Sharp energy estimates and 1D symmetry for nonlinear equations involving fractional Laplacians

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(joint work with Xavier Cabré)

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We consider nonlinear fractional equations of the type:

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^n, \quad 0 < s < 1, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\beta}$ function, for some $\beta > \max\{0, 1-2s\}$.

The fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^s u(x) = C_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

It can also be defined using Fourier transform, in the following way:

$$\widehat{(-\Delta)^s} u(\xi) = |\xi|^{2s} \widehat{u}(\xi).$$

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Case s=1/2

We will realize the non local problem (1) in a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition.

More precisely: u is a solution of $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^n$, if and only if its harmonic extension $v(x,\lambda)$ defined on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$ satisfies the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\frac{\partial v}{\partial \lambda} = f(v) & \text{on } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{2}$$

We define

$$G(u) = \int_u^1 f.$$

If the following conditions holds we call the nonlinearity f of *balanced bistable type* and the potential G of *double well type*:

$$\begin{aligned} &(H1) \quad f \ \text{ is odd;} \\ &(H2) \quad G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1,1); \\ &(H3) \quad f' \ \text{ is decreasing in } (0,1). \end{aligned} \tag{3}$$

For example $G(u) = \frac{1}{4}(1-u^2)^2$.

A conjecture of De Giorgi (1978)

Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution in all of $\mathbb{R}^n$ of the equation

$$-\Delta u = u - u^3,$$

such that $\partial_{x_n} u > 0$. Then, at least if $n \leq 8$, all the level sets $\{u = t\}$ of u are hyperplanes, or equivalently u is of the form

$$u(x) = g(a \cdot x + b) \text{ in } \mathbb{R}^n$$

for some $a \in \mathbb{R}^n$, $|a| = 1$, $b \in \mathbb{R}$.

True for:

1. $n = 2$ (Ghoussoub and Gui, 1998),
2. $n = 3$ (Ambrosio and Cabré, 2000 - Alberti, Ambrosio, and Cabré, 2001),
3. $4 \leq n \leq 8$ if, in addition, $u \to \pm 1$ for $x_n \to \pm\infty$ (Savin 2009),
4. counterexample for $n \geq 9$ (Del Pino, Kowalczyk and Wei).

1-D symmetry for the fractional equation: known results

- In dimension $n = 2$ the 1-D symmetry property of stable solutions for problem (1) with $s = 1/2$ was proven by Cabré and Solá-Morales
- In dimension $n = 2$ and for every $0 < s < 1$, 1-D symmetry property for stable solutions has been proven by Cabré and Sire and by Sire and Valdinoci.

Some definitions

Consider the cylinder

$$C_R = B_R \times (0, R) \subset \mathbb{R}^{n+1}_+,$$

where B_R is the ball of radius R centerd at 0 in $\mathbb{R}^n$.

We consider the energy functional

$$\mathcal{E}_{C_R}(v) = \int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v) dx. \tag{4}$$

Definition

We say that a bounded solution v of (2) is *stable* if the second variation of energy $\delta^2\mathcal{E}/\delta^2\xi$ with respect to perturbations ξ compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, is nonnegative. That is, if

$$Q_v(\xi) := \int_{\mathbb{R}^{n+1}_+} |\nabla \xi|^2 - \int_{\partial\mathbb{R}^{n+1}_+} f'(v)\xi^2 \geq 0 \qquad (5)$$

for every $\xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$.

We say that v is *unstable* if and only if v is not stable.

Definition

We say that a bounded solution $u(x)$ of (1) in $\mathbb{R}^n$ is *stable* (*unstable*) if its harmonic extension $v(x,\lambda)$ is a stable (unstable) solution for the problem (2).

Definition

We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v in $\mathbb{R}^{n+1}_+$ is a *global minimizer* of (2) if

$$\mathcal{E}_{C_R}(v) \le \mathcal{E}_{C_R}(w),$$

for every bounded cylinder $C_R \subset \overline{\mathbb{R}^{n+1}_+}$ and every $C^\infty(\mathbb{R}^{n+1}_+)$ function w such that $w \equiv v$ in $\mathbb{R}^{n+1}_+ \setminus \overline{C}_R$.

Definition

We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (1) if its harmonic extension v is a global minimizer of (2).

Definition

We call *layer solutions* for the problem (1) bounded solutions that are monotone increasing, say from -1 to 1, in one of the x-variables

Remark

We remind that every layer solution is a global minimizer (Cabré and Solá-Morales).

Principal ingredients in the proof of the conjecture of De Giorgi:

- Stability of solutions;
- Estimate for the Dirichlet energy:

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 \leq CR^2 \log R.$$

Principal results

Theorem (Energy estimate for minimizers in dimension n)

Set $c_u = min\{G(s) : \inf v \le s \le \sup v\}$.

Let f *be any* $C^{1,\beta}$ *nonlinearity with* $\beta \in (0,1)$ *and* $u : \mathbb{R}^n \to \mathbb{R}$ *be a bounded global minimizer of (1). Let* v *be the harmonic extension of* u *in* $\mathbb{R}^{n+1}_+$.

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-1} \log R, \tag{6}$$

where $C_R = B_R \times (0, R)$ *and* C *is a constant depending only on* n, $||f||_{C^1}$, *and on* $||u||_{L^\infty(\mathbb{R}^n)}$.

In particular we have that

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda \leq CR^{n-1}\log R. \tag{7}$$

Remark

As a consequence we have that the energy estimate (15) holds for layer solutions of problem (1).

Theorem (Energy estimate for monotone solutions in dimension 3)

Let $n = 3$, f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0,1)$ and u be a bounded solution of (1) such that $\partial_{x_n} u > 0$ in $\mathbb{R}^3$. Let v be its harmonic extension in $\mathbb{R}^4_+$.

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^2 \log R, \tag{8}$$

where C is a constant depending only on $||u||_{L^\infty}$ and on $||f||_{C^1}$.

Theorem (1-D symmetry)

Let $n = 3$, $s = 1/2$ and f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0, 1)$. Let u be either a bounded global minimizer of (1), or a bounded solution monotone in the direction x_n.
Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$, or equivalently the level sets of u are planes.

Some remarks

- Energy estimate (15) is sharp because it is optimal for 1-D solutions (Cabré, Solá-Morales).
- In dimension $n = 1$ energy estimate (15), for layer solutions, has been proved by Cabré and Solá-Morales ; more precisely they give estimates for kinetic and potential energies separately:

$$\int_{C_R} |\nabla v|^2 dx d\lambda \leq C \log R, \quad \int_{-\infty}^{+\infty} G(v(x,0)) dx < \infty.$$

In Theorem 5 we have a weaker estimate because we cannot prove that the potential energy in dimension n is bounded by R^{n-1}.

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Sketch of the proof of Theorem 5

The proof of energy estimates for global minimizer is based on a comparison argument. It can be resumed in 3 steps:

- Construct the comparison function w, which takes the same value of v on $\partial C_R \cap \{\lambda > 0\}$ and thus, such that

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(w),$$

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- use the rescaled $H^{1/2}(\partial C_1) \to H^1(C_1)$ estimate in the cylinder of radius 1 and height 1:

$$\int_{C_1} |\nabla \overline{w}|^2 \leq C||w||^2_{L^2(\partial C_1)} + C \int_{\partial C_1} \int_{\partial C_1} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}},$$

where w is the trace of $\overline{w}$ on ∂C_1,

- give the key estimate

$$\int_{\partial C_R} \int_{\partial C_R} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}} \leq CR^{n-1} \log R.$$

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$$\int_{\partial C_R} \int_{\partial C_R} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}} \leq C R^{n-1} \log R.$$

The comparison function $\overline{w}$ satisfies:

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } C_R \\ \overline{w}(x,0) = 1 & \text{on } B_{R-1} \times \{\lambda = 0\} \\ \overline{w}(x,\lambda) = v(x,\lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}. \end{cases} \tag{9}$$

Sketch of the proof of 1-D symmetry result in dimension 3

1-D symmetry of minimizers and of monotone solutions in dimension 3 follows by our energy estimate and the following Liouville type Theorem due to Moschini:

Proposition (Moschini)

Let $\varphi \in L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^{n+1}_+)$ satisfies

$$\begin{cases} -\sigma div(\varphi^2 \nabla \sigma) \leq 0 & in \ \mathbb{R}^{n+1}_+ \\ -\sigma \partial_\lambda \sigma \leq 0 & on \ \partial \mathbb{R}^{n+1}_+ \end{cases} \tag{10}$$

in the weak sense. If

$$\int_{C_R} (\varphi\sigma)^2 dx \leq CR^2 \log R$$

for some finite constant C independent of R, then σ is constant.

Sketch of the proof of 1-D symmetry result in dimension 3

Suppose $v_{x_3} > 0$; set $\varphi = v_{x_3}$ and for $i = 1, ..., n-1$ fixed, consider the function:

$$\sigma_i = \frac{v_{x_i}}{\varphi}.$$

We prove that σ_i is constant in $\mathbb{R}^{n+1}_+$, using the Liouville result due to Moschini and our energy estimate.

- the function σ_i satisfies

$$\begin{cases} -\sigma_i \mathrm{div}(\varphi^2 \nabla \sigma_i) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\sigma_i \partial_\lambda \sigma_i = 0 & \text{in } \partial\mathbb{R}^{n+1}_+ \end{cases}, \tag{11}$$

- by our energy estimate, we get

$$\int_{C_R} (\varphi\sigma_i)^2 \leq \int_{C_R} |\nabla v|^2 \leq CR^2 \log R,$$

- by Proposition (4) we deduce $\sigma_i = c_i$ is constant then v depends only on λ and the variable parallel to the vector $(c_1, c_2, c_3, 0)$ and then $u(x) = v(x, 0)$ is 1-D.

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Energy estimate for global minimizers of $(-\Delta)^s u = f(u)$, with $0 < s < 1$

Local problem:

u is a solution of

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n, \tag{12}$$

if and only if, v defined on $\mathbb{R}^{n+1}_+ = \{(x,\lambda) : x \in \mathbb{R}^n,\ \lambda > 0\}$, is a solution of the problem

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \\ -\lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda v = f(v). \end{cases} \tag{13}$$

The energy functional associated to problem (13) is given by

$$\mathcal{E}_{s,C_R}(v) = \int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} G(v) dx. \tag{14}$$

Remark

- The weight λ^{1-2s} belongs to the Muckenoupt class A_2, since $-1 < 1-2s < 1$ [theory of Fabes-Kenig-Serapioni];
- problem (13) is invariant under translations in the x_i-directions.

Theorem (Energy estimate for minimizers in dimension n)

Let f be any $C^{1,\beta}$ nonlinearity, with $\beta > \max\{0, 1-2s\}$, and $u : \mathbb{R}^n \to \mathbb{R}$ be a global minimizer of (1). Let v be the s-extension of u in $\mathbb{R}^{n+1}_+$.

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-2s} \quad \text{if } 0 < s < 1/2$$

$$\left(\int_{C_R} \tfrac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-1}\log R \quad \text{if } s = 1/2 \right) \tag{15}$$

$$\int_{C_R} \frac{1}{2}\lambda^{1-2s}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \le CR^{n-1} \quad \text{if } 1/2 < s < 1,$$

where C denotes different positive constants depending only on n, $||f||_{C^1}$, $||u||_{L^\infty(\mathbb{R}^n)}$ and s.

The proof is based on a comparison argument as before. Here a crucial ingredient is the following extension theorem.

Theorem

Let Ω be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial\Omega$ and M a Lipschitz subset of $\partial\Omega$. For $z \in \mathbb{R}^{n+1}$, let $d_M(z)$ denote the Euclidean distance from the point z to the set M. Let w belong to $C(\partial\Omega)$.
Then, there exists an extension $\widetilde{w}$ of w in Ω belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, such that

$$\int_\Omega d_M(z)^{1-2s} |\nabla \widetilde{w}|^2 dz \leq C||w||^2_{L^2(\partial\Omega)} + C \int\int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} d\sigma_z d\sigma_{\overline{z}}$$

$$+C \int\int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} d\sigma_z d\sigma_{\overline{z}}. \qquad (16)$$

The sets B_s and B_w are defined as follows:

$$B_s = \begin{cases} \partial\Omega \times \partial\Omega & \text{if } 0 < s < 1/2 \\ M \times M & \text{if } 1/2 < s < 1, \end{cases} \tag{17}$$

and

$$B_w = \begin{cases} (\partial\Omega \setminus M) \times (\partial\Omega \setminus M) & \text{if } 0 < s < 1/2 \\ (\partial\Omega \setminus M) \times \partial\Omega & \text{if } 1/2 < s < 1. \end{cases} \tag{18}$$

After rescaling, we apply this result for $\Omega = C_1$ and $M = B_1 \times \{0\}$

Theorem (1-D symmetry)

Let $n = 3$, $1/2 \leq s < 1$ and f be any $C^{1,\beta}$ nonlinearity with $\beta > \max\{0, 1-2s\}$. Let u be either a bounded global minimizer of (1), or a bounded solution monotone in the direction x_n.

Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$, or equivalently the level sets of u are planes.

Some open problems:

- 1-D symmetry for $n = 3$ and $0 < s < 1/2$;
- 1-D symmetry for $n > 3$ and $0 < s < 1$;
- critical dimension;
- counterexample in large dimensions.