Asymptotics of the solutions of the stochastic lattice wave equation

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Abstract

We consider the long time limit theorems for the solutions of a discrete wave equation with a weak stochastic forcing. The multiplicative noise conserves the energy and the momentum. We obtain a time-inhomogeneous Ornstein-Uhlenbeck equation for the limit wave function that holds both for square integrable and statistically homogeneous initial data. The limit is understood in the point-wise sense in the former case, and in the weak sense in the latter. On the other hand, the weak limit for square integrable initial data is deterministic.

1 Introduction

Energy transport and dispersion in dynamics of oscillators in a lattice have been investigated in many situations in order to understand macroscopic thermal conductivity properties. Typical example is the Fermi-Pasta-Ulam chain under the Hamiltonian evolution corresponding to a quartic interaction potential

$$\mathcal{H} = \sum_{y=-N}^{N} \left(\frac{\mathfrak{p}_y^2}{2m} + \frac{1}{2}\omega_0^2 \mathfrak{q}_y^2 \right) + \sum_{y=-N+1}^{N} \left[\frac{1}{2}(\mathfrak{q}_y - \mathfrak{q}_{y-1})^2 + \gamma(\mathfrak{q}_y - \mathfrak{q}_{y-1})^4 \right] \tag{1.1}$$

Here $\mathfrak{q}_y$ is the displacement of the y-th particle from its equilibrium position, $\mathfrak{p}_y$ is its momentum and m is the mass. When $\omega_0 \neq 0$, the particle is confined, this breaks translation invariance, and correspondingly the conservation of the total momentum, and we say that the chain is pinned.

When $\gamma = 0$ the Hamiltonian dynamics is given by the discrete in space linear wave equation, and the energy evolution is purely ballistic and dispersive. If $\gamma > 0$ and $\omega_0 \neq 0$, we expect that wave *scattering* due to the presence of the non-linearity gives a finite thermal conductivity and consequently a diffusive macroscopic evolution of the energy. If the chain is

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unpinned, $\omega_0 = 0$, and $\gamma > 0$, long waves scatter rarely, giving rise to a superdiffusive behavior of the energy [12].

The mathematical analysis of the macroscopic behavior of the energy is difficult in the case of deterministic nonlinear dynamics, and recently various models considering stochastic perturbations of the dynamics have been proposed. They simulate, at least qualitatively, the effect of the scattering by the non-linearity. Such noisy perturbations should conserve energy and be local in space [5]. In the unpinned case it is also important that they conserve the momentum [2, 3]. The perturbations considered in these papers are given by a random exchange of momentum so that the total kinetic energy is constant (consequently, the total energy is preserved as well, since the position components are untouched by the noise) and the total momentum is also conserved. This is achieved by adding, to each triple of adjacent particles, a diffusion on the corresponding circle of constant energy and momentum. Another example of a noisy perturbation having similar properties appears in a discontinuous in time model in which momenta of pairs of adjacent particles are exchanged at independent random times that are exponentially distributed.

When the interaction is linear, the thermal diffusivity of the energy in these models can be explicitly computed – it is finite for the pinned model but diverges with the size of the system in the unpinned case (corresponding to superdiffusive energy transport for the unpinned model).

The limit dynamics for the spectral measure of the energy in these stochastic models is investigated in [4], where the noise is also rescaled in such a way that there are only finitely many wave collisions in the unit macroscopic time. In a sense, this weak noise limit is similar to the regime where phonon-Boltzmann equation is valid in weakly nonlinear models (cf. [16]). The dynamics is defined in the following way. Consider the infinite lattice $\mathbb{Z}$ with the Hamiltonian associated to the linear evolution (1.1) ($\gamma = 0$), with $N = \infty$, perturbed by a conservative noise. Formally, it is given by the solution of the stochastic differential equations:

$$\begin{aligned} \dot{\mathfrak{q}}_y(t) &= \mathfrak{p}_y(t) \\ d\mathfrak{p}_y(t) &= \left(\Delta \mathfrak{q}_y - \omega_0^2 \mathfrak{q}_y\right) dt \; + d\eta_y(\epsilon t), \end{aligned} \tag{1.2}$$

where $\Delta \mathfrak{q}_y = \mathfrak{q}_{y+1} + \mathfrak{q}_{y-1} - 2\mathfrak{q}_y$ is the lattice Laplacian. The noise $d\eta_y(\epsilon t)$ will be chosen so as to model random exchange of momenta between the adjacent sites so that the total kinetic energy and momentum of the system are conserved (see (2.1) for the precise form of the noise). The small parameter $\epsilon > 0$ slows down its effect. The total Hamiltonian can be formally written as

$$\mathcal{H}(\mathfrak{q}, \mathfrak{p}) = \sum_{y \in \mathbb{Z}} \frac{\mathfrak{p}_y^2}{2} + \sum_{x,y \in \mathbb{Z}} \alpha_{x-y} \mathfrak{q}_x \mathfrak{q}_y, \tag{1.3}$$

with $\alpha_0 = \frac{1}{2}\omega_0^2 + 2$, $\alpha_{-1} = \alpha_{+1} = -1$, and $\alpha_y = 0$ otherwise. The dispersion relation $\omega(k)$ for this system is

$$\omega(k) = \sqrt{\hat{\alpha}(k)} = \sqrt{\frac{\omega_0^2}{2} + 4\sin^2(\pi k)}, \quad k \in \mathbb{T}. \tag{1.4}$$

The Fourier transform $\hat{\alpha}(k)$ is defined as in (2.3) below. Let us introduce the complex wave function

$$\psi_y(t) := (\check{\omega} * \mathfrak{q})_y(t) + i\mathfrak{p}_y(t), \tag{1.5}$$

where $\check{\omega}_y$ is the inverse Fourier transform of $\omega(k)$. Its Fourier transform is given by

$$\hat{\psi}(t, k) := \omega(k)\hat{\mathfrak{q}}(k, t) + i\hat{\mathfrak{p}}(t, k) \tag{1.6}$$

and satisfies the equation

$$d\hat{\psi}(t,k) := -i\omega(k)\hat{\psi}(t,k)dt + id\hat{\eta}(\epsilon t,k), \tag{1.7}$$

where $d\hat{\eta}(t,k)$ is the Fourier transform of the noise. Due to the conservation properties of the dynamics, if the initial configuration has finite total energy $\mathcal{H}(\mathfrak{q}(0),\mathfrak{p}(0)) < +\infty$, then all the functions introduced in (1.3) and (1.5)-(1.6) are well defined and

$$\mathcal{H}(\mathfrak{q}(t),\mathfrak{p}(t)) = \sum_y |\psi_y(t)|^2 = \int_{\mathbb{T}} |\hat{\psi}(t,k)|^2 dk$$

Therefore we can identify $|\hat{\psi}(t,k)|^2$ with the energy density in the mode space. In the zero noise case, $|\hat{\psi}(t,k)|^2$ is conserved for any $k \in \mathbb{T}$ (i.e. $\partial_t|\hat{\psi}(t,k)|^2 = 0$). The stochastic conservative perturbation mixes the energies between different modes k, and $|\hat{\psi}(t,k)|^2$ becomes a random variable. The evolution of the average energy $\mathcal{E}(t,k) := \mathbb{E}|\hat{\psi}(t,k)|^2$ was considered in [4]. Since the stochastic perturbation is of order ϵ, to have a visible effect of mixing of different modes we have to look at the time scale $\epsilon^{-1}t$. It was shown in [4] that the limit

$$\lim_{\epsilon\to 0} \mathcal{E}\left(\frac{t}{\epsilon},k\right) = \bar{\mathcal{E}}\left(t,k\right) \tag{1.8}$$

exists in the sense of distributions, and is the solution of the linear kinetic equation

$$\partial_t \bar{\mathcal{E}}\left(t,k\right) = \int_{\mathbb{T}} R(k,k')\left[\bar{\mathcal{E}}\left(t,k'\right) - \bar{\mathcal{E}}\left(t,k\right)\right] dk' \tag{1.9}$$

with the initial condition $\bar{\mathcal{E}}\left(0,k\right) = |\hat{\psi}(0,k)|^2$. The scattering kernel $R(k,k')$ is given by (3.2) below.

The goal of the present article is to obtain a direct information on the wave function $\hat{\psi}(t/\epsilon,k)$, as was done in [1] for the Schrödinger equation, and not only for the average energy. It follows from (1.7) that the unperturbed (by noise) evolution of this function is governed by the highly oscillating factor $e^{-i\omega(k)t/\varepsilon}$ (after we rescale the time). It is therefore reasonable to consider, in case of the perturbed system, the compensated wave function of the form

$$\tilde{\psi}^{(\epsilon)}(t,k) := e^{i\omega(k)t/\varepsilon}\hat{\psi}(t/\varepsilon,k).$$

We show that once we compensate for fast oscillations, the wave function converges in law to the solution a Langevin equation driven by (1.9). More precisely, we prove in Theorem 3.1 below, existence of the limit (in law and pointwise in k):

$$\lim_{\varepsilon\to 0} \tilde{\psi}^{(\epsilon)}(t,k) = \tilde{\psi}(t,k). \tag{1.10}$$

The limit $\tilde{\psi}(t,k)$ is a complex valued stochastic process satisfying the linear (time inhomogeneous) Ornstein-Uhlenbeck equation

$$d\tilde{\psi}(t,k) = -\frac{\hat{\beta}(k)}{4}\tilde{\psi}(t,k)dt + \sqrt{\mathcal{R}(t,k)}dw_k(t), \tag{1.11}$$

with the initial condition $\tilde{\psi}(0,k) = \hat{\psi}(0,k)$. Here $\hat{\beta}(k)$ is given by (2.5) below,

$$\mathcal{R}(t,k) = \int_{\mathbb{T}} \bar{\mathcal{E}}(t,k')R(k,k')dk', \tag{1.12}$$

and $\{w_k(t)\}$ is a family of pairwise independent standard complex valued Brownian motions parametrized by $k \in \mathbb{T}$. That is, they are complex valued, jointly Gaussian, centered processes satisfying

$$\mathbb{E}[w_k(t) w_{k'}(s)] = 0 \quad \text{and} \quad \mathbb{E}[w^*_{k'}(t) w_k(s)] = \delta_{k,k'} t \wedge s$$

for all $t, s \geq 0$ and $k, k' \in \mathbb{T}$. Here $\delta_{k,k'} = 0$ for $k \neq k'$ and $\delta_{k,k} = 1$. Equation (1.11) has the explicit solution

$$\tilde{\psi}(t,k) = e^{-\frac{1}{4}\hat{\beta}(k)t}\hat{\psi}(0,k) + \int_0^t e^{-\frac{1}{4}\hat{\beta}(k)(t-s)}\sqrt{\mathcal{R}(s,k)}dw_k(s). \tag{1.13}$$

In particular, we have

$$\mathbb{E}|\tilde{\psi}(t,k)|^2 = e^{-\frac{1}{2}\hat{\beta}(k)t}|\hat{\psi}(0,k)|^2 + \int_0^t e^{-\frac{1}{2}\hat{\beta}(k)(t-s)}\mathcal{R}(s,k)ds$$

which is equivalent to (1.9), since $\bar{\mathcal{E}}(t,k) = \mathbb{E}|\tilde{\psi}(t,k)|^2$ and

$$\hat{\beta}(k) = 2\int_{\mathbb{T}} R(k,k')dk', \tag{1.14}$$

as can be seen by a direct calculation from (2.5) and (3.2).

Initial conditions such that $\int_{\mathbb{T}} |\hat{\psi}(0,k)|^2 dk < \infty$ correspond to a *local* perturbation of the zero temperature equilibrium. We are also interested in the macroscopic evolution of the equilibrium states at a positive temperature $T > 0$, starting with a random distribution given by the Gibbs measure at temperature T. In the mode space this is a centered, complex valued, Gaussian random field with distribution valued $\hat{\psi}(k)$. Its covariance is given by

$$\mathbb{E}[\hat{\psi}^*(k)\hat{\psi}(k')] = T\delta(k-k'), \qquad \mathbb{E}[\hat{\psi}(k)\hat{\psi}(k')] = 0. \tag{1.15}$$

Here $\delta(k-k')$ is Dirac's delta function. The distributions are invariant under the dynamics, due to the conservation of energy. Actually, in Section 3.2 we consider more general class of space homogeneous Gaussian, random initial conditions that are not necessarily stationary in law in time. More precisely, we show (see Theorem 3.3) that if the law of the initial condition is a homogeneous, centered Gaussian field with the covariance given by

$$\mathbb{E}\left[\hat{\psi}(k)^*\hat{\psi}(k')\right] = \mathcal{E}_0(k)\delta(k-k'), \qquad \mathbb{E}\left[\hat{\psi}(k)\hat{\psi}(k')\right] = 0,$$

then the compensated wave function converges in law, as a continuous in time process taking values in an appropriate distribution space, to the solution of the time inhomogeneous stochastic equation:

$$d\tilde{\psi}(t,k) = -\frac{\hat{\beta}(k)}{4}\tilde{\psi}(t,k)dt + \sqrt{\mathcal{R}(t,k)}dW(t,k). \tag{1.16}$$

Here, $\mathcal{R}(t,k)$ is given by (1.12) and $\bar{\mathcal{E}}(t,k)$ is the solution of the deterministic equation (1.9) with the initial condition $\bar{\mathcal{E}}(0,k) = \mathcal{E}_0(k)$, while $dW(t,k)$ is a white noise on $\mathbb{R} \times \mathbb{T}$, a complex valued Gaussian process with the covariance

$$\mathbb{E}[dW(t,k)dW^*(s,k')] = \delta(k-k') \otimes \delta(t-s)dtds$$

and $\mathcal{R}(t,k)$ is given by (1.12). The solution of (1.16) is also explicit: $\tilde{\psi}(t)$ is the distribution

$$\tilde{\psi}(t) = e^{-\hat{\beta}t/4}\hat{\psi} + \int_0^t e^{-\hat{\beta}(t-s)/4}\mathcal{R}^{1/2}(s)dW(s).$$

In particular, in the case of the initial condition distributed according to a Gibbs measure, the solution $\hat{\psi}(t,k)$ of (1.7) has the same law for all times, therefore $\bar{\mathcal{E}}(t,k) = T$ for all $t \geq 0$. In this case, (1.14) shows that $\mathcal{R}(t,k) = \hat{\beta}(k)T/2$. Therefore, as a consequence of (1.16), the limit of the compensated wave function is the solution of the linear infinite dimensional stochastic differential equation:

$$d\tilde{\psi}(t,k) = -\frac{\hat{\beta}(k)}{4}\tilde{\psi}(t,k)dt + \sqrt{\frac{T\hat{\beta}(k)}{2}}dW(t,k). \tag{1.17}$$

In the general case, when $\mathcal{E}_0(k)$ is not constant, we have

$$\lim_{t\to\infty} \bar{\mathcal{E}}(t,k) = \int_{\mathbb{T}} \mathcal{E}_0(k')dk' = T,$$

hence, equation (1.17) describes the asymptotic stationary regime of (1.16) where the temperature is given by the average of the initial energy over all the modes k. Recall that the microscopic noise conserves the total energy and that the resulting temperature T depends only on the law of the initial condition.

Let us also comment on the difference between the square integrable and distribution-valued initial data. While the Ornstein-Uhlenbeck equations (1.11) and (1.16) are essentially identical, the limits hold in a different sense. For the square integrable data, the limit equation holds point-wise in k. If you consider the limit in the sense of distributions (that is, integrated against a test function) for such initial data, it is described simply by attenuation of the initial condition by an exponential factor $e^{-\beta(k)t/4}$ (see part (ii) of Theorem 3.1) – that is, by (1.11) with no stochastic forcing. This result stands in sharp contrast with the case of spatially homogeneous initial data (note that then the energy has to be infinite) when the respective limit in the sense of distributions is stochastic, see (1.16), and fluctuations can not be averaged out by integration.

The results of [4] also concern the Wigner transform, which is the spatially localized energy spectrum. The spatially inhomogeneous version of (1.9) is (after a simultaneous rescaling of space-time by $(\epsilon^{-1}x, \epsilon^{-1}t)$)

$$\partial_t\bar{\mathcal{E}}(t,x,k) + \omega'(k)\partial_x\bar{\mathcal{E}}(t,x,k) = \int_{\mathbb{T}} R(k,k')\left[\bar{\mathcal{E}}\left(t,x,k'\right) - \bar{\mathcal{E}}(t,x,k)\right]dk'. \tag{1.18}$$

This gives the probability distribution at time t of the phonons in the (x,k) space. The behavior at small k of the velocity $\omega'(k)$ is responsible for the superdiffusive behavior of the energy for the unpinned chain (cf. [2, 3, 10, 11]). It would be interesting to understand the relation of an inhomogeneous version of the result of the present paper to this superdiffusive phenomenon.

The organization of the paper is as follows. In Section 2 we introduce the basic notions that shall be used throughout the article. We formulate rigorously the stochastic differential equation for the Fourier transform of the wave function, see (2.8). The equation for the compensated wave function is formulated in (2.13).

The paper is organized as follows. Section 2 contains the precise mathematical formulation of the problem and necessary definitions. We formulate the results for the convergence of compensated wave function in in Section 3, see Theorem 3.1 for square integrable initial data, and Theorem 3.3 for spatially homogeneous, Gaussian initial distributions. The proofs of these results are presented in Sections 4 and 5, respectively.

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2 Preliminaries

2.1 Infinite system of interacting harmonic oscillators

The dynamics of the system of oscillators can be written formally as a system of Itô stochastic differential equations indexed by $y \in \mathbb{Z}$

$$
\begin{aligned}
d\mathfrak{q}_y(t) &= \mathfrak{p}_y(t)dt \qquad (2.1)\\
d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y \, dt - \frac{\epsilon}{2}(\beta * \mathfrak{p}(t))_y \, dt + \sqrt{\epsilon} \sum_{k=-1,0,1} (Y_{y+k}\mathfrak{p}_y(t))dw_{y+k}(t).
\end{aligned}
$$

Here

$$
Y_x := (\mathfrak{p}_x - \mathfrak{p}_{x+1})\partial_{\mathfrak{p}_{x-1}} + (\mathfrak{p}_{x+1} - \mathfrak{p}_{x-1})\partial_{\mathfrak{p}_x} + (\mathfrak{p}_{x-1} - \mathfrak{p}_x)\partial_{\mathfrak{p}_{x+1}} \qquad (2.2)
$$

and $\{w_y(t),\, t \ge 0\}$, $y \in \mathbb{Z}$ is a family of i.i.d. one dimensional, real valued, standard Brownian motions, that are non-anticipative over the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\, t \ge 0\}, \mathbb{P})$. In addition,

$$
\beta_y = \Delta\beta_y^{(0)} := \beta_{y+1}^{(0)} + \beta_{y-1}^{(0)} - 2\beta_y^{(0)}
$$

with

$$
\beta_y^{(0)} = \begin{cases} -4, & y = 0 \\ -1, & y = \pm 1 \\ 0, & \text{if otherwise.} \end{cases}
$$

Recall that the lattice Laplacian of $g : \mathbb{Z} \to \mathbb{C}$ is given by $\Delta g_y := g_{y+1} + g_{y-1} - 2g_y$.

The Fourier transform of a square integrable sequence of complex numbers $\{\gamma_y,\, y \in \mathbb{Z}\}$ is defined as

$$
\hat{\gamma}(k) = \sum_{y \in \mathbb{Z}} \gamma_y e_y(k), \quad k \in \mathbb{T}. \qquad (2.3)
$$

The one dimensional torus $\mathbb{T}$ considered in this article is understood as the interval $[-1/2, 1/2]$ with identified endpoints. The inverse transform is given by

$$
\check{f}_y = \int_{\mathbb{T}} f(k) e_y^*(k) dk, \quad y \in \mathbb{Z} \qquad (2.4)
$$

for any f belonging to $L^2(\mathbb{T})$ - the space of complex valued, square integrable functions. Here

$$
e_y(k) := \exp\{-i2\pi yk\}, \quad y \in \mathbb{Z}
$$

is the standard orthonormal base in $L^2(\mathbb{T})$. A simple calculation shows that

$$\hat{\beta}(k) = 8\sin^2(\pi k)\left[1 + 2\cos^2(\pi k)\right]. \tag{2.5}$$

We assume also (cf [4]) that

a1) $\{\alpha_y,\, y \in \mathbb{Z}\}$ is real valued and there exists $C > 0$ such that $|\alpha_y| \le Ce^{-|y|/C}$ for all $y \in \mathbb{Z}$,

a2) $\hat{\alpha}(k)$ is also real valued and $\hat{\alpha}(k) > 0$ for $k \neq 0$ and in case $\hat{\alpha}(0) = 0$ we have $\hat{\alpha}''(0) > 0$.

The above conditions imply that both functions $y \mapsto \alpha_y$ and $k \mapsto \hat{\alpha}(k)$ are even. In addition, $\hat{\alpha} \in C^\infty(\mathbb{T})$ and in case $\hat{\alpha}(0) = 0$ we have $\hat{\alpha}(k) = k^2\phi(k^2)$ for some strictly positive $\phi \in C^\infty(\mathbb{T})$. Recall that the function $\omega(k) := \sqrt{\hat{\alpha}(k)}$ is the dispersion relation.

2.2 Evolution of the wave function

For a given $m \in \mathbb{R}$ we define the space $H^m(\mathbb{T})$ as the completion of $C^\infty(\mathbb{T})$ under the norm

$$\|f\|_m^2 := \sum_{y \in \mathbb{Z}} (1 + y^2)^m |\tilde{f}_y|^2.$$

We shall denote by $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{T})$. By continuity it extends in an obvious way to $H^m(\mathbb{T}) \times H^{-m}(\mathbb{T})$ for an arbitrary $m \in \mathbb{R}$.

It is convenient to introduce the wave function that, adjusted to the macroscopic time, is given by

$$\psi^{(\epsilon)}(t) := \tilde{\omega} * \mathfrak{q}\left(\frac{t}{\epsilon}\right) + i\mathfrak{p}\left(\frac{t}{\epsilon}\right). \tag{2.6}$$

Here $\{\tilde{\omega}_y,\, y \in \mathbb{Z}\}$ is the inverse Fourier transform of $\omega(k) := \sqrt{\hat{\alpha}(k)}$. We shall consider the Fourier transform of the wave function

$$\hat{\psi}^{(\epsilon)}(t, k) := \omega(k)\hat{\mathfrak{q}}\left(\frac{t}{\epsilon}, k\right) + i\hat{\mathfrak{p}}\left(\frac{t}{\epsilon}, k\right). \tag{2.7}$$

Using (2.1) as a motivation, we obtain formally, by considering the Fourier transform of (2.1), that

$$\begin{aligned} &d\hat{\psi}^{(\epsilon)}(t) = A[\hat{\psi}^{(\epsilon)}(t)]dt + Q[\hat{\psi}^{(\epsilon)}(t)]dW(t), \\ &\hat{\psi}^{(\epsilon)}(0) = \hat{\psi}, \end{aligned} \tag{2.8}$$

where $\hat{\psi} \in L^2(\mathbb{T})$, and mapping $A : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by

$$A[f](k) := -\frac{i}{\epsilon}\omega(k)f(k) - \frac{\hat{\beta}(k)}{4}[f_1(k) - f_{-1}(k)], \quad \forall\, f \in L^2(\mathbb{T}). \tag{2.9}$$

Here

$$f_1(k) := f(k) \quad \text{and} \quad f_{-1}(k) := f^*(-k). \tag{2.10}$$

In addition, $Q[g] : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a linear mapping that for any $g \in L^2(\mathbb{T})$ is given by

$$Q[g](f)(k) := i\int_{\mathbb{T}} r(k, k')[g_1(k - k') - g_{-1}(k - k')]f(k')dk', \quad \forall\, f \in L^2(\mathbb{T}), \tag{2.11}$$

where

$$r(k,k') := \sin(2\pi k) + \sin[2\pi(k-k')] + \sin[2\pi(k'-2k)]$$
$$= 4\sin(\pi k)\sin[\pi(k-k')]\sin\left[(2k-k')\pi\right], \quad k,k' \in \mathbb{T}.$$

The cylindrical Wiener process on $L^2(\mathbb{T})$ appearing in (2.8) is $dW(t) := \sum_{y\in\mathbb{Z}} e_y dw_y(t)$.

It can be easily checked that $\sum_{y\in\mathbb{Z}} \|Q[g](e_y)\|_{L^2}^2 \le C\|g\|_{L^2}^2$ for some $C>0$ and all $g \in L^2(\mathbb{T})$ so $Q[g]$ is Hilbert-Schmidt, which ensures that

$$Q[\hat{\psi}^{(\epsilon)}(t)]dW(t) := \sum_{y\in\mathbb{Z}} Q[\hat{\psi}^{(\epsilon)}(t)](e_y)dw_y(t)$$

is summable in $L^2(\mathbb{T})$, both in the L^2 and a.s. sense. It is also obvious that the mapping A is Lipschitz. Using Theorem 7.4, p. 186, of [6] one concludes therefore that there exists an $L^2(\mathbb{T})$-valued, adapted process $\{\hat{\psi}^{(\epsilon)}(t),\, t\ge 0\}$ that is a unique solution to (2.8), understood in the mild sense. In addition, see Section 2 of [4], the total energy is conserved:

$$\|\hat{\psi}^{(\epsilon)}(t)\|_{L^2} = \text{const}, \quad \forall\, t \ge 0 \tag{2.12}$$

for a.s. realization of Brownian motions and an initial condition from $L^2(\mathbb{T})$.

2.3 Compensated wave function

Let us define the compensated wave function

$$\tilde{\psi}^{(\epsilon)}(t,k) := \hat{\psi}^{(\epsilon)}(t,k)\exp\left\{it\frac{\omega(k)}{\epsilon}\right\}.$$

From (2.8) we obtain the following equation

$$d\tilde{\psi}^{(\epsilon)}(t,k) = \mathcal{A}\left[\frac{t}{\epsilon}, \tilde{\psi}^{(\epsilon)}(t)\right](k)dt + d\tilde{\mathcal{M}}_t^{(\epsilon)}(k),$$
$$\tilde{\psi}^{(\epsilon)}(0) = \hat{\psi}, \tag{2.13}$$

where $\hat{\psi} \in L^2(\mathbb{T})$, $\mathcal{A}[t,\cdot] : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is Lipschitz

$$\mathcal{A}[t,f](k) := -\frac{\hat{\beta}(k)}{4}\left[f(k) - \exp\left\{2i\omega(k)t\right\} f^*(-k)\right]. \tag{2.14}$$

The martingale term is

$$d\tilde{\mathcal{M}}_t^{(\epsilon)} := \tilde{Q}\left[\frac{t}{\epsilon}, \tilde{\psi}^{(\epsilon)}(t)\right]dW(t), \tag{2.15}$$

where for any $g \in L^2(\mathbb{T})$ and $t \ge 0$, the operator $\tilde{Q}[t,g] : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, given by

$$\tilde{Q}[t,g](f)(k) := i\sum_{\sigma=\pm1}\sigma\int_{\mathbb{T}} r(k,k')g_\sigma(k-k')f(k')\exp\left\{i[\omega(k)-\sigma\omega(k-k')]t\right\}dk', \tag{2.16}$$

is Hilbert-Schmidt. Its Hilbert-Schmidt norm satisfies

$$\|\tilde{Q}[t,g_1] - \tilde{Q}[t,g_2]\|_{(HS)} \le C\|g_1 - g_2\|_{L^2(\mathbb{T})}, \quad \forall\, t \ge 0,\, g_1, g_2 \in L^2(\mathbb{T}),$$

for some constant $C>0$. In addition, it can be verified by a direct computation that $t \mapsto \tilde{A}[t,\cdot]$ is Lipschitz, uniformly in t. The solution of (2.13) exists in the mild sense, see [6], Theorem 7.4, p. 186, and is unique. Moreover, if the initial condition belongs to $C(\mathbb{T})$ so does $\tilde{\psi}^{(\epsilon)}(t)$ for any $t \ge 0$ and a.s. realization of the Wiener process.

3 Convergence of the compensated process

3.1 Square integrable initial data

Before formulating the result we introduce some auxiliaries. First, for any $k_1^0, k_2^0 \in \mathbb{T}$, $\sigma_1, \sigma_2 \in \{-1, 1\}$ let us denote

$$\mathcal{K}^{\sigma_1,\sigma_2}(k_1^0, k_2^0) := [k : \omega(k_1^0) - \omega(k_1^0 - k) = \sigma_1[\omega(k_2^0) - \sigma_2\omega(k_2^0 - k)]]$$

and

$$\mathcal{K}_0(k_1^0, k_2^0) := [k : \omega(k_1^0) + \omega(k_1^0 - k) = \omega(k_2^0) + \omega(k_2^0 - k)].$$

Let also

$$\mathcal{K}(k_1^0, k_2^0) = \bigcup_{\sigma_1,\sigma_2=\pm 1} \mathcal{K}^{\sigma_1,\sigma_2}(k_1^0, k_2^0) \cup \mathcal{K}_0(k_1^0, k_2^0).$$

We shall require that:

Condition ω) for any $k_1^0 \neq k_2^0$ the one dimensional Lebesgue measure $m_1(\mathcal{K}(k_1^0, k_2^0)) = 0$.

Define the scattering operator $\mathcal{L} : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ by

$$\mathcal{L}f(k) := \int_{\mathbb{T}} R(k, k')[f(k') - f(k)]dk', \quad f \in L^1(\mathbb{T}), \tag{3.1}$$

where the scattering kernel is given by

$$\begin{aligned} R(k, k') &:= r^2(k, k - k') + r^2(k, k + k') \\ &= 16\sin^2(\pi k)\sin^2(\pi k')\left\{\sin^2\left[\pi(k + k')\right] + \sin^2\left[\pi(k - k')\right]\right\}. \end{aligned} \tag{3.2}$$

Suppose that $\hat{\psi} \in L^2(\mathbb{T})$. Let

$$\mathcal{R}(t, k) := \int_{\mathbb{T}} R(k, k')\bar{\mathcal{E}}(t, k')dk', \tag{3.3}$$

where $\bar{\mathcal{E}}(t, k)$ is the solution of an equation

$$\bar{\mathcal{E}}(t, k) = |\hat{\psi}(k)|^2 + \int_0^t \mathcal{L}\bar{\mathcal{E}}(s, k)ds. \tag{3.4}$$

Assume also that $\{w_k(t),\, t \geq 0\}$ is a family of pairwise independent standard, one dimensional, complex valued Brownian motions indexed by $k \in \mathbb{T}$. Our first principal result can be stated as follows.

Theorem 3.1 *(i) Suppose that $\hat{\psi} \in C(\mathbb{T})$ and $k_1, \ldots, k_n \in \mathbb{T}$ for a given integer $n \geq 1$. Then $\{(\tilde{\psi}^{(\epsilon)}(t, k_1), \ldots, \tilde{\psi}^{(\epsilon)}(t, k_n)),\, t \geq 0\}$ converge in law over $C([0, +\infty); \mathbb{C}^n)$, as $\epsilon \to 0+$, to $\{(\tilde{\psi}(t, k_1), \ldots, \tilde{\psi}(t, k_n)),\, t \geq 0\}$, where $\{\tilde{\psi}(t, k),\, t \geq 0\}$ is a complex valued, non-homogeneous in time Ornstein-Uhlenbeck process that is the solution of the equation*

$$\begin{aligned} d\tilde{\psi}(t, k) &= -\frac{\hat{\beta}(k)}{4}\tilde{\psi}(t, k)dt + \mathcal{R}^{1/2}(t, k)dw_k(t), \\ \tilde{\psi}(0, k) &= \hat{\psi}(k). \end{aligned} \tag{3.5}$$

(ii) If $\hat{\psi} \in L^2(\mathbb{T})$ then for any $f \in L^2(\mathbb{T})$ and $t_* > 0$ we have

$$\lim_{\epsilon \to 0+} \sup_{t \in [0,t_*]} \left| \langle \tilde{\psi}^{(\epsilon)}(t) - \bar{\psi}(t), f \rangle \right| = 0 \tag{3.6}$$

in probability. Here $\bar{\psi}(t)$ is given by

$$\bar{\psi}(t,k) := \hat{\psi}_0(k) \exp \left\{ -\frac{t\hat{\beta}(k)}{4} \right\}. \tag{3.7}$$

Remark. Note that $\mathcal{L}1 = 0$ and

$$\langle \mathcal{L}f, f \rangle = -\frac{1}{2} \int_{\mathbb{T}^2} R(k,k')[f(k') - f(k)]^2 dk dk' \le 0, \quad \forall f \in L^2(\mathbb{T}).$$

Since the operator $\mathcal{L}$ is symmetric and compact on $L^2(\mathbb{T})$, and 0 is its simple eigenvalue we conclude that $\mathcal{R}(t,k) \to (\hat{\beta}(k)/2)T$, where $T = \|\hat{\psi}_0\|_{L^2(\mathbb{T})}$, as $t \to +\infty$, uniformly on $\mathbb{T}$. As a result we obtain that

$$\lim_{t \to +\infty} \mathbb{E} \left| \tilde{\psi}(t,k) - \tilde{\psi}_s(t,k) \right|^2 = 0, \tag{3.8}$$

where $\tilde{\psi}_s(t,k)$ is a time homogeneous Ornstein-Uhlenbeck process given by

$$\begin{aligned} d\tilde{\psi}_s(t,k) &= -\frac{\hat{\beta}(k)}{4} \tilde{\psi}_s(t,k) dt + \sqrt{\frac{\hat{\beta}(k)T}{2}} dw_k(t), \\ \tilde{\psi}_s(0,k) &= \hat{\psi}(k). \end{aligned} \tag{3.9}$$

Let us also comment briefly on condition ω). An important fact used in the proof of part (i) of Theorem 3.1, is that the energy $|\tilde{\psi}^{(\epsilon)}(t)|^2$ converges in probability, weakly to $\bar{\mathcal{E}}(t)$, as $\epsilon \to 0+$. The fact that the mean $\mathbb{E}\langle |\tilde{\psi}^{(\epsilon)}(t)|^2, f \rangle$ converges to $\langle \bar{\mathcal{E}}(t), f \rangle$ has already been shown in [4]. It is clear from (2.13) that $\langle |\tilde{\psi}^{(\epsilon)}(t)|^2, f \rangle$ is a semimartingale, for any test function f. Condition ω) is used to prove that the martingale part of the semimartingale vanishes, see Lemma 4.6 below. This, in turn implies convergence in probability. The following simple criterion is useful for verification of condition ω), e.g. for dispersion relation $\omega(k)$ of the form (1.4).

Lemma 3.2 *Suppose that the dispersion relation satisfies the following condition: the equation*

$$\omega''(k) = 0 \tag{3.10}$$

has no solution in $\mathbb{T} \backslash \{0\}$. Then, for any (k_1^0, k_2^0) such that $k_1^0 \neq k_2^0$ we have $m_1(\mathcal{K}(k_1^0, k_2^0)) = 0$. In consequence the hypothesis ω) holds.

Proof. From the assumptions made we know that $\omega \in C^\infty(\mathbb{T} \setminus \{0\})$. Fix (k_1^0, k_2^0) such that $k_1^0 \neq k_2^0$. We only prove that the corresponding section of $\mathcal{K}_1$:

$$\mathcal{K}^{1,1}(k_1^0, k_2^0) := [k : \omega(k_1^0) - \omega(k_2^0 - k) = \omega(k_2^0) - \omega(k_1^0 - k)]$$

is of null Lebesgue measure. To simplify our considerations we assume that $\sigma_1 = \sigma_2 = 1$. The remaining cases can be dealt with similarly. Suppose on the contrary that the Lebesgue measure of the set is positive. We can find an increasing sequence of $\{\ell_n,\, n\ge 1\}\subset \mathcal{K}^{1,1}(k_1^0,k_2^0)$ so that the signs of $\{k_2^0-\ell_n,\, n\ge 1\}$ and $\{k_1^0-\ell_n,\, n\ge 1\}$ are definite. In consequence, we get

$$\omega(k_2^0-\ell_n)-\omega(k_2^0-\ell_{n+1})=\omega(k_1^0-\ell_n)-\omega(k_1^0-\ell_{n+1}).$$

Since $k_1^0\neq k_2^0$ and $\{\ell_n,\, n\ge 1\}$ is converging we conclude the existence of $\ell'\neq\ell''$ such that

$$\omega'(\ell')=\omega'(\ell'').$$

When the signs of the arguments are the same we conclude easily the contradiction with (3.10). Assume therefore that they are opposite and $\ell'<\ell''$. Since $\omega'(k)$ (understood as a function on $\mathbb{R}$) is 1-periodic and differentiable except at integer lattice points, the above implies that

$$\omega'(1+\ell')=\omega'(\ell'')$$

and $0<\ell''<1+\ell'<1$. Hence, we conclude the existence of a point $k\in\mathbb{T}\setminus\{0\}$ where (3.10) holds. This is also a contradiction. Therefore, $m_1(\mathcal{K}^{1,1}(k_1^0,k_2^0))=0$ for all $k_1^0\neq k_2^0$. $\Box$

3.2 Statistically homogeneous initial data

For a given non-negative $m\ge 0$, we assume that the initial data $\hat\psi$ is an $H^{-m}(\mathbb{T})$ valued Gaussian random element. More precisely, suppose that $\mathcal{E}_0(\cdot)$ is a non-negative function from $C(\mathbb{T})$, $\{\xi_y,\, y\in\mathbb{Z}\}$ are i.i.d. complex Gaussian random variables such that $\mathbb{E}\xi_0=0$ and $\mathbb{E}|\xi_0|^2=1$, and

$$\hat\psi(k)=\sum_{y\in\mathbb{Z}}\xi_y\mathcal{E}_0^{1/2}(k)e_y(k). \tag{3.11}$$

It is supported in $H^{-m}(\mathbb{T})$, provided that $m>1/2$. Observe that the covariance form equals

$$\mathcal{C}(J_1,J_2):=\mathbb{E}\left[\langle J_1,\hat\psi\rangle\langle J_2,\hat\psi\rangle^*\right]=\int_{\mathbb{T}}\mathcal{E}_0(k)J_1(k)J_2^*(k)dk \tag{3.12}$$

for any $J_1,J_2\in C^\infty(\mathbb{T})$. The Gibbs equilibrium states described in the introduction correspond to $\mathcal{E}_0(k)\equiv$ const.

Suppose that $\omega(\cdot)\in C_b^\infty(\mathbb{T}\setminus\{0\})$ is bounded with all derivatives on $\mathbb{T}\setminus\{0\}$ and such that $\omega'(0-)$ and $\omega'(0+)$ are defined. Then, for any $m<3/2$ and $t_*>0$ there exists $C>0$ such that operators $\mathcal{A}[t,\cdot]$, given by (2.14), satisfy

$$\|\mathcal{A}[t,g_1]-\mathcal{A}[t,g_2]\|_{H^{-m}(\mathbb{T})}\le C\|g_1-g_2\|_{H^{-m}(\mathbb{T})},\quad \forall\, t\in[0,t_*].$$

Likewise, the Hilbert-Schmidt norm of $\tilde Q[t,g]:L^2(\mathbb{T})\to H^{-m}(\mathbb{T})$ satisfies

$$\|\tilde Q[t,g_1]-\tilde Q[t,g_2]\|_{(HS)}\le C\|g_1-g_2\|_{H^{-m}(\mathbb{T})},\quad \forall\, g_1,g_2\in H^{-m}(\mathbb{T})$$

for $t\in[0,t_*]$. Using again the results of [6], it is easy to conclude that equation (2.13) has a unique mild solution $\{\tilde\psi^{(\epsilon)}(t),\, t\ge 0\}$ whose realizations belong to $C([0,+\infty);H^{-m}(\mathbb{T}))$.

Let $\mathcal{R}(t,k)$ be given by (3.3) with $\bar{\mathcal{E}}(t,k)$ the solution of (3.4) satisfying $\bar{\mathcal{E}}(0,k)=\mathcal{E}_0(k)$. Since the operator $f(k)\mapsto\mathcal{R}^{1/2}(t,k)f(k)$ is Hilbert-Schmidt, when considered from $L^2(\mathbb{T})$ to $H^{-m}(\mathbb{T})$, and $f(k)\mapsto-(\hat\beta(k)/4)f(k)$ is bounded on $H^{-m}(\mathbb{T})$, the equation

$$\begin{aligned} d\bar\psi_*(t,k) &= -\frac{\hat\beta(k)}{4}\bar\psi_*(t,k)dt+\mathcal{R}^{1/2}(t,k)dW(t,k),\\ \bar\psi_*(0,k) &= \hat\psi(k) \end{aligned} \tag{3.13}$$

has a unique $H^{-m}(\mathbb{T})$-valued mild solution, see Theorem 7.4, p. 186 of [6]. It is given by the formula

$$\bar{\psi}_*(t,k) = e^{-\hat{\beta}(k)t/4}\hat{\psi} + \int_0^t e^{-\hat{\beta}(k)(t-s)/4}\mathcal{R}^{1/2}(s,k)dW(s,k).$$

We denote by $H_w^{-m}(\mathbb{T})$ the Hilbert space equipped with the weak topology. Our main result is as follows.

Theorem 3.3 *Suppose that* $m \in (1/2, 3/2)$ *and condition* $\omega)$ *holds. Then, under the above assumptions, the processes* $\{\tilde{\psi}^{(\epsilon)}(t),\, t \geq 0\}$ *converge in law over* $C([0,+\infty), H_w^{-m}(\mathbb{T}))$*, as* $\epsilon \to 0+$*, to* $\{\bar{\psi}_*(t),\, t \geq 0\}$.

Remark. As in the remark made after Theorem 3.1 we can also conclude that

$$\lim_{t\to+\infty} \mathbb{E}\left|\langle \bar{\psi}_*(t) - \bar{\psi}_s(t), f\rangle\right|^2 = 0, \tag{3.14}$$

where $\bar{\psi}_s(t)$ is a time homogeneous, distribution valued Ornstein-Uhlenbeck process given by

$$\begin{aligned} d\bar{\psi}_s(t,k) &= -\frac{\hat{\beta}(k)}{4}\bar{\psi}_s(t,k)dt + \sqrt{\frac{\hat{\beta}(k)T}{2}}dW(t,k), \\ \bar{\psi}_s(0,k) &= \hat{\psi}(k), \end{aligned} \tag{3.15}$$

where $T = \|\mathcal{E}_0\|_{L^1(\mathbb{T})}$.

4 Proof of Theorem 3.1

We explain the idea of the proof in the case $n = 1$ (that is, the process $\hat{\psi}(t,k)$ for a fixed k), the independence of the compensated wave function for various k is handled in the same manner. Since the coefficients appearing in the stochastic differential equation describing the evolution of $\tilde{\psi}^{(\epsilon)}(t)$ (see (2.13)) are of the order $O(1)$, it is easy to conclude that for each k the laws of processes $\{\tilde{\psi}^{(\epsilon)}(t,k),\, t \geq 0\}$ are tight over $C([0,+\infty);\mathbb{C})$, as $\epsilon \to 0+$. In order to identify the weak limit, thus proving part i) of the theorem, we have to deal with the rapidly oscillating terms. First, we show that the rapidly oscillating part of the bounded variation term in (2.13) (with the factor $\exp\{2i\omega(k)t/\epsilon\}$ in (2.14)) vanishes in the limit, because of the following result.

Proposition 4.1 *For given* $t_* > 0$, $a \geq 0$, $k, k' \in \mathbb{T}$, $k \neq 0$ *and function* $f \in C^1[0,t_*]$ *we have*

$$\lim_{\epsilon\to 0+} \mathbb{E}\left|\sup_{t\in[0,t_*]} \int_0^t \exp\left\{-i\frac{as}{\epsilon}\right\} f(s)\hat{\psi}_\epsilon(s,k)ds\right| = 0. \tag{4.1}$$

This result is a part of Corollary 4.5 below.

Next, we show that the limit of the martingale part $\tilde{\mathcal{M}}_t^{(\epsilon)}(k)$ in (2.13) is a complex Gaussian martingale with the quadratic variation equal to $\int_0^t \mathcal{R}(s,k)ds$. To this purpose we prove the following convergence of the quadratic variation:

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left|\langle \tilde{\mathcal{M}}^{(\epsilon)}(k)\rangle_t - \int_0^t \mathcal{R}(s,k)ds\right| = 0 \tag{4.2}$$

in probability, for any $t_* > 0$. This is done in Proposition 4.2. The method of proof of (4.2) is as follows. The terms of the following form appear in the quadratic variation of $\tilde{\mathcal{M}}_t^{(\epsilon)}(k)$:

$$
\begin{aligned}
\mathcal{V}_\epsilon^{(0)}(t) &:= \int_0^t \langle |\hat{\psi}^{(\epsilon)}(s)|^2, f\rangle ds, \\
\mathcal{V}_\epsilon^{(1)}(t) &:= \int_0^t ds \int_{\mathbb{T}} \hat{\psi}^{(\epsilon)}(s,k)\hat{\psi}^{(\epsilon)}(s,-k) f^*(k) dk, \\
\mathcal{V}_\epsilon^{(2)}(t) &:= \int_0^t \langle [\hat{\psi}^{(\epsilon)}(s)]^2, f\rangle ds.
\end{aligned}
\tag{4.3}
$$

Here $f(k)$ is a certain explicit function related to the scattering kernel. As $\hat{\psi}^{(\epsilon)}(t,k)$ (without the compensation) is rapidly oscillating as $e^{-i\omega(k)t/\epsilon}$, we expect that only $\mathcal{V}_\epsilon^{(0)}(t)$ has a non-trivial limit. This term contains no oscillation and is essentially the time integral of scattered energy $|\hat{\psi}^{(\epsilon)}(t,k)|^2$. It has been shown in [4] that the expectation of the energy converges to the solution of (1.9). We need to strengthen this result to convergence in probability.

The proof of part ii) uses the same ideas. Integrating against a test function results in the formula for the quadratic variation, see (4.35), containing only terms with fast oscillating factors, so the stochastic part vanishes in the limit.

We now turn to the proof of the theorem. An application of the Itô formula to (2.8) yields, see Theorem 4.17 of [6],

$$
d|\hat{\psi}^{(\epsilon)}(t,k)|^2 = [I_\epsilon(t,k) + II_\epsilon(t,k)]\, dt + d\mathcal{M}_t^{(\epsilon)}(k), \tag{4.4}
$$

where

$$
\begin{aligned}
I_\epsilon(t,k) &:= (A[\hat{\psi}^{(\epsilon)}(t)])^*(k)\, \hat{\psi}^{(\epsilon)}(t,k) + (\hat{\psi}^{(\epsilon)})^*(t,k)\, A[\hat{\psi}^{(\epsilon)}(t)](k), \\
II_\epsilon(t,k) &:= \sum_{y\in\mathbb{Z}} (Q[\hat{\psi}^{(\epsilon)}(t)](e_y))^*(k)\, Q[\hat{\psi}^{(\epsilon)}(t)](e_y)(k),
\end{aligned}
$$

and $\mathcal{M}_t^{(\epsilon)}$ is an $\mathcal{F}_t$-adapted local martingale, given by $\mathcal{M}_t^{(\epsilon)} = \mathcal{M}_t^{(1,\epsilon)} + \mathcal{M}_t^{(2,\epsilon)}$, with

$$
\begin{aligned}
\mathcal{M}_t^{(1,\epsilon)}(k) &:= \int_0^t (Q[\hat{\psi}^{(\epsilon)}(s)]dW(s))^*(k)\, \hat{\psi}^{(\epsilon)}(s,k), \\
\mathcal{M}_t^{(2,\epsilon)}(k) &:= \int_0^t (\hat{\psi}^{(\epsilon)})^*(s,k)\, (Q[\hat{\psi}^{(\epsilon)}(s)]dW(s))(k).
\end{aligned}
\tag{4.5}
$$

We obtain from (2.9) that

$$
I_\epsilon(t,k) = -\frac{\hat{\beta}(k)}{2}|\hat{\psi}^{(\epsilon)}(t,k)|^2 - \frac{\hat{\beta}(k)}{4}\hat{\psi}_2^{(\epsilon)}(t,k), \tag{4.6}
$$

where

$$
\hat{\psi}_2^{(\epsilon)}(t,k) := \hat{\psi}^{(\epsilon)}(t,k)\hat{\psi}^{(\epsilon)}(t,-k) + (\hat{\psi}^{(\epsilon)})^*(t,k)(\hat{\psi}^{(\epsilon)})^*(t,-k),
$$

while equation (2.11) yields

$$
II_\epsilon(t,k) = \int_{\mathbb{T}} R(k,k')|\hat{\psi}^{(\epsilon)}(t,k')|^2 dk' + \frac{1}{2}\int_{\mathbb{T}} R(k,k')\hat{\psi}_2^{(\epsilon)}(t,k')dk'. \tag{4.7}
$$

We can derive analogous equations for $d[\hat\psi^{(\epsilon)}(t,k)]^2$ and $d[\hat\psi^{(\epsilon)}(t,k)\hat\psi^{(\epsilon)}(t,-k)]$. The corresponding terms shall be denoted by $I_\epsilon^{(i)}(t,k)$, $II_\epsilon^{(i)}(t,k)$ and the martingale

$$\mathcal{N}_{t,i}^{(\epsilon)}(k) = \mathcal{N}_{t,i}^{(1,\epsilon)}(k) + \mathcal{N}_{t,i}^{(2,\epsilon)}(k), \quad i = 1, 2.$$

We have

$$I_\epsilon^{(1)}(t,k) = -\frac{2i\omega(k)}{\epsilon}[\hat\psi^{(\epsilon)}(t,k)]^2 + \mathcal{P}_1[\hat\psi^{(\epsilon)}(t), (\hat\psi^{(\epsilon)})^*(t)],$$
$$I_\epsilon^{(2)}(t,k) = -\frac{2i\omega(k)}{\epsilon}\hat\psi_2^{(\epsilon)}(t,k) + \mathcal{P}_2[\hat\psi^{(\epsilon)}(t), (\hat\psi^{(\epsilon)})^*(t)],$$

and

$$II_\epsilon^{(i)}(t,k) = \mathcal{Q}_i[\hat\psi^{(\epsilon)}(t), (\hat\psi^{(\epsilon)})^*(t)], \quad i = 1, 2, \tag{4.8}$$

where $\mathcal{P}_i, \mathcal{Q}_i$ are second degree polynomials in $\hat\psi^{(\epsilon)}(t), (\hat\psi^{(\epsilon)})^*(t)$.

Proposition 4.2 *Let $f \in L^\infty(\mathbb{T})$, and $\mathcal{V}_\epsilon^{(j)}(t)$, $j = 0, 1, 2$ be defined by (4.3). Then for any $t_* > 0$ we have*

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left| \mathcal{V}_\epsilon^{(0)}(t) - \int_0^t \langle \bar{\mathcal{E}}(s), f\rangle ds \right| = 0 \tag{4.9}$$

and

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left| \mathcal{V}_\epsilon^{(i)}(t) \right| = 0, \quad i = 1, 2 \tag{4.10}$$

in probability.

Proof. The proof of this proposition shall be obtained at the end of a series of lemmas. We start with the following.

Lemma 4.3 *For any $p \in [2, +\infty)$ there exists $C > 0$ such that*

$$\sup_{\epsilon\in(0,1]} \mathbb{E}\left[\sup_{t\in[0,t_*]} \|\hat\psi^{(\epsilon)}(t)\|^p_{L^p(\mathbb{T})} \right] \le Ce^{Ct_*}\|\hat\psi\|^p_{L^p(\mathbb{T})}, \quad \forall\, t_* > 0. \tag{4.11}$$

Proof. Let

$$T_t^{(\epsilon)}\hat\psi(k) := \exp\left\{ -i\frac{\omega(k)t}{\epsilon} \right\} \hat\psi(k), \quad \hat\psi \in L^p(\mathbb{T}),\ t \in \mathbb{R}.$$

We obviously have

$$\|T_t^{(\epsilon)}\hat\psi\|_{L^p(\mathbb{T})} = \|\hat\psi\|_{L^p(\mathbb{T})}, \quad \forall\, t \ge 0,\ p \in [1, +\infty]. \tag{4.12}$$

Using the Duhamel formula, the solution of (2.8) can be written as

$$\hat\psi^{(\epsilon)}(t,k) = \hat\psi(k) + \int_0^t T_{t-s}^{(\epsilon)} B[\hat\psi^{(\epsilon)}(s)](k) ds + \sum_{y\in\mathbb{Z}} \int_0^t T_{t-s}^{(\epsilon)} Q[\hat\psi^{(\epsilon)}(s)](e_y)(k) dw_y(s). \tag{4.13}$$

Hence, for a given $\epsilon \in (0,1]$ and $t_0 > 0$ to be adjusted later on, we can write

$$\mathbb{E}\left[\sup_{t\in[0,t_0]} |\hat{\psi}^{(\epsilon)}(t,k)|^p\right] \le C\left\{|\hat{\psi}(k)|^p + t_0^{p-1}\int_0^{t_0}\mathbb{E}|\hat{\psi}^{(\epsilon)}(s,k)|^p ds\right.$$
$$\left. + \mathbb{E}\left\{\sup_{t\in[0,t_0]}\left|\int_0^t \sum_{y\in\mathbb{Z}} T_{-s}^{(\epsilon)} Q[\hat{\psi}^{(\epsilon)}(s)](e_y)(k)dw_y(s)\right|^p\right\}\right\}. \tag{4.14}$$

To estimate the martingale term on the right hand side we use Burkholder-Davis-Gundy inequality which allows to bound it by

$$4^{p/2}\mathbb{E}\left(\int_0^{t_0}\int_{\mathbb{T}} R(k,k')|\hat{\psi}^{(\epsilon)}(s,k-k')|^2 dk' ds\right)^{p/2} \le C_1 t_0^{p/2-1}\int_0^{t_0}\mathbb{E}\|\hat{\psi}^{(\epsilon)}(s)\|^p_{L^p(\mathbb{T})}ds, \tag{4.15}$$

for some constant $C_1 > 0$. Choosing t_0 sufficiently small, so that $Ct_0^p + CC_1 t_0^{p/2} < 1/2$, we conclude that

$$\mathbb{E}\left\{\sup_{t\in[0,t_0]}\|\hat{\psi}^{(\epsilon)}(t)\|^p_{L^p(\mathbb{T})}\right\} \le 2C\|\hat{\psi}\|^p_{L^p(\mathbb{T})}. \tag{4.16}$$

The argument leading to (4.16) can be used on each of the intervals $[jt_0,(j+1)t_0)$ for any $j \ge 1$ and yields

$$\mathbb{E}\left\{\sup_{t\in[jt_0,(j+1)t_0]}\|\hat{\psi}^{(\epsilon)}(t)\|^p_{L^p(\mathbb{T})}\right\} \le C\mathbb{E}\|\hat{\psi}^{(\epsilon)}(jt_0)\|^p_{L^p(\mathbb{T})} \le C\mathbb{E}\left\{\sup_{t\in[(j-1)t_0,jt_0]}\|\hat{\psi}^{(\epsilon)}(t)\|^p_{L^p(\mathbb{T})}\right\}, \tag{4.17}$$

for some constant $C > 0$ independent of j and $\epsilon \in (0,1]$. Hence, after j iterations of the above estimate, we conclude

$$\mathbb{E}\left\{\sup_{t\in[jt_0,(j+1)t_0]}\|\hat{\psi}^{(\epsilon)}(t)\|^p_{L^p(\mathbb{T})}\right\} \le C^j\|\hat{\psi}\|^p_{L^p(\mathbb{T})} \tag{4.18}$$

and (4.11) follows. □

Combining the above result with estimates (4.14) and (4.15) we conclude the following.

Corollary 4.4 *For a given $p \in [1,+\infty)$ there exists $C > 0$ such that*

$$\sup_{\epsilon\in(0,1],k\in\mathbb{T}}\mathbb{E}\left[\sup_{t\in[0,t_*]}|\hat{\psi}^{(\epsilon)}(t,k)|^p\right] \le Ce^{Ct_*}\|\hat{\psi}\|_{L^p(\mathbb{T})}, \quad \forall\, t_* > 0.$$

From (2.8) and the above corollary we conclude the following.

Corollary 4.5 *For given $t_* > 0$, $a \ge 0$, $k, k' \in \mathbb{T}$, $k \ne 0$ and function $f \in C^1[0,t_*]$ we have*

$$\lim_{\epsilon\to 0+}\mathbb{E}\left|\sup_{t\in[0,t_*]}\int_0^t \exp\left\{-i\frac{as}{\epsilon}\right\} f(s)\hat{\psi}_\epsilon(s,k)ds\right| = 0 \tag{4.19}$$

and

$$\lim_{\epsilon\to 0+}\mathbb{E}\left\{\sup_{t\in[0,t_*]}\left|\int_0^t \hat{\psi}^{(\epsilon)}(s,k)\hat{\psi}^{(\epsilon)}(s,k')ds\right|\right\} = 0. \tag{4.20}$$

In addition if $\omega(k)\neq\omega(k')$ we have

$$\lim_{\epsilon\to 0+}\mathbb{E}\left\{\sup_{t\in[0,t_*]}\left|\int_0^t \hat{\psi}^{(\epsilon)}(s,k)(\hat{\psi}^{(\epsilon)})^*(s,k')ds\right|\right\}=0. \tag{4.21}$$

Proof. Using (2.8) we obtain

$$\exp\left\{-i\frac{at}{\epsilon}\right\}f(t)\hat{\psi}_\epsilon(t,k)-f(0)\hat{\psi}(k)=-i\frac{a+\omega(k)}{\epsilon}\int_0^t\exp\left\{-i\frac{sa}{\epsilon}\right\}f(s)\hat{\psi}_\epsilon(s,k)ds$$
$$+\int_0^t\mathcal{P}[\hat{\psi}_\epsilon(s),(\hat{\psi}_\epsilon)^*(s)](k)ds+\int_0^t\sum_{y\in\mathbb{Z}}\mathcal{Q}_y[\hat{\psi}_\epsilon(s),(\hat{\psi}_\epsilon)^*(s)](k)w_y(ds), \tag{4.22}$$

where $\mathcal{P}$, $\mathcal{Q}_y$ are first degree polynomials in $\hat{\psi}_\epsilon(s)$, $(\hat{\psi}_\epsilon)^*(s)$ with bounded coefficients and such that

$$\sup_{s\in[0,t_*]}\sum_{y\in\mathbb{Z}}|\mathcal{Q}_y[\hat{\psi}_\epsilon(s),(\hat{\psi}_\epsilon)^*(s)](k)|^2\le C\|\hat{\psi}_\epsilon(s)\|^2_{L^2(\mathbb{T})}.$$

Dividing both sides of (4.22) by $(\omega(k)+a)/\epsilon$ (possible since this factor is strictly positive) we calculate

$$\int_0^t\exp\left\{-i\frac{sa}{\epsilon}\right\}f(s)\hat{\psi}_\epsilon(s,k)ds.$$

Using Lemma 4.3 together with Corollary 4.4 we can easily conclude (4.19).

The proof of (4.20) is analogous. We use the Itô formula to express $d[\hat{\psi}^{(\epsilon)}(s,k)\hat{\psi}^{(\epsilon)}(s,k')]$ and $d[\hat{\psi}^{(\epsilon)}(s,k)(\hat{\psi}^{(\epsilon)})^*(s,k')]$. Then, we repeat the argument used above. □

The following lemma shall be crucial for us.

Lemma 4.6 *For any $f\in L^2(\mathbb{T})$, $t_*>0$ we have*

$$\lim_{\epsilon\to 0+}\mathbb{E}\left[\sup_{t\in[0,t_*]}\left|\langle\mathcal{M}_t^{(i,\epsilon)},f\rangle\right|^2\right]=0 \tag{4.23}$$

and

$$\lim_{\epsilon\to 0+}\mathbb{E}\left[\sup_{t\in[0,t_*]}\left|\langle\mathcal{N}_{t,j}^{(i,\epsilon)},f\rangle\right|^2\right]=0,\quad i,j=1,2. \tag{4.24}$$

Proof. We can write

$$\mathbb{E}\left|\langle\mathcal{M}_t^{(1,\epsilon)},f\rangle\right|^2\le 2\left\{\sum_{j\in\mathbb{Z}}\int_0^t ds\,\mathbb{E}\left|\int_{\mathbb{T}^2}r(k,k')f^*(k)(\hat{\psi}^{(\epsilon)})^*(s,k-k')e_j^*(k')\hat{\psi}^{(\epsilon)}(s,k)d\mathbf{k}\right|^2\right.$$
$$\left.+\sum_{j\in\mathbb{Z}}\int_0^t ds\mathbb{E}\left|\int_{\mathbb{T}^2}r(k,k')f^*(k)(\hat{\psi}^{(\epsilon)})(s,k'-k)e_j^*(k')\hat{\psi}^{(\epsilon)}(s,k)d\mathbf{k}\right|^2\right\} \tag{4.25}$$

Here, for abbreviation sake, we wrote $d\mathbf{k}=dkdk'$. The estimate of $\mathbb{E}\left|\langle\mathcal{M}_t^{(2,\epsilon)},f\rangle\right|^2$ is very similar except f^* on the right hand side should be replaced by f. Using Parseval identity we can further transform the right hand side of (4.25) into

$$
\begin{aligned}
&2\int_0^t ds\int_{\mathbb{T}^3} r(k,k')r(k_1,k')f^*(k)f(k_1)\\
&\times\Big\{\mathbb{E}\Big[(\hat\psi^{(\epsilon)})^*(s,k-k')\hat\psi^{(\epsilon)}(s,k)\hat\psi^{(\epsilon)}(s,k_1-k')(\hat\psi^{(\epsilon)})^*(s,k_1)\Big]\\
&+\mathbb{E}\Big[\hat\psi^{(\epsilon)}(s,k-k')\hat\psi^{(\epsilon)}(s,k)(\hat\psi^{(\epsilon)})^*(s,k_1-k')(\hat\psi^{(\epsilon)})^*(s,k_1)\Big]\Big\}\,d\mathbf{k},
\end{aligned}
\tag{4.26}
$$

where $d\mathbf{k}=dkdk_1dk'$.

Consider the term of (4.26) corresponding to the first expectation (the other can be dealt with in a similar fashion). Recall that

$$
\mathcal{K}_1:=[(k,k',k_1):\omega(k)+\omega(k'-k_1)=\omega(k')+\omega(k-k_1)].
$$

We claim that for $\mathbf{k}=(k,k',k_1)\notin\mathcal{K}_1$ we have

$$
\lim_{\epsilon\to0+}\int_0^t \Psi^{(\epsilon)}(s,\mathbf{k})ds=0, \tag{4.27}
$$

where

$$
\Psi^{(\epsilon)}(s,\mathbf{k}):=\mathbb{E}\Big[(\hat\psi^{(\epsilon)})^*(s,k-k')\hat\psi^{(\epsilon)}(s,k_1-k')\hat\psi^{(\epsilon)}(s,k)(\hat\psi^{(\epsilon)})^*(s,k_1)\Big].
$$

Using (2.8) and Itô formula we conclude that

$$
\begin{aligned}
&\frac{i}{\epsilon}\left[\omega(k-k')+\omega(k_1)-\omega(k_1-k')-\omega(k)\right]\int_0^t \Psi^{(\epsilon)}(s,\mathbf{k})ds\\
&=\Psi^{(\epsilon)}(t,\mathbf{k})-\Psi^{(\epsilon)}(0,\mathbf{k})+\int_0^t \mathcal{P}[\hat\psi^{(\epsilon)}(s),(\hat\psi^{(\epsilon)})^*(s)](\mathbf{k})ds,
\end{aligned}
\tag{4.28}
$$

where $\mathcal{P}$ is a fourth degree polynomial formed over the wave function $\hat\psi^{(\epsilon)}(s)$, $(\hat\psi^{(\epsilon)})^*(s)$. Dividing both sides of (4.28) by the factor in front of the integral on the left hand side and subsequently using Corollary 4.4 with $p=4$ we conclude (4.27). The lemma then follows, provided we can substantiate the following interchange of the limit with integral

$$
\begin{aligned}
&\lim_{\epsilon\to0+}\int_0^t ds\int_{\mathbb{T}^3} r(k,k')r(k_1,k')f^*(k)f(k_1)\Psi^{(\epsilon)}(s,\mathbf{k})d\mathbf{k}\\
&=\int_{\mathbb{T}^3} r(k,k')r(k_1,k')f^*(k)f(k_1)d\mathbf{k}\left\{\lim_{\epsilon\to0+}\int_0^t \Psi^{(\epsilon)}(s,\mathbf{k})ds\right\}.
\end{aligned}
$$

The latter however is a consequence of the Lebesgue dominated convergence theorem and Corollary 4.4. This ends the proof of (4.23). The proof of (4.24) is analogous. □

Going back to the proof of Proposition 4.2 we conclude that (4.10) holds, thanks to (4.20) and the Lebesgue dominated convergence theorem.

On the other hand from the Itô formula for $d|\hat\psi^{(\epsilon)}(t,k)|^2$ we obtain

$$
\langle|\hat\psi^{(\epsilon)}(t)|^2,f\rangle-\langle|\hat\psi(0)|^2,f\rangle=\int_0^t\langle\mathcal{L}|\hat\psi^{(\epsilon)}(s)|^2,f\rangle ds+\frac12\int_0^t\langle\mathcal{L}\hat\psi_2^{(\epsilon)}(s),f\rangle ds+\langle\mathcal{M}_t^{(\epsilon)},f\rangle.
$$

Denote by $\{Q_\epsilon,\ \epsilon\in(0,1]\}$ the family of the laws of $\{|\hat\psi^{(\epsilon)}(t)|^2,\ t\ge0\}$ over $C([0,+\infty),L^2_w(\mathbb{T}))$. Here $L^2_w(\mathbb{T})$ stands for the space $L^2(\mathbb{T})$ equipped with the weak topology.

Using Lemma 4.3 we conclude from the above equality that for any $t_* > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}\left|\langle|\hat{\psi}^{(\epsilon)}(t)|^2, f\rangle - \langle|\hat{\psi}^{(\epsilon)}(s)|^2, f\rangle\right|^4 \leq C(t-s)^2, \quad \forall\, \epsilon \in (0,1],\, t,s \in [0,t_*].$$

This implies tightness of the family of the laws of $\{\langle|\hat{\psi}^{(\epsilon)}(t)|^2, f\rangle,\, t \geq 0\}$, as $\epsilon \to 0+$ over $C[0,+\infty)$. From the above and estimate (4.11) we conclude weak pre-compactness of Q_ϵ, $\epsilon \in (0,1]$, see Theorem 3.1, p. 276 of [8]. Thanks again to Lemma 4.6 and the already proved formula (4.10) we conclude that the limiting law is a δ-type measure supported on $\bar{\mathcal{E}}(t)$ – the solution of (3.4). This, in particular, implies that

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left|\langle|\hat{\psi}^{(\epsilon)}(t)|^2 - \bar{\mathcal{E}}(t), f\rangle\right| = 0$$

in probability. Hence (4.9) follows. □

With the results proved above in hand, we return to the proof of Theorem 3.1. Assume first that $n = 1$ and we consider the process $\tilde{\psi}^{(\epsilon)}(t,k)$ evaluated at a single k. From (2.13) and Corollary 4.4 we conclude easily that for any $t_* > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}|\tilde{\psi}^{(\epsilon)}(t,k) - \tilde{\psi}^{(\epsilon)}(s,k)|^4 \leq C(t-s)^2, \quad \forall\, \epsilon \in (0,1],\, s,t \in [0,t_*].$$

This implies tightness of the laws of $\{\tilde{\psi}^{(\epsilon)}(t,k),\, t \geq 0\}$ over $C[0,+\infty)$.

In the next step we identify the limiting law P_k of $\{\tilde{\psi}^{(\epsilon)}(t,k),\, t \geq 0\}$ over $C[0,+\infty)$. Denote by $\Pi_t(f) := f(t)$, $f \in C[0,+\infty)$ the canonical coordinate map.

Consider the quadratic variation of the complex valued martingale given by (2.15)

$$\langle\tilde{\mathcal{M}}^{(\epsilon)}(k), (\tilde{\mathcal{M}}^{(\epsilon)})^*(k)\rangle_t = \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_{\mathbb{T}} r^2(k,k')\psi^{(\epsilon)}_{\sigma_1}(t,k-k')(\psi^{(\epsilon)}_{\sigma_2})^*(t,k-k')dk', \quad (4.29)$$

and

$$\langle\tilde{\mathcal{M}}^{(\epsilon)}(k), \tilde{\mathcal{M}}^{(\epsilon)}(k)\rangle_t = \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_{\mathbb{T}} r^2(k,k')\psi^{(\epsilon)}_{\sigma_1}(t,k-k')\psi^{(\epsilon)}_{\sigma_2}(t,k-k')dk'. \quad (4.30)$$

Using Proposition 4.2 we conclude that

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left|\langle\tilde{\mathcal{M}}^{(\epsilon)}(k), (\tilde{\mathcal{M}}^{(\epsilon)})^*(k)\rangle_t - \int_0^t \mathcal{R}(s,k)ds\right| = 0$$

and

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} \left|\langle\tilde{\mathcal{M}}^{(\epsilon)}(k), \tilde{\mathcal{M}}^{(\epsilon)}(k)\rangle_t\right| = 0.$$

Then by virtue of Theorem 5.4 of [7] we conclude that $\{\tilde{\mathcal{M}}^{(\epsilon)}_t,\, t \geq 0\}$ converge in law over $C[0,+\infty)$ to a complex valued Gaussian process $\{\tilde{\mathcal{M}}_t,\, t \geq 0\}$ given by

$$\tilde{\mathcal{M}}_t(k) := \int_0^t \mathcal{R}^{1/2}(s,k)w(ds), \quad (4.31)$$

where $\{w(t),\, t \geq 0\}$ is a complex valued standard Brownian motion.

Assume now that $k \neq 0$ and P_k is a limiting law of $\{\tilde{\psi}^{(\epsilon)}(t,k),\, t \geq 0\}$ obtained from a certain sequence $\epsilon_n \to 0+$. Denote by Π_t the coordinate mapping, given by $\Pi_t(g) := g(t)$ for $g \in C[0,+\infty)$.We conclude from (2.13) and (4.19) that

$$\Pi_t + \frac{\hat{\beta}(k)}{4}\int_0^t \Pi_s ds, \quad t \geq 0$$

is a P_k-martingale whose law coincides with that of the process described by (4.31). The conclusion extends also to the case when $k = 0$ and $\omega(0) > 0$. If, on the other hand, $\omega(0) = 0$ we have $\hat{\beta}(0) = 0$ and $\mathcal{R}^{1/2}(s,0) = 0$ and therefore $\Pi_t \equiv \Pi_0$ a.s.

Suppose now that $k_1, \ldots, k_n \in \mathbb{T}$ are pairwise distinct. Denote by Q_ϵ the law of

$$\{(\tilde{\psi}^{(\epsilon)}(t,k_1), \ldots, \tilde{\psi}^{(\epsilon)}(t,k_n)),\, t \geq 0\}$$

over $C([0,+\infty),\mathbb{C}^n)$. Then, following the argument made in the previous part of the proof, we conclude

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]}\left|\langle \tilde{\mathcal{M}}^{(\epsilon)}(k_i), (\tilde{\mathcal{M}}^{(\epsilon)})^*(k_j)\rangle_t - \delta_{i,j}\int_0^t \mathcal{R}(s,k_i)ds\right| = 0$$

and

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]}\left|\langle \tilde{\mathcal{M}}^{(\epsilon)}(k_i), \tilde{\mathcal{M}}^{(\epsilon)}(k_j)\rangle_t\right| = 0, \quad \forall\, i,j = 1,\ldots,n.$$

Combining this observation with (4.19) we obtain from equation (2.13) that any limiting point of the family of laws of Q_{ϵ_n} as $\epsilon_n \to 0+$ is a measure $P_{k_1,\ldots,k_n}$ such that

$$\mathcal{M}_t = (\mathcal{M}_t^{(1)}, \ldots, \mathcal{M}_t^{(n)}) := \Pi_t + \frac{\hat{\beta}(k)}{4}\int_0^t \Pi_s ds, \quad t \geq 0$$

is $\mathbb{C}^n$-valued martingale, whose quadratic covariation is given by

$$\langle \mathcal{M}^{(i)}, (\mathcal{M}^{(j)})^*\rangle_t = \delta_{i,j}\int_0^t \mathcal{R}(s,k_j)ds$$

and

$$\langle \mathcal{M}^{(i)}, (\mathcal{M}^{(j)})\rangle_t = 0, \quad \forall\, i,j = 1,\ldots,n.$$

This of course implies that $P_{k_1,\ldots,k_n} = P_{k_1}\otimes\ldots\otimes P_{k_n}$.

Let $f \in L^2(\mathbb{T})$. We shall prove that

$$\lim_{\epsilon\to 0+}\mathbb{E}|\langle \tilde{\mathcal{M}}_t^{(\epsilon)}, f\rangle|^2 = 0. \tag{4.32}$$

Assuming this result we show how to finish the proof of part (ii). Denote

$$\delta\psi^{(\epsilon)}(t) := \tilde{\psi}^{(\epsilon)}(t) - \bar{\psi}(t).$$

Using Lemma 4.3 and Theorem 3.1, p. 276 of [8] we can conclude weak pre-compactness of P_ϵ, $\epsilon \in (0,1]$ – the family of the laws of $\{\delta\psi^{(\epsilon)}(t),\, t \geq 0\}$ – in $C([0,+\infty), L^2_w(\mathbb{T}))$. With the help of Corollary 4.20 and (4.32) we conclude that the limiting measure, as $\epsilon \to 0+$, is supported on the solution of the equation

$$\langle g(t), f\rangle - \frac{1}{4}\int_0^t \langle \hat{\beta} g(s), f\rangle ds = 0, \quad \forall\, f \in L^2(\mathbb{T}).$$

This of course shows that it is the δ-measure supported on $g(t) \equiv 0$. Hence, in particular we get

$$\lim_{\epsilon\to 0+} \sup_{t\in[0,t_*]} |\langle \delta\psi^{(\epsilon)}(t), f\rangle| = 0 \tag{4.33}$$

in probability and (3.6) follows.

Coming back to the proof of (4.32) note that by the definition of the martingale $\tilde{\mathcal{M}}_t^{(\epsilon)}$, see (2.15), we only need to show that

$$\lim_{\epsilon\to 0+} \mathbb{E}\left|\int_0^t \sum_{y\in\mathbb{Z}} \int_{\mathbb{T}^2} \exp\left\{is\frac{\omega(k)}{\epsilon}\right\} r(k,k') f^*(k)\ \hat{\psi}_\sigma^{(\epsilon)}(s, k-k') e_y(k') w_y(ds)\right|^2 = 0 \tag{4.34}$$

for $\sigma = \pm 1$. We consider only the case $\sigma = 1$, the other one can be dealt in a similar manner. The expression under the limit in (4.34) equals

$$\int\limits_0^t \int\limits_{\mathbb{T}^3} \exp\left[is\frac{\omega(k)-\omega(k_1)}{\epsilon}\right] r(k,k') r(k_1,k') f^*(k) f(k_1) \mathbb{E}\left[\hat{\psi}^{(\epsilon)}(s,k-k')(\hat{\psi}^{(\epsilon)})^*(s,k_1-k')\right] ds d\mathbf{k}, \tag{4.35}$$

with $d\mathbf{k} = dkdk_1dk'$. Observe that

$$\begin{aligned}
&\exp\left\{it\frac{\omega(k)-\omega(k_1)}{\epsilon}\right\} \mathbb{E}\left[\hat{\psi}^{(\epsilon)}(t,k-k')(\hat{\psi}^{(\epsilon)})^*(t,k_1-k')\right] - \mathbb{E}\left[\hat{\psi}(k-k')\hat{\psi}^*(k_1-k')\right] \\
&= \int_0^t \frac{d}{ds}\left\{\exp\left\{is\frac{\omega(k)-\omega(k_1)}{\epsilon}\right\} \mathbb{E}\left[\hat{\psi}^{(\epsilon)}(s,k-k')(\hat{\psi}^{(\epsilon)})^*(s,k_1-k')\right]\right\} ds \\
&= \frac{i}{\epsilon}\left[\omega(k) - \omega(k-k') - \omega(k_1) + \omega(k_1-k')\right] \\
&\times \int_0^t \exp\left\{is\frac{\omega(k)-\omega(k_1)}{\epsilon}\right\} \mathbb{E}\left[\hat{\psi}^{(\epsilon)}(s,k-k')(\hat{\psi}^{(\epsilon)})^*(s,k_1-k')\right] ds \\
&+ \int_0^t \mathcal{P}\left[\hat{\psi}^{(\epsilon)}(s), (\hat{\psi}^{(\epsilon)})^*(s)\right](k,k')ds,
\end{aligned}$$

where $\mathcal{P}$ is a second degree polynomial in $\hat{\psi}^{(\epsilon)}(s)$, $(\hat{\psi}^{(\epsilon)})^*(s)$. Using Corollary 4.4 and an argument identical with the one used in the proof of Lemma 4.6 we conclude that

$$\lim_{\epsilon\to 0+} \int_0^t \exp\left\{is\frac{\omega(k)-\omega(k_1)}{\epsilon}\right\} \mathbb{E}\left[\hat{\psi}^{(\epsilon)}(s,k-k')(\hat{\psi}^{(\epsilon)})^*(s,k_1-k')\right] ds = 0$$

for all $k \neq k_1$. Equality in (4.34) is a consequence of the Lebesgue dominated convergence theorem.

5 Spatially homogeneous initial data

Tightness of the family of laws $\{\tilde{\psi}^{(\epsilon)}(t),\, t \ge 0\}$, in the space of continuous functionals taking values in a space of distributions is again due to the fact that the evolution equation (2.13) contains no terms that are large in magnitude. This is done in Sections 5.1 and 5.2. However, we have no estimates of the $H^{-m}(\mathbb{T})$ norm of $\tilde{\psi}^{(\epsilon)}(t)$ analogous to the ones in Lemma 4.3, that have played an important role in the limit identification argument of Section 4 for square integrable data. Therefore, instead of considering the quadratic variation of the martingale term as we did in the proof of Theorem 3.1, for the proof of Theorem 3.3 we identify the limit of all moments of $\tilde{\psi}^{(\epsilon)}(t)$. Accordingly, we first write equations for time evolution of an arbitrary moment of $\tilde{\psi}^{(\epsilon)}(t)$ in Section 5.3. Using standard averaging argument we show (see Proposition 5.3) the convergence of moments, as $\epsilon \to 0+$, to a solution of the limiting equation obtained simply by discarding the oscillatory terms from the moment equation. Finally in Section 5.5 we prove that the solutions of the limiting equation coincide with the respective moments of the non-homogeneous Ornstein-Uhlenbeck equation (3.13) concluding in this way the proof of Theorem 3.3.

5.1 Properties of spatially homogeneous solutions of (2.8)

Recall that the initial data $\hat{\psi}$ is random and takes values in the Hilbert space of distributions $H^{-m}(\mathbb{T})$ for some $m \in (1/2, 3/2)$. In fact, in Sections 5.1-5.3 we shall not make any use of the assumption that the data is Gaussian and we use only the fact that $\mathbb{E}\|\hat{\psi}\|_{-m}^2 < +\infty$. Consider the random field $\{\psi_y := \langle \hat{\psi}, e_y \rangle,\, y \in \mathbb{Z}\}$. Note that

$$\sum_{y \in \mathbb{Z}} (1 + y^2)^{-m} \mathbb{E}|\psi_y|^2 = \mathbb{E}\|\hat{\psi}\|_{-m}^2 < +\infty,$$

and $\mathbb{E}\psi_0 = 0$. Moreover, $\{\psi_{y+z},\, y \in \mathbb{Z}\}$ and $\{\psi_y,\, y \in \mathbb{Z}\}$ have identical laws for all $z \in \mathbb{Z}$. The latter is equivalent to the fact that $\hat{\psi}(k)$ and $e_z(k)\hat{\psi}(k)$ are identically distributed in $H^{-m}(\mathbb{T})$ for any $z \in \mathbb{Z}$.

Since the covariance function of the field

$$S_{x-y} := \mathbb{E}[\psi_x \psi_y^*], \quad \forall\, x, y \in \mathbb{Z}$$

is positive definite, there exists a finite measure $\hat{E}(dk)$ such that

$$S_x = \int_{\mathbb{T}} e^{ixk} \hat{E}(dk), \quad \forall\, x \in \mathbb{Z}.$$

We assume that $\hat{E}(dk) = \mathcal{E}_0(k)dk$ for some non-negative density $\mathcal{E}_0 \in C(\mathbb{T})$. In addition, we also suppose that the covariance function depends sufficiently fast in space so that

$$\sum_{x \in \mathbb{Z}} |\mathbb{E}[\psi_x \psi_0]| < +\infty. \tag{5.1}$$

When the field ψ_x is a complex valued Gaussian, the quantity in the left side vanishes. Assumption (5.1) implies, in particular, that

$$\mathcal{Y} = \sum_{x \in \mathbb{Z}} e_x \mathbb{E}[\psi_x \psi_0]$$

belongs to $C(\mathbb{T})$.

We note that the translation invariance of the solution persists in time. Indeed, let $\psi_x^{(\epsilon)}(t) := \langle \hat\psi^{(\epsilon)}(t), e_x\rangle$ and $z \in \mathbb{Z}$. A direct computation shows that $e_z\hat\psi^{(\epsilon)}(t)$ is also a solution of (2.13). Since the laws of the initial conditions $e_z\hat\psi$ and that of $\hat\psi$ are identical, we conclude from the uniqueness in law of solutions that the same holds for the processes $\{e_z\hat\psi^{(\epsilon)}(t),\, t \ge 0\}$ and $\{\hat\psi^{(\epsilon)}(t),\, t \ge 0\}$. In consequence, the laws of $\{\psi_x^{(\epsilon)}(t),\, x \in \mathbb{Z}\}$ and that of $\{\psi_{x+z}^{(\epsilon)}(t),\, x \in \mathbb{Z}\}$ are identical for any $z \in \mathbb{Z}$. We can now define the correlation functions

$$S_{t,x}^{(\epsilon)} = \mathbb{E}\left[\psi_x^{(\epsilon)}(t)(\psi_0^{(\epsilon)})^*(t)\right] \quad \text{and} \quad Y_{t,x}^{(\epsilon)} = \mathbb{E}\left[\psi_x^{(\epsilon)}(t)\psi_0^{(\epsilon)}(t)\right]$$

and introduce two distributions on $H^{-m}(\mathbb{T})$

$$\langle f, \hat S_t^{(\epsilon)}\rangle := \sum_{x\in\mathbb{Z}} \check f_x (S_{t,x}^{(\epsilon)})^* \quad \text{and} \quad \langle f, \hat Y_t^{(\epsilon)}\rangle := \sum_{x\in\mathbb{Z}} \check f_x (Y_{t,x}^{(\epsilon)})^*.$$

We recall the following result of [4].

Proposition 5.1 *For any $\epsilon \in (0,1]$ and $t \ge 0$ we have $\hat S_t^{(\epsilon)}, \hat Y_t^{(\epsilon)} \in L^1(\mathbb{T})$. Moreover,*

(1) $\hat S_t^{(\epsilon)}$ is non-negative, and for any $t_* > 0$

$$\sup_{\epsilon\in(0,1]}\sup_{t\in[0,t_*]} (\|\hat S_t^{(\epsilon)}\|_{L^1(\mathbb{T})} + \|\hat Y_t^{(\epsilon)}\|_{L^1(\mathbb{T})}) < +\infty, \tag{5.2}$$

(2) for any $f \in L^\infty(\mathbb{T})$ we have

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]} \left|\langle \hat S_t^{(\epsilon)} - \bar{\mathcal{E}}(t), f\rangle\right| = 0, \tag{5.3}$$

where $\bar{\mathcal{E}}(t)$ is given by (3.4) with the initial condition replaced by $\mathcal{E}_0(k)$

(3) for any f such that $f\omega^{-1} \in L^\infty(\mathbb{T})$ we have

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]} \left|\int_0^t \langle \hat Y_s^{(\epsilon)}, f\rangle ds\right| = 0. \tag{5.4}$$

Proof. Parts 1) and 2) of the lemma are contained in Lemma 12 and Theorem 10 of [4], respectively. Part 3) follows easily from part 1) and the arguments used in the proof of Corollary 4.5. □

5.2 Tightness of solutions of (2.13)

Given $f \in H^m(\mathbb{T})$, we denote by Q_ϵ and $Q_{\epsilon,f}$ the laws of the processes $\{\hat\psi^{(\epsilon)}(t),\, t \ge 0\}$ and $\{\langle f, \hat\psi^{(\epsilon)}(t)\rangle,\, t \ge 0\}$ over $C([0,+\infty), H_w^{-m}(\mathbb{T}))$ and $C([0,+\infty), \mathbb{C})$, respectively, and by $\{\tilde Q_\epsilon,\, \epsilon \in (0,1]\}$ the family of laws of $\{\tilde\psi^{(\epsilon)}(t),\, t \ge 0\}$ over $C([0,+\infty), H_w^{-m}(\mathbb{T}))$. According to [15], see Remark R1, p. 997, to verify the tightness of $\tilde Q_\epsilon$, it suffices to show the following two conditions:

(UC) for any $\sigma, M, t_* > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}\left[\sup_{t\in[0,t_*]} |\langle\tilde{\psi}^{(\epsilon)}(t), f\rangle| \geq M\right] < \sigma, \quad \forall \|f\|_m < \delta, \quad \epsilon \in (0,1],$$

and

(FDT) for any $f \in H^m(\mathbb{T})$ the family of the laws of the processes $\{\langle\tilde{\psi}^{(\epsilon)}(t), f\rangle, t \in [0,t_*]\}$, $\epsilon \in (0,1]$ is tight over $C[0,t_*]$.

As in (3.12) we conclude that for any $f_1, f_2 \in H^m(\mathbb{T})$, where $m > 1/2$, the covariance

$$\mathbb{E}\left[\langle f_1, \hat{\psi}_t^{(\epsilon)}\rangle\langle f_2, \hat{\psi}_t^{(\epsilon)}\rangle^*\right] = \int_{\mathbb{T}} \hat{S}_t^{(\epsilon)}(k) f_1(k) f_2^*(k) dk. \tag{5.5}$$

From (2.13) and Doob's inequality there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,t_*]} |\langle\tilde{\psi}^{(\epsilon)}(t), f\rangle|^2\right] \leq C\left\{\mathbb{E}|\langle\hat{\psi}, f\rangle|^2 + \int_0^{t_*} \mathbb{E}\left|\left\langle \mathcal{A}\left[\frac{t}{\epsilon}, \tilde{\psi}^{(\epsilon)}(t)\right], f\right\rangle\right|^2 dt + \mathbb{E}\left|\left\langle \tilde{\mathcal{M}}_{t_*}^{(\epsilon)}, f\right\rangle\right|^2\right\}. \tag{5.6}$$

Using (5.5), (5.2) and the definitions of $\mathcal{A}[t/\epsilon, \cdot]$, and the martingale $\tilde{\mathcal{M}}_t^{(\epsilon)}$ (see (2.14) and (2.15)) we conclude that the right hand side of (5.6) can be estimated from above by $C\|f\|_\infty^2$, which can be made less than $\sigma > 0$, provided we choose $\delta > 0$ sufficiently small.

To show condition (FDT) consider $\tilde{Q}_{\epsilon,f}^{(M)}$ – the law of the stopped process

$$\{(\langle\tilde{\psi}^{(\epsilon)}(t\wedge\tau_M^{(\epsilon)}), f\rangle, \langle\tilde{\psi}^{(\epsilon)}(t\wedge\tau_M^{(\epsilon)}), f_0\rangle)\, t \in [0,t_*]\}$$

over $C([0,t_*];\mathbb{C}^2)$. Here $f_0(k) := f(-k)$ and

$$\tau_M^{(\epsilon)} := \inf[t \in [0,t_*] : |\langle\tilde{\psi}^{(\epsilon)}(t), f\rangle|^2 + |\langle\tilde{\psi}^{(\epsilon)}(t), f_0\rangle|^2 \geq M^2].$$

We adopt the convention that $\tau_M := t_*$ if the set is empty. Thanks to (UC) we conclude that $\lim_{M\to+\infty} \tau_M^{(\epsilon)} = t_*$, a.s. for each $\epsilon \in (0,1]$. Denote also by $\tilde{Q}_{\epsilon,f}$ the law of the process without the stopping condition.

From (2.13) we conclude that for a fixed M and an arbitrary non-negative function $\phi : \mathbb{C}^2 \to \mathbb{R}$, of class $C_c^1(\mathbb{R}^4)$, one can choose a constant K_ϕ, independent of spatial translations of ϕ, such that

$$\phi(\langle\tilde{\psi}^{(\epsilon)}(t\wedge\tau_M^{(\epsilon)}), f\rangle, \langle\tilde{\psi}^{(\epsilon)}(t\wedge\tau_M^{(\epsilon)}), f_0\rangle) + K_\phi t,\ t \in [0,t_*]$$

is a non-negative submartingale. This proves tightness of $\{\tilde{Q}_{\epsilon,f}^{(M)}, \epsilon \in (0,1]\}$ for a fixed M, by virtue of Theorem 1.4.3 of [17]. Since for any $\sigma > 0$ one can find a sufficiently large $M > 0$ such that B_M – the ball centered at 0 and of radius M in $C([0,t_*];\mathbb{C}^2)$ – satisfies

$$\tilde{Q}_{\epsilon,f}^{(M)}(B_M^c) + \tilde{Q}_{\epsilon,f}(B_M^c) < \sigma$$

and

$$\tilde{Q}_{\epsilon,f}^{(M)}(B_M \cap A) = \tilde{Q}_{\epsilon,f}(B_M \cap A)$$

for all Borel measurable subsets A of $C([0,t_*];\mathbb{C}^2)$, we conclude tightness of $\{\tilde{Q}_{\epsilon,f}, \epsilon \in (0,1]\}$, see step (vi) of the proof of Theorem 3 of [9] for details of this argument.

5.3 Evolution of moments

To describe the evolution of moments we rewrite equation (2.13) in a more compact form. Denote by $\mathbf{C}(t,k) = [C_{ij}(t,\mathbf{k})]$, $i,j = \pm 1$, the 2×2 hermitian matrix

$$\mathbf{C}(t,k) := \begin{bmatrix} C_{1,1} & C_{1,-1} \\ C_{-1,1} & C_{-1,-1} \end{bmatrix},$$

with the entries

$$C_{p,q}(t,k) := \frac{pq\hat{\beta}(k)}{4} \exp\left\{ip\omega(k)(1-pq)t\right\}.$$

Let also $\mathbf{Q}(t,k,k') = [Q_{pq}(t,k,k')]$, $p,q = \pm 1$, be the 2×2 matrix

$$Q_{p,q}(t,k,k') := ipqr(k,k-k')e^{ip[\omega(k)-pq\omega(k')]t}$$

and $W(t,k) := \sum_y e_y(k)w_y(t)$. Let us recall that $\tilde{\psi}^{(\epsilon)}_{-1}(t,k) = \tilde{\psi}^{(\epsilon)*}(t,-k)$. Then, equation for

$$\Psi^{(\epsilon)}(t,k) = \begin{bmatrix} \tilde{\psi}^{(\epsilon)}(t,k) \\ \\ \tilde{\psi}^{(\epsilon)}_{-1}(t,k) \end{bmatrix}$$

is

$$\begin{aligned} & d\Psi^{(\epsilon)}(t,k) = -\mathbf{C}\left(\frac{t}{\epsilon},k\right)\Psi^{(\epsilon)}(t,k)dt + \int_{\mathbb{T}}\mathbf{Q}\left(\frac{t}{\epsilon},k,k-k'\right)\Psi^{(\epsilon)}(t,k-k')W(dt,k')dk', \\ & \Psi^{(\epsilon)}(0,k) = \Psi(k), \end{aligned} \tag{5.7}$$

with the initial data

$$\Psi(k) = \begin{bmatrix} \hat{\psi}(k) \\ \\ \hat{\psi}_{-1}(k) \end{bmatrix}.$$

Let $\{\mathbf{S}_\epsilon(s,t,k),\ s,t\in\mathbb{R}\}$ be the Hermitian matrices solving the deterministic system of ODE's

$$\begin{aligned} & \frac{d\mathbf{S}_\epsilon(s,t,k)}{dt} = -\mathbf{C}\left(\frac{t}{\epsilon},k\right)\mathbf{S}_\epsilon(s,t,k) \\ & \mathbf{S}_\epsilon(s,s,k) = I_2. \end{aligned}$$

Here I_2 is the 2×2 identity matrix. Existence and uniqueness of solutions to (5.7) in the strong sense (thus implying the result in the mild, or weak sense as well) follows from an argument used in Chapter 6 of [6] (because the generators for the evolution family $\mathbf{S}_\epsilon(s,t)$ are bounded), see Proposition 6.4 there. Although the case considered here differs slightly because the coefficients are time dependent, this does not influence the results.

Given a nonnegative integer $p\ge 1$, define a tensor valued distribution on $H^{-m/p}(\mathbb{T}^p)$

$$\hat{M}^{(\epsilon)}(t) := \left[\hat{M}^{(\epsilon)}_{\mathbf{i}}(t)\right], \quad \mathbf{i} = (i_1,\ldots,i_p)\in\{-1,1\}^p,$$

by

$$\hat{M}^{(\epsilon)}_{\mathbf{i}}(t) = \mathbb{E}\left[\tilde{\psi}^{(\epsilon)}_{i_1}(t)\otimes\ldots\otimes\tilde{\psi}^{(\epsilon)}_{i_p}(t)\right].$$

Note that also

$$\hat{M}_{\mathbf{i}}^{(\epsilon)}(0) = \hat{M}_{\mathbf{i}} := \mathbb{E}\left[\hat{\psi}_{i_1} \otimes \ldots \otimes \hat{\psi}_{i_p}\right] \tag{5.8}$$

For a given multi-index $\mathbf{i}$ we define the multi-indices $\mathbf{i}_\ell(j) = (i'_1, \ldots, i'_p)$, $\mathbf{i}_{\ell,m}(j_1, j_2) = (i''_1, \ldots, i''_p)$ given by: $i'_q = i_q$ for $q \neq \ell$ and $i'_\ell = j$, and $i''_q = i_q$ for $q \neq \ell, m$ and $i''_\ell = j_1$, $i''_m = j_2$. Denote by $\mathcal{M}(\mathbb{T}^p)$ the space of all complex valued Borel measures ν on $\mathbb{T}^p$ whose total variation norm $\|\nu\|_{\mathrm{TV}}$ is finite.

Proposition 5.2 *The following are true:*

1) *$\hat{M}^{(\epsilon)}(t)$ is the unique solution in $H^{-m/p}(\mathbb{T}^p)$ of the system of equations*

$$\begin{aligned}\frac{d}{dt}\hat{M}_{\mathbf{i}}^{(\epsilon)}(t, \mathbf{k}) = &-\sum_{\ell=1}^{p}\sum_{j=\pm 1} C_{i_\ell, j}\left(\frac{t}{\epsilon}, k_\ell\right)\hat{M}_{\mathbf{i}_\ell(j)}^{(\epsilon)}(t, \mathbf{k}) \\ &+ \sum_{1 \le \ell < m \le p}\sum_{j_1, j_2 = \pm 1}\int_{\mathbb{T}} \mathcal{R}_{i_\ell, i_m}^{j_1, j_2}\left(\frac{t}{\epsilon}, k_\ell, k_m, k'\right)\hat{M}_{\mathbf{i}_{\ell,m}(j_1,j_2)}^{(\epsilon)}(t, \mathbf{k}'_{\ell,m})dk',\end{aligned} \tag{5.9}$$

with $\mathbf{i} \in \{-1, 1\}^p$ and the initial data given by (5.8). Here

$$\mathcal{R}_{i_\ell, i_m}^{j_1, j_2}\left(\frac{t}{\epsilon}, k_\ell, k_m, k'\right) := Q_{i_\ell, j_1}\left(\frac{t}{\epsilon}, k_\ell, k'_\ell\right) Q_{i_m, j_2}\left(\frac{t}{\epsilon}, k_m, k'_m\right)$$

and $\mathbf{k}'_{\ell,m} = (k'_1, \ldots, k'_p)$, where $k'_p := k_p$ for $p \neq \ell, m$ and $k'_\ell := k_\ell - k'$, $k'_m := k_m + k'$.

2) *If the initial condition is from $\mathcal{M}(\mathbb{T}^p)$ then the solution also belongs to $\mathcal{M}(\mathbb{T}^p)$ and for any $t_* > 0$*

$$M_*(T) := \sum_{\mathbf{i} \in \{-1,1\}^p}\sup_{\epsilon \in (0,1]}\sup_{t \in [0, t_*]}\|\hat{M}_{\mathbf{i}}^{(\epsilon)}(t)\|_{\mathrm{TV}} < +\infty. \tag{5.10}$$

Proof. The fact that $\hat{M}^{(\epsilon)}(t)$ is a solution of (5.9) follows by an application of Itô formula and equation (5.7). Since the operators appearing on the right hand side of the equation in question are uniformly Lipschitz, on any compact time interval, both in $H^{-m/p}(\mathbb{T}^p)$ and $\mathcal{M}(\mathbb{T}^p)$ the proof of uniqueness of solutions in these spaces is standard. Estimate (5.10) follows by an application of Gronwall's inequality. □

5.4 Asymptotics of even moments

Let us now describe the limit moment equations. Assume that $p = 2n$ is even, then for any $1 \le \ell < m \le 2n$ let $D_{\ell,m} := [\mathbf{k} \in \mathbb{T}^{2n} : k_\ell = -k_m]$. We define a bounded linear operator $\mathcal{R}_{\ell,m} : \mathcal{M}(\mathbb{T}^{2n}) \to \mathcal{M}(\mathbb{T}^{2n})$ by

$$\int_{\mathbb{T}^{2n}} f d\mathcal{R}_{\ell,m}\nu := \int_{\mathbb{T}} dk\left\{\int_{D_{\ell,m}} r^2(k, k - k'_\ell) f(S(\mathbf{k}', k))\nu(d\mathbf{k}')\right\}$$

for any bounded, measurable $f : \mathbb{T}^{2n} \to \mathbb{C}$ and $\nu \in \mathcal{M}(\mathbb{T}^{2n})$. Here $\mathbf{k}' = (k'_1, \ldots, k'_{2n})$ and $S : \mathbb{T}^{2n+1} \to \mathbb{T}^{2n}$ is given by $(k_1, \ldots, k_{2n}) = S(\mathbf{k}', k)$ if $k_j = k'_j$ for $j \notin \{\ell, m\}$ and $k_\ell = k$, $k_m = -k$.

Suppose that the components of the tensor $\hat{M} = [\hat{M}_{\mathbf{i}}]$ belong to $\mathcal{M}(\mathbb{T}^{2n})$. Similarly to part 1) of Proposition 5.2 we conclude that the initial value problem

$$\frac{d}{dt}\hat{M}_{\mathbf{i}}(t) = -\frac{1}{4}\left(\sum_{\ell=1}^{2n}\hat{\beta}(k_\ell)\right)\hat{M}_{\mathbf{i}}(t) + \sum_{1\le\ell<m\le 2n}\sum_{j=\pm 1}\mathcal{R}_{\ell,m}\hat{M}_{\mathbf{i}_{\ell,m}(j,-j)}(t),$$
$$\hat{M}(0) = \hat{M}. \tag{5.11}$$

possesses a unique solution in $C([0,+\infty),\mathcal{M}(\mathbb{T}^{2n}))$.

Any partition of the set $\{1,\ldots,2n\}$ into a disjoint set of pairs is called a pairing. Define

$$\mu(d\mathbf{k}) = \sum_{\mathcal{F}}\prod_{(\ell,m)\in\mathcal{F}}\delta(k_\ell + k_m)d\mathbf{k},$$

where $d\mathbf{k} = dk_1\ldots dk_{2n}$ and the summation extends over all possible pairings of $\{1,\ldots,2n\}$. The measure is supported in $\mathbb{H} := \bigcup_{\mathcal{F}}\mathbb{H}(\mathcal{F})$ where

$$\mathbb{H}(\mathcal{F}) := [\mathbf{k} : k_\ell + k_m = 0,\, \forall\,(\ell,m)\in\mathcal{F}].$$

Suppose that the components of the tensor $\rho(\mathbf{k}) = [\rho_{\mathbf{i}}(\mathbf{k})]$, $\mathbf{i}\in\{-1,1\}^{2n}$ belong to $L^1(\mu)$. Consider the initial value problem

$$\frac{d}{dt}\rho_{\mathbf{i}}(t,\mathbf{k}) = -\frac{1}{4}\left(\sum_{\ell=1}^{2n}\hat{\beta}(k_\ell)\right)\rho_{\mathbf{i}}(t,\mathbf{k})$$
$$+\sum_{1\le\ell<m\le 2n}\sum_{j=\pm1}\int_{\mathbb{T}} r^2(k_\ell, k_\ell - k')1_{D_{\ell,m}}(\mathbf{k})\rho_{\mathbf{i}_{\ell,m}(j,-j)}(t,\mathbf{k}'_{\ell,m})dk',$$
$$\rho_{\mathbf{i}}(0,\mathbf{k}) = \rho_{\mathbf{i}}(\mathbf{k}),\ \mathbf{i}\in\{-1,1\}^{2n}, \tag{5.12}$$

with $\mathbf{k}'_{\ell,m} := (k_1,\ldots,k_{\ell-1},k',\ldots,k_{m-1},-k',\ldots,k_{2n})$. It is straightforward to conclude that the above system possesses a unique continuous solution $\rho(t,\mathbf{k}) = [\rho_{\mathbf{i}}(t,\mathbf{k})]$ whose components belong to $L^1(\mu)$. The next proposition gives the convergence of even moments to the solution of (5.11).

Proposition 5.3 *Suppose that all the components of the tensor $[\hat{M}_{\mathbf{i}}(d\mathbf{k})]$ are absolutely continuous with respect to μ, i.e. $\hat{M}_{\mathbf{i}}(d\mathbf{k}) = \rho_{\mathbf{i}}(\mathbf{k})\mu(d\mathbf{k})$, and the dispersion relation satisfies hypothesis ω). Then, the following are true:*

1) *$\hat{M}_{\mathbf{i}}(t,d\mathbf{k})$ is absolutely continuous with respect to $\mu(d\mathbf{k})$ and*

$$\hat{M}_{\mathbf{i}}(t,d\mathbf{k}) = \rho_{\mathbf{i}}(t,\mathbf{k})\mu(d\mathbf{k}),\quad \forall\,\mathbf{i}\in\{-1,1\}^{2n} \tag{5.13}$$

where $\{\rho_{\mathbf{i}}(t),\, t\ge 0\}$ satisfy (5.12).

2) *For any $T > 0$ there exists a constant $C > 0$ such that*

$$\lim_{\epsilon\to 0+}\sum_{\mathbf{i}\in\{-1,1\}^{2n}}\sup_{t\in[0,t_*]}\|\hat{M}_{\mathbf{i}}^{(\epsilon)}(t) - \hat{M}_{\mathbf{i}}(t)\|_{\mathrm{TV}} = 0. \tag{5.14}$$

Proof. The conclusion of part 1) follows from uniqueness of solutions of (5.11) and (5.12), and the fact that the right hand side of (5.13) defines a solution of (5.11). From (5.9) and (5.11) we conclude that

$$
\begin{aligned}
\|\hat{M}_{\mathbf{i}}^{(\epsilon)}(t)-\hat{M}_{\mathbf{i}}(t)\|_{\mathrm{TV}} &\le \sum_{\ell=1}^{2n}\sum_{j=\pm1}\int_0^t \left\|C_{i_\ell,j}\left(\frac{s}{\epsilon}\right)[\hat{M}^{(\epsilon)}_{\mathbf{i}_\ell(j)}(s)-\hat{M}_{\mathbf{i}_\ell(j)}(s)]\right\|_{\mathrm{TV}} ds\\
&+\sum_{1\le\ell<m\le 2n}\sum_{j_1,j_2=\pm1}\int_0^t \left\|\mathcal{R}^{j_1,j_2}_{i_\ell,i_m}\left(\frac{s}{\epsilon}\right)[\hat{M}^{(\epsilon)}_{\mathbf{i}_{\ell,m}(j_1,j_2)}(s)-\hat{M}_{\mathbf{i}_{\ell,m}(j_1,j_2)}(s)]\right\|_{\mathrm{TV}} ds\\
&+\sum_{\ell=1}^{2n}\sum_{j=\pm1}\left|\int_0^t\int_{\mathbb{T}^{2n}} E_{i_\ell,j}\left(\frac{s}{\epsilon},k_\ell\right)\rho_{\mathbf{i}_\ell(j)}(s,\mathbf{k})ds\mu(d\mathbf{k})\right|\\
&+\sum_{1\le\ell<m\le 2n}\sum_{j_1,j_2=\pm1}\left|\int_0^t\int_{\mathbb{T}^{2n+1}}\tilde{\mathcal{R}}^{j_1,j_2}_{i_\ell,i_m}\left(\frac{s}{\epsilon},\mathbf{k},k'\right)\rho_{\mathbf{i}_{\ell,m}(j_1,j_2)}(s,\mathbf{k})ds\mu(d\mathbf{k})dk'\right|.
\end{aligned}
$$

The matrix $\mathbf{E}(t,k)=[E_{p,q}(t,k)]$, $p,q=\pm1$ is given by

$$
\mathbf{E}(t,k):=\mathbf{C}(t,k)-(\hat{\beta}(k)/4)\mathbf{I}_2, \tag{5.15}
$$

where $\mathbf{I}_2$ is the 2×2 identity matrix. In addition,

$$
\tilde{\mathcal{R}}^{j_1,j_2}_{i_\ell,i_m}\left(\frac{s}{\epsilon},\mathbf{k},k'\right):=\mathcal{R}^{j_1,j_2}_{i_\ell,i_m}\left(\frac{s}{\epsilon},k_\ell,k_m,k'\right)-\delta_{i_\ell}^{-i_m}\delta_{j_1}^{-j_2}r^2(k_\ell,k_\ell-k')1_{D_{\ell,m}}(\mathbf{k}).
$$

Denote the terms appearing on the right hand side of (5.15) by $I(t)$, $II(t)$, $III(t)$ and $IV(t)$ respectively. It is easy to see that

$$
I(t)+II(t)\le C\int_0^t \sup_{\mathbf{i}\in\{-1,1\}^{2n}}\left\|\hat{M}_{\mathbf{i}}^{(\epsilon)}(s)-\hat{M}_{\mathbf{i}}(s)\right\|_{\mathrm{TV}} ds \tag{5.16}
$$

for some constant $C>0$. To estimates the term III we need to bound terms of the form

$$
\left|\int_0^t\int_{\mathbb{T}^{2n}}\hat{\beta}(k_\ell)\exp\left\{2i\omega(k_\ell)\frac{s}{\epsilon}\right\}\rho_{\mathbf{i}}(s,\mathbf{k})ds\mu(d\mathbf{k})\right|
$$

for some ℓ and $\mathbf{i}$. Integrating by parts we obtain that the expression above can be bounded from above by

$$
\begin{aligned}
&\epsilon\left|\int_{\mathbb{T}^{2n}}\frac{\hat{\beta}(k_\ell)}{2i\omega(k_\ell)}\left[\exp\left\{2i\omega(k_\ell)\frac{t}{\epsilon}\right\}-1\right]\rho_{\mathbf{i}}(t,\mathbf{k})1_{D_{\ell,m}}(\mathbf{k})\mu(d\mathbf{k})\right|\\
&+\epsilon\left|\int_0^t\int_{\mathbb{T}^{2n}}\frac{\hat{\beta}(k_\ell)}{2i\omega(k_\ell)}\left[\exp\left\{2i\omega(k_\ell)\frac{t}{\epsilon}\right\}-1\right]\frac{d}{ds}\rho_{\mathbf{i}}(s,\mathbf{k})1_{D_{\ell,m}}(\mathbf{k})ds\mu(d\mathbf{k})\right|.
\end{aligned}
$$

The first term can be easily estimated by $C\epsilon$, due to the fact that $\sup_{k\in\mathbb{T}}\hat{\beta}(k)\omega^{-1}(k)<+\infty$. To estimate the second term, we use equation (5.11). As a result,, we conclude that for any $t_*>0$ we can find a constant $C(t_*)>0$ such that

$$
\sup_{t\in[0,t_*]} III(t)\le C(t_*)\epsilon. \tag{5.17}
$$

Finally we show that

$$\lim_{\epsilon\to0+}\sup_{t\in[0,t_*]} \mathit{IV}(t)=0. \tag{5.18}$$

It implies the conclusion of part 2) of the proposition, via an application of the Gronwall's inequality.

We write $\mathit{IV}(t)=\mathit{IV}_1(t)+\mathit{IV}_2(t)$, where the terms $\mathit{IV}_i(t)$, $i=1,2$ correspond to the integration over $D_{\ell,m}$ and its complement. In the latter case, we have to deal with terms of the form

$$\Big|\int_0^t\int_{\mathbb{T}^{2n+1}} 1_{[k_\ell\neq -k_m]} r(k_\ell,k')r(k_m,-k')\rho_{\mathbf{i}}(s,\mathbf{k})$$
$$\times\prod_{j=1}^2 \exp\left\{i\sigma_1^{(j)}[\omega(k_\ell^{(j)})+\sigma_2^{(j)}\omega(k_\ell^{(j)}+(-1)^j k')]\frac{s}{\epsilon}\right\} ds\mu(d\mathbf{k})dk'\Big|$$

for some $\mathbf{i}\in\{-1,1\}^{2n}$, $\sigma_p^{(j)}\in\{-1,1\}$. Here $k_\ell^{(1)}=k_\ell$ and $k_\ell^{(2)}=k_m$. Using integration by parts over the s variable we can estimate the supremum of the above expression over $t\in[0,t_*]$ by the sum of

$$\begin{aligned} I_\epsilon := &\int_{\mathbb{T}^{2n+1}} \mu(d\mathbf{k})dk' 1_{[k_\ell\neq -k_m]}|r(k_\ell,k')r(k_m,-k')|\sup_{t\in[0,t_*]}|\rho_{\mathbf{i}}(t,\mathbf{k})| \\ &\times\epsilon\left|\sum_{j=1}^2 \sigma_1^{(j)}[\omega(k_\ell^{(j)})+\sigma_2^{(j)}\omega(k_\ell^{(j)}+(-1)^j k')]\right|^{-1} \\ &\times\sup_{t\in[0,t_*]}\prod_{j=1}^2\left|\exp\left\{i\sigma_1^{(j)}[\omega(k_\ell^{(j)})+\ si_2^{(j)}\omega(k_\ell^{(j)}+(-1)^j k')]\frac{t}{\epsilon}\right\}-1\right|, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} J_\epsilon := &\int_0^T ds\left|\int_{\mathbb{T}^{2n+1}} \mu(d\mathbf{k})dk' 1_{[k_\ell\neq -k_m]} r(k_\ell,k')r(k_m,-k')\frac{d}{ds}\rho_{\mathbf{i}}(s,\mathbf{k})\right. \\ &\times\epsilon\left\{\sum_{j=1}^2 \sigma_1^{(j)}[\omega(k_\ell^{(j)})+\sigma_2^{(j)}\omega(k_\ell^{(j)}+(-1)^j k')]\right\}^{-1} \\ &\left.\times\prod_{j=1}^2\left\{\exp\left\{i\sigma_1^{(j)}[\omega(k_\ell^{(j)})+\sigma_2^{(j)}\omega(k_\ell^{(j)}+(-1)^j k')]\frac{s}{\epsilon}\right\}-1\right\}\right|. \end{aligned} \tag{5.20}$$

Using (5.12) and Gronwall's inequality, we conclude that

$$\int_{\mathbb{T}^{2n}}\sup_{t\in[0,t_*]}|\rho_{\mathbf{i}}(t,\mathbf{k})|d\mathbf{k}<+\infty.$$

Using condition ω) we conclude therefore, by virtue of Lebesgue dominated convergence theorem, that $\lim_{\epsilon\to0+} I_\epsilon=0$. Likewise, we conclude that $\lim_{\epsilon\to0+} J_\epsilon=0$. Part 2) of the proposition follows then from another application of Gronwall's inequality after substituting for $\rho'_{\mathbf{i}}(s,\mathbf{k})$ from (5.12). Summarizing, we have shown so far that

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]} IV_2(t)=0.$$

We are left therefore with estimates of the term

$$IV_1(t):=\sum_{1\le\ell<m\le 2n}\sum_{j_1,j_2=\pm 1}\left|\int_0^t\int_{\mathbb{T}^{2n+1}}1_{D_{\ell,m}}(\mathbf{k})\right. \tag{5.21}$$
$$\left.\times\tilde{\mathcal{R}}^{j_1,j_2}_{i_\ell,i_m}\left(\frac{s}{\epsilon},\mathbf{k},k'\right)\rho_{\mathbf{i}_{\ell,m}(j_1,j_2)}(s,\mathbf{k})ds\mu(d\mathbf{k})dk'\right|.$$

The non-vanishing terms appearing in the above sum are of the form

$$\left|\int_0^t\int_{\mathbb{T}^{2n+1}}r^2(k_\ell,k_\ell-k')1_{D_{\ell,m}}(\mathbf{k})\prod_{j=1}^{2}\exp\left\{i\sigma_1^{(j)}[\omega(k_\ell)+\sigma_2^{(j)}\omega(k_\ell-k')]\frac{s}{\epsilon}\right\}ds\mu(d\mathbf{k})dk'\right|,$$

with $(\sigma_1^{(1)},\sigma_2^{(1)})\neq-(\sigma_1^{(2)},\sigma_2^{(2)})$ and $\sigma_p^{(j)}\in\{-1,1\}$. To these terms we can apply the integration by parts argument as before, to conclude that

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,t_*]} IV_1(t)=0.$$

Summarizing, we have shown that (5.18) holds, and the proof of part 2 of the proposition is therefore complete. □

5.5 Proof of Theorem 3.3

In this section, and in this section only, we make use of the assumption that $\hat\psi$ is Gaussian. We show that the limiting measure for $\tilde Q_\epsilon$, as $\epsilon\to 0+$, coincides with the law $\tilde Q$ of the process given (3.13) by proving that for any $N\ge 1$, $0\le t_1<\ldots<t_N$, any non-negative integers ℓ_j,m_j, test functions $f_j,g_j\in H^m(\mathbb{T})$, $j=1,\ldots,N$ we have

$$\lim_{\epsilon\to 0+}\mathbb{E}\left[\prod_{j=1}^{N}[\langle\tilde\psi^{(\epsilon)}(t_j),f_j\rangle^{\ell_j}(\langle\tilde\psi^{(\epsilon)}(t_j),g_j\rangle^*)^{m_j}]\right]=\mathbb{E}\left[\prod_{j=1}^{N}[\langle\bar\psi(t_j),f_j\rangle^{\ell_j}(\langle\bar\psi(t_j),g_j\rangle^*)^{m_j}]\right]. \tag{5.22}$$

To simplify the notation, we prove (5.22) only in the case $N=1$. The general case can be handled in the same manner, using Markov property of the process $\{\tilde\psi^{(\epsilon)}(t),\,t\ge 0\}$, at the expense of some additional complications in the notation. We recall (see Section 3.2) that the initial data $\{\hat\psi(k),\,k\in\mathbb{T}\}$ is a δ-correlated Gaussian random field given by (3.11). Therefore, for the odd moments we have

$$\hat M^{(\epsilon)}_{\mathbf{i}}(0)=0,\quad\forall\,\mathbf{i}\in\{-1,1\}^{2n-1},$$

where $n\ge 1$ is an integer. By uniqueness of solutions of (5.9) we conclude that in this case $\hat M^{(\epsilon)}(t)\equiv 0$ for all $t\ge 0$. When $\mathbf{i}\in\{-1,1\}^{2n}$ we can use the conclusion (5.14) of Proposition 5.3. Define

$$\bar M^{(2n)}(t):=\left[\bar M^{(2n)}_{\mathbf{i}}(t)\right],\quad \mathbf{i}=(i_1,\ldots,i_{2n})\in\{-1,1\}^{2n},$$

where

$$\bar{M}_{\mathbf{i}}^{(2n)}(t) = \mathbb{E}\left[\bar{\psi}_{i_1}(t)\otimes\ldots\otimes\bar{\psi}_{i_{2n}}(t)\right]$$

and $\bar{\psi}_1(t) = \bar{\psi}(t)$ is the solution of (3.13) and $\bar{\psi}_{-1}(t,k) = \bar{\psi}^*(t,-k)$. The conclusion of Theorem 3.3 will follow provided that we show that $\bar{M}^{(2n)}(t)$, satisfies (5.11). Note that for $n=1$ we obtain that

$$\bar{M}_{i_1,i_2}^{(2)}(t,d\mathbf{k}) = \delta_{i_1,-i_2}\bar{\mathcal{E}}(t,k_1)\delta(k_1+k_2)dk_1dk_2.$$

From (3.13) and Itô formula we conclude that

$$\begin{aligned}&\frac{d}{dt}\bar{M}_{\mathbf{i}}^{(2n)}(t) = -\frac{1}{4}\left(\sum_{\ell=1}^{2n}\hat{\beta}\left(k_\ell\right)\right)\bar{M}_{\mathbf{i}}^{(2n)}(t) - \sum_{1\le\ell<m\le 2n}\mathcal{R}(t,k_\ell)\bar{M}_{\mathbf{i}_{\ell,m}}^{(2n-2)}(t)\otimes_{\ell,m}\Delta,\\&\bar{M}(0) = \hat{M}.\end{aligned} \tag{5.23}$$

Here $\bar{M}_{\mathbf{i}_{\ell,m}}^{(2n-2)}(t)$ is the $2n-2$-nd order moment obtained from $\bar{M}_{\mathbf{i}}^{(2n)}(t)$ by omitting $\bar{\psi}_{i_\ell}(t)$ and $\bar{\psi}_{i_m}(t)$ and for any measure ν on $\mathbb{T}^{2n-2}$, $1\le\ell<m\le 2n$ we denote by $\nu\otimes_{\ell,m}\Delta$ a measure on $\mathbb{T}^{2n}$ given by

$$\int_{\mathbb{T}^{2n}} fd(\nu\otimes_{\ell,m}\Delta) = \int_{\mathbb{T}^{2n-2}} d\mathbf{k}\int_{\mathbb{T}} dk' f(k_1,\ldots,k_{\ell-1},k,\ldots,k_{m-1},-k',\ldots,k_{2n-2})$$

for all $f\in C(\mathbb{T}^{2n})$. Since

$$\begin{aligned}\mathcal{R}(t,k_\ell) &= \int_{\mathbb{T}} R(k_\ell,k')\bar{\mathcal{E}}(t,k')dk' = \int_{\mathbb{T}}[r^2(k_\ell,k_\ell-k')+r^2(k_\ell,k_\ell+k')]\bar{\mathcal{E}}(t,k')dk'\\&= \sum_{j=\pm 1}\int_{\mathbb{T}^2} r^2(k_\ell,k_\ell-k')\mathbb{E}\left[\bar{\psi}_j(t,k')\otimes\bar{\psi}_{-j}(t,k'')\right]dk'dk''\end{aligned}$$

and $(\bar{\psi}_{i_1}(t),\ldots,\bar{\psi}_{i_{2n}}(t))$ is jointly Gaussian, we infer that the last term on the right hand side of the first equation in (5.23) can be rewritten as being equal to the last term on the right hand side of the first equation of (5.11). Thus the conclusion of Theorem 3.3 has been shown.

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