u + \frac{1}{\rho} \nabla_x \cdot [\mu D_x u] \\
&\quad + \tfrac{d-1}{d} \frac{\tau_1}{\rho} \Delta_x \nabla_x \theta + \frac{A^\rho}{\rho} : \nabla_x^2 \rho + \frac{A^\theta}{\rho} : \nabla_x^2 \theta \\
&\quad + \frac{B^\rho}{\rho} \cdot \nabla_x \rho + \frac{B^\theta}{\rho} \cdot \nabla_x \theta - \frac{1}{\rho} \nabla_x (\rho\theta) - u \cdot \nabla_x u \,, \\
\partial_t \theta &= -\epsilon\, \Delta_x^2 \theta + \tfrac{2}{d} \frac{1}{\rho} \nabla_x \cdot [\kappa \nabla_x \theta] + \tfrac{4(d-1)}{d^2} \frac{\tau_4}{\rho} \Delta_x \nabla_x \cdot u \\
&\quad + \tfrac{2}{d} \frac{A^u}{\rho} \vdots \nabla_x^2 u + \tfrac{1}{d} \frac{\tau_1 + 2\tau_5}{\rho} D_x u : \nabla_x^2 \theta + \tfrac{2}{d} \frac{\tau_6}{\rho} D_x u : \nabla_x^2 \rho \\
&\quad + \tfrac{2}{d} \frac{B^u}{\rho} : \nabla_x u + \tfrac{1}{d} \frac{\tau_2}{\rho} \nabla_x \theta \cdot D_x u \cdot \nabla_x \theta + \tfrac{1}{d} \frac{\tau_3}{\rho} \nabla_x \rho \cdot D_x u \cdot \nabla_x \theta \\
&\quad + \tfrac{1}{d} \frac{\mu}{\rho} |D_x u|^2 - \tfrac{2}{d} \theta \nabla_x \cdot u - u \cdot \nabla_x \theta \,.
\end{aligned}
\tag{11}
$$

Associated Linear System

By replacing certain (ρ, u, θ) above by a given state $(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ we obtain

$$
\begin{aligned}
\partial_t \tilde{\rho} &= -\epsilon \, \Delta_x^2 \tilde{\rho} - \widehat{\rho} \nabla_x \cdot \tilde{u} - \widehat{u} \cdot \nabla_x \tilde{\rho} \,, \\
\partial_t \tilde{u} &= -\epsilon \, \Delta_x^2 \tilde{u} + \frac{1}{\widehat{\rho}} \nabla_x \cdot \left[\mu(\widehat{\theta}) D_x \tilde{u} \right] \\
&\quad + \widehat{\tau}_1 \Delta_x \nabla_x \tilde{\theta} + \widehat{A}^\rho : \nabla_x^2 \tilde{\rho} + \widehat{A}^\theta : \nabla_x^2 \tilde{\theta} \\
&\quad + \widehat{B}^\rho \cdot \nabla_x \tilde{\rho} + \widehat{B}^\theta \cdot \nabla_x \tilde{\theta} - \nabla_x \tilde{\theta} - \frac{\widehat{\theta}}{\widehat{\rho}} \nabla_x \tilde{\rho} - \widehat{u} \cdot \nabla_x \tilde{u} \,, \\
\partial_t \tilde{\theta} &= -\epsilon \, \Delta_x^2 \tilde{\theta} + \tfrac{2}{d} \frac{1}{\widehat{\rho}} \nabla_x \cdot \left[\kappa(\widehat{\theta}) \nabla_x \tilde{\theta} \right] \\
&\quad + \widehat{\tau}_4 \Delta_x \nabla_x \cdot \tilde{u} + \widehat{A}^u \vdots \nabla_x^2 \tilde{u} + \widehat{\tau}_5 D_x \widehat{u} : \nabla_x^2 \tilde{\theta} + \widehat{\tau}_6 D_x \widehat{u} : \nabla_x^2 \tilde{\rho} \\
&\quad + \widehat{B}^u : \nabla_x \tilde{u} + \widehat{\tau}_2 \nabla_x \widehat{\theta} \cdot D_x \tilde{u} \cdot \nabla_x \widehat{\theta} + \widehat{\tau}_3 \nabla_x \widehat{\rho} \cdot D_x \tilde{u} \cdot \nabla_x \widehat{\theta} \\
&\quad + \tfrac{1}{d} \frac{\mu(\widehat{\theta})}{\widehat{\rho}} D_x \widehat{u} : D_x \tilde{u} - \tfrac{2}{d} \widehat{\theta} \nabla_x \cdot \tilde{u} - \widehat{u} \cdot \nabla_x \tilde{\theta} \,,
\end{aligned}
\tag{12}
$$

where

$$\begin{aligned}
\widehat{\tau}_1 &= \tfrac{d-1}{d}\frac{\tau_1(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{\tau}_2 &= \tfrac{1}{d}\frac{\tau_2(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,,\\
\widehat{\tau}_3 &= \tfrac{1}{d}\frac{\tau_3(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{\tau}_4 &= \tfrac{4(d-1)}{d^2}\frac{\tau_4(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,,\\
\widehat{\tau}_5 &= \tfrac{1}{d}\frac{\tau_1(\widehat{\rho},\widehat{\theta})+2\tau_5(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{\tau}_6 &= \tfrac{2}{d}\frac{\tau_6(\widehat{\rho},\widehat{\theta})}{\widehat{\rho}}\,,\\
\widehat{A}^\rho &= \frac{A^\rho(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{B}^\rho &= \frac{B^\rho(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,,\\
\widehat{A}^\theta &= \frac{A^\theta(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{B}^\theta &= \frac{B^\theta(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,,\\
\widehat{A}^u &= \tfrac{2}{d}\frac{A^u(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,, & \widehat{B}^u &= \tfrac{2}{d}\frac{B^u(\widehat{\rho},\widehat{\theta},\nabla_x\widehat{\rho},\nabla_x\widehat{\theta})}{\widehat{\rho}}\,.
\end{aligned}\tag{13}$$

The associated linear system (12) is satisfied by $(\tilde{\rho},\tilde{u},\tilde{\theta}) = (\rho-\bar{\rho}, u, \theta-\bar{\theta})$ whenever $(\widehat{\rho},\widehat{u},\widehat{\theta}) = (\rho, u, \theta)$ solves the regularized DNS system (11).

Notation. We will use Ψ_m to denote any pseudo-differential operator (ΨDO) of order m whenever its specific form is not important. We will denote the space of all m^{th} order symbols as S^m. For a symbol $p(\xi) \in S^m$, let $|p|_{S^m}^{(j)}$ be the seminorm defined as

$$|p|_{S^m}^{(j)} = \sup_{\alpha,\beta} \left\{ \|\langle\xi\rangle^{-m+\alpha} \partial_\xi^\alpha \partial_x^\beta p(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} : |\alpha+\beta| \leq j \right\} ,$$

where $\langle\xi\rangle = (1+|\xi|^2)^{-1/2}$. The following theorem is a classical result for ΨDO's.

Thm: Let $m, s \in \mathbb{R}$. Let $p(\xi) \in S^m$ be the symbol of the pseudo-differential operator Ψ_p. Then $\Psi_p : H^{m+s}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ is a bounded linear operator. Moreover, there exist $N = N(d, m, s) \in \mathbb{N}$ and $c = c(d, m, s)$ such that

$$\|\Psi_p f\|_{H^s} \leq c|p|_{S^m}^{(N)} \|f\|_{H^{m+s}} . \tag{14}$$

With this notation we see that the linear system (12) has the form

$$
\begin{aligned}
\partial_t \tilde{\rho} &= -\epsilon\, \Delta_x^2 \tilde{\rho} - \widehat{\rho} \nabla_x \cdot \tilde{u} - \widehat{u} \cdot \nabla_x \tilde{\rho}\,, \\
\partial_t \tilde{u} &= -\epsilon\, \Delta_x^2 \tilde{u} + \frac{1}{\widehat{\rho}} \nabla_x \cdot \left[\mu(\widehat{\theta}) D_x \tilde{u} \right] + \widehat{\tau}_1 \Delta_x \nabla_x \tilde{\theta} \\
&\quad + \Psi_2(\tilde{\rho}, \tilde{\theta}) + \Psi_1(\tilde{\rho}, \tilde{\theta}) - \widehat{u} \cdot \nabla_x \tilde{u}\,, \\
\partial_t \tilde{\theta} &= -\epsilon\, \Delta_x^2 \tilde{\theta} + \tfrac{2}{d} \frac{1}{\widehat{\rho}} \nabla_x \cdot \left[\kappa(\widehat{\theta}) \nabla_x \tilde{\theta} \right] + \widehat{\tau}_4 \Delta_x \nabla_x \cdot \tilde{u} \\
&\quad + \Psi_2(\tilde{\rho}, \tilde{u}, \tilde{\theta}) + \Psi_1 \tilde{u} - \widehat{u} \cdot \nabla_x \tilde{\theta}\,,
\end{aligned}
\tag{15}
$$

where Ψ_1 and Ψ_2 are first and second order ΨDOs whose coefficients depend algebraically upon $(\widehat{\rho}, \widehat{\theta}, \nabla_x \widehat{\rho}, D_x \widehat{u}, \nabla_x \widehat{\theta})$.

The key point here is that all derivatives of solutions to the nonlinear system satisfy equations of this form.

The earlier discussion of degeneracies suggests that we decompose u into its divergence free part Pu and its gradient part Qu. System (15) is thereby decomposed into its nondispersive part

$$\begin{aligned} \partial_t \rho &= -\epsilon \Delta_x^2 \rho - \widehat{u} \cdot \nabla_x \rho + \Psi_1 Qu \,, \\ \partial_t Pu &= -\epsilon \Delta_x^2 Pu + \widehat{\mu} \Delta_x Pu + \Psi_2(\rho, \theta) + \Psi_1(\rho, u, \theta) \,, \end{aligned} \tag{16}$$

and its strictly dispersive part

$$\begin{aligned} \partial_t Qu &= -\epsilon \, \Delta_x^2 Qu + \widehat{\mu} \left(\Delta_x + \tfrac{d-2}{d} \nabla_x \nabla_x \cdot \right) Qu \\ &\quad + \nabla_x \left(\widehat{\tau}_1 \Delta_x \theta \right) + \Psi_2(\rho, \theta) + \Psi_1(\rho, u, \theta) \,, \\ \partial_t \theta &= -\epsilon \, \Delta_x^2 \theta + \widehat{\kappa} \Delta_x \theta \\ &\quad + \widehat{\tau}_4 \Delta_x \nabla_x \cdot Qu + \Psi_2(\rho, Pu, Qu) + \Psi_1(\rho, u, \theta) \,, \end{aligned} \tag{17}$$

where $\widehat{\mu} = \mu(\widehat{\theta})/\widehat{\rho}$ and $\widehat{\kappa} = \frac{2}{d} \kappa(\widehat{\theta})/\widehat{\rho}$. Notice that these two parts couple through the lower order terms $\Psi_2(\rho, Pu, Qu, \theta)$ and $\Psi_1(\rho, Pu, Qu, \theta)$.

We drop the tildes on $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ and write the regularized system (15) as

$$\partial_t(\rho, u, \theta) = -\epsilon \Delta_x^2(\rho, u, \theta) + \mathcal{L}(\widehat{\rho}, \widehat{u}, \widehat{\theta})(\rho, u, \theta)\,, \tag{18}$$

where the linear operator $\mathcal{L}$ is defined through (15) and has the form

$$\begin{aligned}
\mathcal{L}(\widehat{\rho}, \widehat{u}, \widehat{\theta})(\rho, u, \theta) &= \begin{pmatrix} \mathcal{L}_1(\widehat{\rho}, \widehat{u}, \widehat{\theta})(\rho, u, \theta) \\ \mathcal{L}_2(\widehat{\rho}, \widehat{u}, \widehat{\theta})(\rho, u, \theta) \\ \mathcal{L}_3(\widehat{\rho}, \widehat{u}, \widehat{\theta})(\rho, u, \theta) \end{pmatrix} \\
&= \begin{pmatrix} \Psi_1(\rho, u) \\ \Psi_D u + \Psi_3 \theta + \Psi_2(\rho, \theta) + \Psi_1(\rho, u, \theta) \\ \Psi_D \theta + \Psi_3 u + \Psi_2(\rho, u, \theta) + \Psi_1(u, \theta) \end{pmatrix} .
\end{aligned}$$

Outline of Proof

- Derive an estimate for the linear system

$$\partial_t U = -\epsilon \Delta_x^2 U + \mathcal{L}(\widehat{U}) U \, .$$

- Use this estimate to get an a-priori estimate for the nonlinear system

$$\partial_t U^\epsilon = -\epsilon \Delta_x^2 U^\epsilon + \mathcal{L}(U^\epsilon) U^\epsilon \, .$$

- Use a fixed-point argument to obtain $O(\epsilon^3)$ existence, and extend it to $O(\epsilon^0)$ existence by the a-piori estimate.

- Let $\epsilon \to 0$ and show that $U^\epsilon \to U$ where U satisfies the DNS system

$$\partial_t U = \mathcal{L}(U) U \, .$$

In order to obtain bounds on the solutions of linear system (15) we make the following assumptions on $(\widehat{\rho}, \widehat{u}, \widehat{\theta})$. These assumptions are the key to choosing the proper space for our well-posedness result.

$\mathcal{A}_1$. *Asymptotic flatness*. There exists constants $c_A, T_1 > 0$ such that $\forall (x,t) \in \mathbb{R}^d \times [0, T_1]$,

$$|\partial_t(\widehat{\rho}, \widehat{u}, \widehat{\theta})| + |\nabla_x(\widehat{\rho}, \widehat{u}, \widehat{\theta})| + |\partial_t \nabla_x(\widehat{\rho}, \widehat{u}, \widehat{\theta})| \leq \frac{c_A}{\langle x \rangle^2} \tag{19}$$

with $\langle x \rangle^2 = 1 + |x|^2$.

$\mathcal{A}_2$. *Regularity*. There exists $T_2 > 0$ such that $(\widehat{\rho}, \widehat{u}, \widehat{\theta}) \in C_b^{N+1}(\mathbb{R}^d \times [0, T_2])$ for N sufficiently large such that the proofs involving the ΨDO's can be carried out. Here $C_b^{N+1}(\mathbb{R}^d \times [0, T_2])$ is the set of functions that have continuous bounded derivatives up to order $N+1$. Again use c_A to denote the uniform upper bound of the coefficients of Ψ_2 in $C_b^N(\mathbb{R}^d \times [0, T_2])$.

$\mathcal{A}_3$. *Lower bounds*. There exists a constant $\alpha^{in} > 0$ such that $\widehat{\rho}, \widehat{\theta} \geq \alpha^{in} > 0$. This together with the uniform bounds on $\widehat{\rho}, \widehat{\theta}$ guarantees the existence of a constant $\tau_0 > 0$ such that $\frac{1}{\tau_0} \geq \widehat{\tau}_1/\widehat{\tau}_4 \geq \tau_0 > 0$.

$\mathcal{A}_4$. *Nontrapping condition*. Let $h^{in}(x,\xi) = \sqrt{\widehat{\tau}_1(x,0)\widehat{\tau}_4(x,0)}\,|\xi|^3$ as defined in (8) and $H_{h^{in}}$ be the corresponding Hamiltonian flow. Then $H_{h^{in}}$ is non-trapping, that is, if $(X,\Xi)(t;x,\xi)$ is a solution to

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \nabla_\xi h^{in}(X,\Xi)\,, \qquad\qquad X(0) = x\,,$$
$$\frac{\mathrm{d}\Xi}{\mathrm{d}t} = -\nabla_x h^{in}(X,\Xi)\,, \qquad\qquad \Xi(0) = \xi\,,$$

then for any $(x,\xi) \neq (0,0)$,

$$|X(t)| \to \infty \quad \text{as} \quad t \to \pm\infty\,.$$

Our linear estimate is the following.

Theorem. 1 *Let* $(\widehat{\rho}, \widehat{u}, \widehat{\theta}) \in C([0,T]; H^\infty)$ *be functions that satisfy assumptions* $\mathcal{A}_1 - \mathcal{A}_4$*. Then for every solution* $(\rho, u, \theta) \in C([0,T]; H^\infty)$ *of the linear system there exists* $T > 0$ *depending on the constants* c_0*,* c_A*, and* α^{in} *in the assumptions and* $c > 0$ *depending on* C^{in} *and* α^{in} *such that the following bound holds :*

$$\sup_{[0,T]} \left(\|\rho\|_{H^1}^2 + \|(u,\theta)\|_{L^2}^2 \right)(t) + \int_0^T \|\nabla_x (u,\theta)\|_{L^2}^2(s)\, \mathrm{d}s \\ \leq c \left(\|\rho^{in} - \bar{\rho}\|_{H^1}^2 + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{L^2}^2 \right) .$$

Both c *and* T *are independent of* ϵ*.*

Remark. All spatial derivatives satisfy similar bounds because they each satisfy a similar linear system.

Low Mach Number Limits: Results and Open Problems

- Boussinesq-balance incompressible Navier-Stokes system (L-Sun-Trivisa, SIAM J. Math. Analysis **44** (2012), 1760-1807)

- Sone's ghost effect system (L-Sun-Trivisa, preprint 2009)

- Dominant-balance incompressible Navier-Stokes system (open)

Well-Posedness Open Problems

- Domains with boundaries
- Momentum flux with $(D_x u)^2$ terms

Thank You!