Stochastic nonlinear Schrödinger equations and modulation of solitary waves

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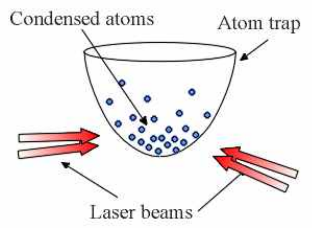
Deterministic and stochastic front propagation
Banff, mars 22-26, 2010

Optically confined Bose-Einstein condensates

Stamper-Kurn et al., Phys. Rev. Lett, 1998

Advantages

- Obtain different geometrical configurations
- study magnetic properties of atoms (trapping not limited to specific magnetic states)



Drawbacks

ex : fluctuations of the laser intensity

$\rightsquigarrow$ introduce stochasticity in the dynamical behavior of the condensate, which has to be taken into account in real situations

Dynamics of BEC under regular variations of trap parameters : widely studied

Castin and Dum, Phys. Rev. Lett. 1996

Kagan, et. al ; Phys. Rev. A, 1996

Ripoll, Perez-Garcia, Phys. Rev. A, 1999, etc...

Mean field theory : fluctuations of laser field intensity regarded as modulations of the harmonic trap $\rightsquigarrow$ NLS equation (Gross-Pitaevskii) with harmonic potential and noise

Randomly varying optical trap potential

Abdullaev, Baizakov, Konotop, in Nonlinearity and disorder, 2001

2D radially symmetric Gross-Pitaevskii equation :

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + (1+e(t))r^2\psi + \chi|\psi|^2\psi - i\gamma\psi$$

in dimensionless variables, with

$\chi = \pm 1$ (related to the sign of the s-wave scattering length a of atoms)

γ damping coefficient (thermal cloud)

$e(t) = \frac{E(t)-E_0}{E_0}$ with $E(t) =$ laser field intensity, mean value E_0.

Assume $e(t)$ is δ-correlated with zero mean : $\langle e(t)\rangle = 0$, $\langle e(t)e(t')\rangle = \sigma^2\delta(t-t')$

Mathematical description of the equation

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $W(t)$ real valued standard Brownian motion ; $e(t) = \varepsilon \dot{W}(t)$
- ψ macroscopic wave function (complex valued)

Use of a Stratonovich product :

$$id\psi + (\Delta\psi - |x|^2\psi)dt - \chi|\psi|^2\psi dt + i\gamma\psi dt = \varepsilon|x|^2\psi \circ dW$$

- Conservation of the squared L^2 norm (total number of atoms) in the absence of damping (W is real valued)
- limit case of processes with non vanishing correlation length

Equivalent equation in Itô form :

$$id\psi+(\Delta\psi-|x|^2\psi)dt-\chi|\psi|^2\psi dt+i\gamma\psi dt = \varepsilon|x|^2\psi dW - i\frac{\varepsilon^2}{2}|x|^4\psi dt$$

More generally :

Consider the equation

$$id\psi+(\Delta\psi-|x|^2\psi)dt-\chi|\psi|^{2\alpha}\psi dt+i\gamma\psi dt+i\frac{\varepsilon^2}{2}|x|^4\psi dt=\varepsilon|x|^2\psi dW$$

$\alpha > 0$, $\gamma \geq 0$, $x \in \mathbf{R}^d$, $d = 1$ or 2.

We consider solutions with paths having finite "energy" almost surely (Σ : energy space)

$$H(\psi)=\frac{1}{2}\int(|\nabla\psi|^2+|x|^2|\psi|^2)dx+\frac{\chi}{2\alpha+2}\int|\psi|^{2\alpha+2}dx$$

H : hamiltonian for the corresponding deterministic equation (without damping) : combination of the energy of the wave packet, and mean square width of the atomic cloud

Existence result (up to now...)

Theorem : dB, Fukuizumi, 2007

Assume $\alpha > 0$, $\gamma \geq 0$ and $\chi = \pm 1$. Assume $\psi_0 \in \Sigma$ if $d = 1$, or $\psi_0 \in \Sigma^2$ and $1/2 \leq \alpha \leq 1$ if $d = 2$. Then there exist a stopping time $\tau^*(\psi_0)$ and a unique solution $\psi^\varepsilon(t)$ of

$$id\psi + (\Delta\psi - |x|^2\psi)dt - \chi|\psi|^{2\alpha}\psi dt + i\gamma\psi dt + i\frac{\varepsilon^2}{2}|x|^4\psi dt = \varepsilon|x|^2\psi dW$$

with $\psi^\varepsilon(0) = \psi_0$, such that $\psi^\varepsilon \in C([0,\tau];\Sigma)$ for any $\tau < \tau^*(\psi_0)$, and ψ^ε is adapted w.r. to the filtration generated by W.
Moreover, we have almost surely,

$$\tau^*(\psi_0,\omega) = +\infty \quad \text{or} \quad \limsup_{t \nearrow \tau^*(\psi_0,\omega)} |\psi^\varepsilon(t)|_\Sigma = +\infty.$$

where :

$\psi_0 \in \Sigma^2$ if $\psi_0 \in L^2$, $\Delta\psi_0 \in L^2$ and $|x|^2\psi_0 \in L^2$

Moreover :

$\chi = +1$ or $\chi = -1$ and $\alpha < 2/d$

or $\chi = -1$, $\alpha = 2/d$ and $|\psi_0|_{L^2}^{4/d} < 1/C_\alpha$

$\rightsquigarrow$ the solution in Σ exists for all t i.e. $\tau^*(\psi_0) = +\infty$ a.s.

- Pathwise conservation of L^2 norm
- Energy equality (Itô formula) for all $\tau < \tau^*(\psi_0)$ a.s.

$$H(\psi(\tau)) = H(\psi_0) - 2\varepsilon \mathrm{Im} \int_0^\tau \int x.\nabla\psi\bar{\psi} dx dW(s) + 2\varepsilon^2 \int_0^\tau |x\psi(s)|_{L^2}^2 ds$$

From now on :

$\chi = -1$ (attractive condensate), $\gamma = 0$ (no damping)

Standing wave of the deterministic equation

Two parameter family of solutions ($\varepsilon = 0$)

$$\psi_{\mu,\theta}(t,x) = e^{i(\mu t+\theta)}\phi_\mu(x), \quad \theta, \mu \in \mathbf{R}$$

ϕ_μ localized profile, positive, radially symmetric (ground state), critical point of

$$S_\mu(u) = H(u) + \frac{\mu}{2}|u|_{L^2}^2$$

ϕ_μ exists and is unique provided

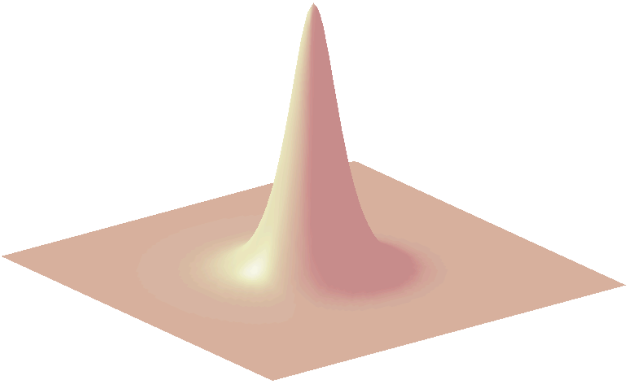
$$\mu \geq -\inf\left\{|\nabla u|_{L^2}^2 + |xu|_{L^2}^2, u \in \Sigma, |u|_{L^2} = 1\right\} = -d$$

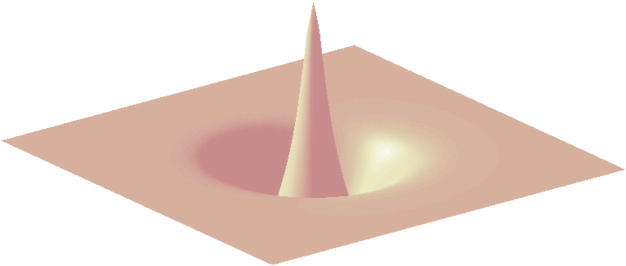
Moreover as $\mu \to -d$,

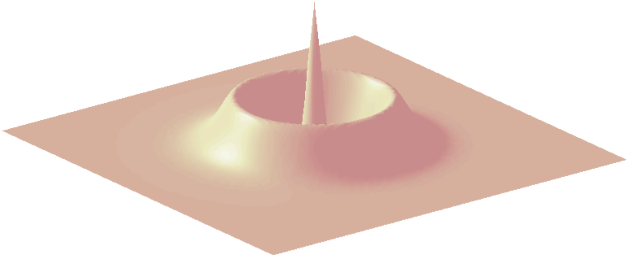
$$\phi_\mu \sim (\alpha+1)^{d/4\alpha}\pi^{d/4}(\mu+d)^{1/2\alpha}\Phi_0$$

where Φ_0 ground state of $-\Delta + x^2$ (first Hermite function in 1-D)

Di Menza, Hadj Selem, 2009







Dynamics of the deterministic equation

The family $\{e^{i\theta}\phi_\mu, \theta \in \mathbf{R}\}$ is stable for μ close to $-d$ (Fukuizumi, Ohta, D.I.E. 2003)

Consider the functional S_μ as a Lyapunov functional ; then $S''_\mu(\phi_\mu) = \mathcal{L}_\mu$ satisfies

$$\langle \mathcal{L}_\mu \mathbf{v}, \mathbf{v} \rangle \geq \nu \|\mathbf{v}\|^2_\Sigma$$

for any $\mathbf{v} = (v_1, v_2)^t$, $v_i \in \Sigma$, with $(\mathbf{v}, \phi_\mu) = (\mathbf{v}, i\phi_\mu) = 0$.

Let ψ be a solution of the equation (with $\varepsilon = \gamma = 0$) and write

$$\psi(t,x) = e^{i\theta(t)}[\phi_\mu(x) + v(t,x)]$$

with $\theta(t)$ such that $(v_2, \phi_\mu) = (v, i\phi_\mu) = 0$;
then

$$\begin{aligned} S_\mu(e^{-i\theta(t)}\psi(t,x)) - S_\mu(\phi_\mu) &= S_\mu(\psi(0,x)) - S_\mu(\phi_\mu) \\ &= S'_\mu(\phi_\mu)v + \frac{1}{2}(S''_\mu(\phi_\mu)v, v) + o(|v|^2_\Sigma) \end{aligned}$$

In general, $(v, \phi_\mu) = (v_1, \phi_\mu) \neq 0$, but conservation of L^2 norm $\rightsquigarrow (v, \phi_\mu) = O(|v|^2_\Sigma)$

Dynamics of the stochastic equation

dB, Fukuizumi, 2009 Let $\psi^\varepsilon(0,x) = \phi_{\mu_0}(x)$ (μ_0 close to $-d$, $\alpha \geq 1/2$) ; in order to use the stability of the standing wave of the deterministic equation, write the solution ψ^ε of the stochastic equation as

$$\psi^\varepsilon(t,x) = e^{i\theta^\varepsilon(t)}(\phi_{\mu^\varepsilon(t)}(x) + \varepsilon\eta^\varepsilon(t,x))$$

with $\theta^\varepsilon(t)$ and $\mu^\varepsilon(t)$ random modulation parameters, chosen such that for all t, $(\eta^\varepsilon(t), \phi_{\mu_0}) = (\eta^\varepsilon(t), i\phi_{\mu_0}) = 0$.
This decomposition holds as long as $\|\varepsilon\eta^\varepsilon\|_\Sigma \leq \delta$ and $|\mu^\varepsilon(t) - \mu_0| \leq \delta$ for $\delta > 0$ sufficiently small.

Let :

$$\tau_\delta^\varepsilon = \inf\{t > 0,\ \|\varepsilon\eta^\varepsilon\|_{H^1} \geq \delta \text{ or } |c^\varepsilon(t) - c_0| \geq \delta\}$$

then : $\exists\ C(\alpha, \mu_0) > 0$, such that $\forall T > 0$, $\exists\varepsilon_0 > 0$ such that $\forall\varepsilon \leq \varepsilon_0$,

$$\mathbf{P}(\tau_\delta^\varepsilon < T) \leq \exp(-\frac{C}{\varepsilon^2 T})$$

Change of the orthogonality conditions

Spectral projection on the generalized null-space of $J\mathcal{L}_{\mu_0}$:

defined for $w = w_1 + iw_2$ by

$$P_{\mu_0} w = (\partial_\mu \phi_{\mu_0}, \phi_{\mu_0})^{-1} [(w_1, \phi_{\mu_0}) \partial_\mu \phi_{\mu_0} + i(w_2, \partial_\mu \phi_{\mu_0}) \phi_{\mu_0}]$$

$\rightsquigarrow$ preceding orthogonality conditions do not imply $P_{\mu_0} \eta^\varepsilon = 0$

Change in the orthogonality conditions :

by setting $\tilde{\theta}^\varepsilon(t) = \theta^\varepsilon(t) - \varepsilon h(t)$, with $h(t)$ a well chosen Itô process (driven by W) one gets

$$\psi^\varepsilon(t,x) = e^{i\tilde{\theta}^\varepsilon(t)} (\phi_{\mu^\varepsilon(t)}(x) + \varepsilon \tilde{\eta}^\varepsilon(t,x))$$

for $t \leq \tau_\alpha^\varepsilon$ and with : $P_{\mu_0} \tilde{\eta}^\varepsilon = O(\varepsilon)$

The modulation parameters

- At first order in ε, the equations for the modulation parameters are given by

$$\begin{cases} d\mu^\varepsilon(t) = o(\varepsilon), \\ d\tilde{\theta}^\varepsilon(t) = \mu_0 dt - \varepsilon \dfrac{(|x|^2 \phi_{\mu_0}, \partial_\mu \phi_{\mu_0})}{(\phi_{\mu_0}, \partial_\mu \phi_{\mu_0})} dW + o(\varepsilon). \end{cases}$$

- This shows in particular that at first order the noise **does not** act on the frequency of the standing wave, but only on its phase.
- Note that coupling with $\tilde{\eta}^\varepsilon$ only at next order (due to the change in the orthogonality conditions)

Central limit Theorem

Theorem : dB, Fukuizumi, 2009 Assume $d = 1$ and $\alpha \geq 1$, or $d = 2$ and $\alpha = 1$. Then, for any $T > 0$, the process $(\tilde{\eta}^\varepsilon(t))_{t\in[0,T\wedge\tau_\alpha^\varepsilon]}$ converges in probability, as ε goes to zero, to a process $\tilde{\eta}$ satisfying

$$d\tilde{\eta} = J\mathcal{L}_{\mu_0}\tilde{\eta}dt - (I - P_{\mu_0})\begin{pmatrix} 0 \\ |x|^2\phi_{\mu_0} \end{pmatrix} dW,$$

with $\tilde{\eta}(0) = 0$, where P_{μ_0} is the spectral projection onto the generalized null space of $J\mathcal{L}_{\mu_0}$. The convergence holds in $C([0,\tau_\delta^\varepsilon \wedge T], L^2)$.

Moreover, $\tilde{\eta}$ satisfies for any $T > 0$ the estimate

$$\mathbf{E}\left(\sup_{t\leq T} |\tilde{\eta}(t)|_\Sigma^2\right) \leq CT \tag{1}$$

for some constant $C > 0$.

Asymptotics on the frequency

Assume $d = 1$ and $\alpha > 1$; as the frequency decreases to -1, the operator $J\mathcal{L}_{\mu_0}$ converges to

$$J \begin{pmatrix} -\partial_x^2 + x^2 + 1 & 0 \\ 0 & -\partial_x^2 + x^2 + 1 \end{pmatrix}$$

which has simple, purely imaginary eigenvalues $\xi_k^\pm = \pm 2i(k+1)$, and a corresponding complete system of eigenfunctions

Let $\tilde{\eta}_k^\pm$ be the corresponding component of the process $\tilde{\eta}$; then as μ_0 goes to -1

$$\mathbf{E}(|\tilde{\eta}_2^\pm(t)|^2) = \frac{\sqrt{\pi}}{4}(\alpha+1)^{\frac{1}{2\alpha}}(\mu_0+1)^{1/\alpha}t + O((\mu_0+1)^{\kappa+1/\alpha}t),$$
$$\mathbf{E}(|\tilde{\eta}_k^\pm(t)|^2) = O((\mu_0+1)^{\kappa+1/\alpha}t) \quad \text{for } k \neq 2.$$

with $\kappa = \min\{1 - 1/\alpha, 1/2\alpha\} > 0$.

Conclusion

- We have considered small multiplicative, time white noise perturbations of a NLS equation with confining potential and standing wave (ground state) as initial data
- The time scale on which the solution stays in a neighborhood of the randomly modulated standing wave is ε^{-2}.
- We obtained at order one a simple behaviour for the modulation parameters
- A central limit theorem holds, i.e. the order one part of the remaining term converges as ε goes to 0 to a centered Gaussian process
- As the frequency tends to its minimal value, this latter process "concentrates" in the third eigenmode
- Open problem : parameters of dark solitons (positive scattering length) : much less invariances in the equation