Existence and Non-existence of Traveling Fronts in Disordered Media

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Transition fronts for reaction-diffusion equations

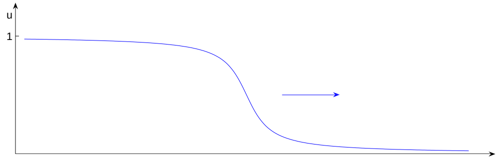
We study transition fronts for the reaction-diffusion PDE

$$u_t = \Delta u + f(x,u)$$

on $\mathbb{R} \times \mathbb{R}$ with $f(x,0) = f(x,1) = 0$.

Transition front (generalized traveling front) is a solution $u(t,x) \in [0,1]$ global in time and satisfying for each $t \in \mathbb{R}$,

$$\lim_{x \to -\infty} u(t,x) = 1 \qquad \text{and} \qquad \lim_{x \to \infty} u(t,x) = 0.$$



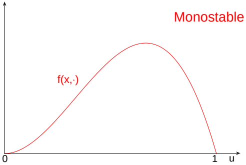
- Defined by Berestycki-Hamel. Also Matano, Shen
- This front moves to the right. Also a front moving left.
- Fronts model invasions (combustion, ecology, genetics)

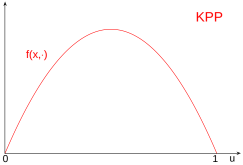
Reaction functions in $u_t = \Delta u + f(x,u)$

Reaction function $f : \mathbb{R} \times [0,1] \to [0,\infty)$ is non-negative Lipschitz with $f(x,0) = f(x,1) = 0$ and ignition temperature

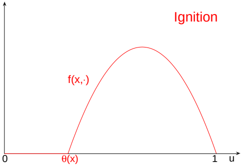
$$\theta(x) = \inf \left\{ u \,\middle|\, f(x,u) > 0 \right\}$$

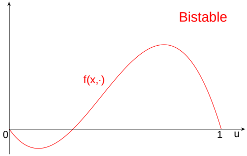
- Monostable: $\inf_x \theta(x) = 0$ (KPP: $f(x,u) \le \frac{\partial f}{\partial u}(x,0)u$)





- Ignition: $\inf_x \theta(x) > 0$

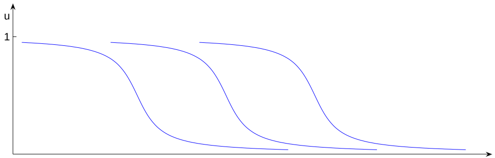




Homogeneous media: Traveling fronts

$$u_t = \Delta u + f(u)$$

A traveling front is a solution $u(t,x) = U(x - ct)$ such that $U(-\infty) = 1$ and $U(\infty) = 0$ (constant profile U and speed c).



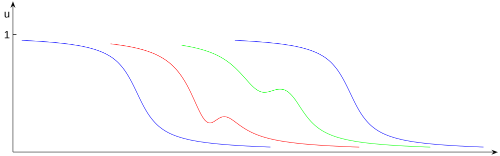
- (U, c) solve $U'' + cU' + f(U) = 0$ (gives $c > 0$)
- Ignition reactions: unique front speed $c_f^* > 0$
- Monostable reactions: minimal front speed $c_f^* > 0$ and all $c \in [c_f^*, \infty)$ are achieved (but c_f^* most physical)
- KPP: $c_f^* = 2\sqrt{f'(0)}$ (Kolmogorov-Petrovskii-Piskunov)

General solutions of the PDE propagate with speed c_f^*.

Periodic media: Pulsating fronts

$$u_t = \Delta u + f(x, u)$$

Assume that f is 1-periodic in x. A pulsating front with speed $c > 0$ is a solution of the form $u(t, x) = U(x - ct, x \bmod 1)$ such that uniformly in the second argument, $U(-\infty, x \bmod 1) = 1$ and $U(\infty, x \bmod 1) = 0$.



- Time-periodic in a moving frame: $u(t + \frac{1}{c}, x + 1) = u(t, x)$
- (U, c) solve a degenerate elliptic equation
- Under mild conditions on f there is again unique/minimal front speed $c_f^* > 0$ for ignition/monostable reactions (Xin, Berestycki-Hamel)

Fronts in general inhomogeneous media

In general inhomogeneous media no special forms exist.

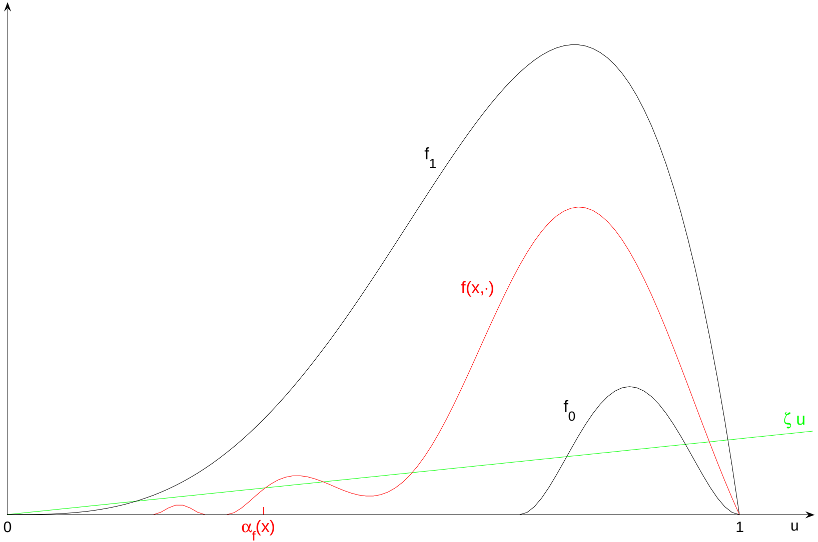
Assume:

- $f(x,u)$ is Lipschitz and $f_0(u) \le f(x,u) \le f_1(u)$ for some reactions $f_0(u) \le f_1(u)$ such that f_0 is ignition and f_1 is ignition or monostable.
- $f_1'(0) < (c^*_{f_0})^2/4$ (true if f_1 is ignition)
  - This is equivalent to $2\sqrt{f_1'(0)} < c^*_{f_0}$ (front is "pushed")
- For some $\zeta < (c^*_{f_0})^2/4$ the function $f(x,\cdot)$ is bounded away from zero (uniformly in x) on the interval $[\alpha_f(x), 1-\varepsilon]$, with

$$\alpha_f(x) = \inf\{u \in (0,1) \,|\, f(x,u) > \zeta u\}$$

  - I.e., f cannot vanish after becoming large (except at $u = 1$)
- These conditions are "qualitatively necessary" for existence of fronts

Fronts in general inhomogeneous media



Fronts in general inhomogeneous media

Theorem (Z.)

Assume the above hypotheses.
(i) There exists a transition front u_+ for

$$u_t = \Delta u + f(x,u)$$

moving to the right, with $(u_+)_t > 0$ (and u_- moving to the left).
(ii) If f_1 is ignition, then $u_\pm$ are unique up to time shifts and general solutions with exponentially decaying initial data converge in L^∞_x to time shifts of $u_\pm$ (global attractors).

- Proved by Nolen-Ryzhik-Mellet-Roquejoffre-Sire in the case $f(x,u) = a(x)g(u)$ with $a(x) \in [a_0, a_1] \subset (0,\infty)$ and g ignition reaction (constant positive ignition temperature).
- Extends to cylindrical domains $D \subset \mathbb{R}^n$ (and includes periodic case of Berestycki-Hamel, Xin)
- Bistable reaction case studied by Shen, Vakulenko-Volpert

Non-existence of fronts for $u_t = \Delta u + f(x,u)$

If f_1 is KPP, then $c^*_{f_0} < c^*_{f_1} = 2\sqrt{f_1'(0)}$, so $f_1'(0) < (c^*_{f_0})^2/4$ fails.

Let f be a KPP reaction and assume

- $a(x) = \frac{\partial f}{\partial u}(x,0) > 0$ (e.g., $f(x,u) = a(x)u(1-u)$)
- $\lambda = \sup \sigma(\Delta + a(x))$
- $\psi =$ principal eigenfunction of $\Delta + a(x)$ (if λ is eigenvalue)

Theorem (Nolen-Roquejoffre-Ryzhik-Z.)

Assume that $a(x) \geq 1$ (so $\lambda \geq 1$) and $\lim_{x\to\pm\infty} a(x) = 1$.
(i) If $\lambda > 2$, then there is a unique entire solution (up to a time shift) strictly between 0 and 1. It satisfies $u(t,x) = e^{\lambda t}\psi(x)$ for $t \ll -1$ (the bump). In particular, no transition front exists.
(ii) If $\lambda < 2$, then there exists a (right-moving) transition front for each speed $c \in (2, \frac{\lambda}{\sqrt{\lambda-1}})$. If $\lambda \in (1,2)$, the bump also exists.

- First general result of non-existence of fronts (based on an unpublished ignition-KPP example by Roquejoffre-Z.)

Proof of (i): non-existence of front for $u_t = \Delta u + f(x,u)$

Lemma

For each $\kappa \in (2, \frac{\lambda}{\sqrt{\lambda-1}})$ *there is* C_κ *such that for* $(t,x) \in \mathbb{R}^- \times \mathbb{R}$,

$$u(t,x) \le C_\kappa e^{|x|-\kappa|t|} u(0,0)$$

Suffices to show $u(t,x) \lesssim e^{\sqrt{\lambda-1}(|x|-\kappa|t|)} u(0,0)$ for $|x| \le \kappa|t|$. Assume the contrary (by Harnack also for any y near x) and consider $x < 0$. Let $\beta = \frac{|x|}{2\sqrt{\lambda-1}|t|} \le \frac{\kappa}{2\sqrt{\lambda-1}} < 1$. Then

$$u(t+\beta|t|, 0) \gtrsim e^{\beta|t|} e^{-\frac{|x|^2}{4\beta|t|}} e^{\sqrt{\lambda-1}(|x|-\kappa|t|)} u(0,0) = e^{(\lambda\beta - \sqrt{\lambda-1}\kappa)|t|} u(0,0)$$

if $u_t = \Delta u + u$. Still holds, with $e^{(1-\varepsilon)\beta|t|}$, because $2\beta|t| < |x|$. Same estimate for any y near 0, so if $\psi(0) = \|\psi\|_\infty \le 1$, then

$$u(0,0) \ge e^{\lambda(1-\beta)|t|} e^{(\lambda\beta - \sqrt{\lambda-1}\kappa - \varepsilon\beta)|t|} u(0,0) = e^{(\lambda - \sqrt{\lambda-1}\kappa - \varepsilon\beta)|t|} u(0,0)$$

This is a contradiction if $\varepsilon > 0$ is small.

Proof of (i): non-existence of front for $u_t = \Delta u + f(x,u)$

So for $(t,x) \in \mathbb{R}^- \times \mathbb{R}^-$ we have

$$u(t,x) \leq C_\kappa e^{-x+\kappa t} u(0,0)$$

Assume $a(x) - 1$ is supported on $\mathbb{R}^+$, pick any $\tau < 0$, and let

$$v^{(\tau)}(t,x) = C_\kappa e^{-x+(\kappa-2)\tau+2t} u(0,0) + C_\kappa e^{x+2t} u(0,0).$$

Then $v^{(\tau)}$ solves

$$v_t^{(\tau)} = \Delta v^{(\tau)} + v^{(\tau)} \geq \Delta v^{(\tau)} + f(x, v^{(\tau)})$$

on $\mathbb{R} \times \mathbb{R}^-$, with $v^{(\tau)}(\tau,x) \geq u(\tau,x)$ for $x < 0$ and $v^{(\tau)}(t,0) \geq u(t,0)$ for $t \in [\tau,0]$. So for $(t,x) \in \mathbb{R}^- \times \mathbb{R}^-$,

$$u(t,x) \leq \lim_{\tau \to -\infty} v^{(\tau)}(t,x) = C_\kappa e^{-|x|+2t} u(0,0)$$

Same for $x \geq 0$, so u is a bump.

Proof of (ii): existence of fronts for $u_t = \Delta u + f(x,u)$

Assume a compactly supported and $f(x,u) = a(x)u$ for $u \le \theta$.

For $\gamma \in (\lambda, 2)$ let ϕ_γ be the generalized eigenfunction of $\Delta + a(x)$ with eigenvalue γ and $\phi_\gamma(x) = e^{-\sqrt{\gamma-1}x}$ for $x \gg 1$.
Then $\phi_\gamma > 0$ and $\phi_\gamma(x) \approx \alpha_\gamma e^{-\sqrt{\gamma-1}x}$ for $x \ll -1$ (with $\alpha_\gamma > 0$).

$$v(t,x) = e^{\gamma t}\phi_\gamma(x)$$

solves $v_t = \Delta v + a(x)v$ so v is a supersolution of the original PDE, "moving" with speed $c = \gamma/\sqrt{\gamma - 1}$ for $|x| \gg 1$.

Let $\varepsilon > 0$ be small and $\varepsilon' = \left(\sqrt{1 + \frac{\varepsilon}{\gamma-1}} - 1\right)\gamma$, so that $\varepsilon' > \varepsilon$ by $\frac{\gamma}{2(\gamma-1)} > 1$. Then

$$w(t,x) = e^{\gamma t}\phi_\gamma(x) - Ae^{(\gamma+\varepsilon')t}\phi_{\gamma+\varepsilon}(x)$$

"moves" with speed c, has a "constant" in t maximum, and is a subsolution where $w \ge 0$ if $A \gg 1$ (so that $\sup w \le \theta$).