We consider the following question: Let $f$ and $g$ be weight $k$ cusp forms with $f(z) = \sum a(n)e^{2\pi inz}$ and $g(z) = \sum b(m)e^{2\pi imz}$ with $a(m) = A(m)m^{(k-1)/2}$ and $b(m) = B(m)m^{(k-1)/2}$ Pick $h \geq 1$ and consider $\sum m \leq X A(m)B(m+h)$ the general approach leads to an estimate of $\ll hX^{1+\epsilon}$ and more interestingly we have

$$= \sum_{j \geq 1} c_j(h)X^{1/2+it_j} + \text{Error}$$

with $c_j(h)$ the $h$th coefficient of the Maass form with eigenvalue of the Laplacian equal to $\frac{1}{2} + it_j$.

When these estimates are used the cancellation in $h$ is typically ignored. Recently, however, Blomer and Harcos used this cancellation. Bump-Hoffstein-etc. used multiple Dirichlet series to study $L$-functions and this study hopes to arrive at multiple Dirichlet series for these forms.

We consider $\sum m \geq 1 \frac{A(m)B(m+h)}{m^s}$

Selberg introduced Poincaré series to study these sums. The series he introduced take the shape

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e^{2\pi ih\gamma z} \text{Im}(\gamma z)^s.$$ 

The inner product of this with $f$ and $g$ leads to the series

$$\sum_{m \geq 1} \frac{A(m)B(m-h)}{m^s}$$

but we want to have a sum on $h$ to get double dirichlet series of the form

$$\sum_{m,h \geq 1} \frac{A(m)B(m+h)}{m^sh^w}.$$ 

To get at this we try to use the series

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e^{-2\pi ih\gamma z} \text{Im}(\gamma z)^s.$$ 

This however is problematic as it introduces exponential growth at the cusp and the inner product is no longer convergent. This leads to the following modification

$$\sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e^{-2\pi ih\gamma z} \text{Im}(\gamma z)^s \Psi_Y(\text{Im}(\gamma z)),$$ 

where $\Psi_Y$ is a suitable weight function.
where

\[ \Psi_Y(z) = \begin{cases} 1 & Y^{-1} \leq y \leq Y \\ 0 & \text{else} \end{cases} \]

Letting \( Y \to \infty \) yields integrals of the following shape

\[ \int_{hY^{-1}}^{hY} y^{s-1/2} e^{y} K_{it_j} (2\pi |m| y) \frac{dy}{y}. \]

If \( \frac{1}{2} < \text{Re} (s) < 1 \) this converges absolutely as \( Y \to \infty \).

We care about the convergence at \( s = 0 \) and so we try instead to introduce a convergence factor in the exponential term introducing the following series

\[ \sum_{\gamma \in \Gamma \setminus \Gamma} e^{-2\pi i h \gamma z (1-\delta)} \text{Im} (\gamma z)^s \Psi_Y (\text{Im} (\gamma z)). \]

This yields to integrals of the following shape

\[ \int_{hY^{-1}}^{hY} y^{s-1/2} e^{y(1-\delta)} K_{it_j} (2\pi |m| y) \frac{dy}{y}. \]

Which have even better convergence properties.

With these series in order we seek a spectral expansion. Unfortunately the spectral expansion has convergence properties of its own. Let

\[ D(s, h) := \sum_{m \geq 1} \frac{A(m) B(m + h)}{m^s}. \]

Up to powers of 2 and \( \pi \), after we let \( Y \to \infty \) and \( \delta \to 0 \) we have \( D(s, h) \) is equal to

\[ \frac{\Gamma(1-s)}{\Gamma(s + k - 1)} \sum_{j} \lambda_j(h) \frac{\Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j) \Gamma(\frac{1}{2} - it_j)}{h^{s-\frac{1}{2}} \Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j)} \rho_j(1) \left( \langle f g \text{Im} (\cdot)^k, u_j \rangle + \text{continuous} \right) \]

Where \( \{u_j\} \) is an orthonormal basis of Maass forms and \( \rho_j(1) \) is the first Fourier coefficient of \( u_j \), which is \( L^2 \) normalized.

If \( s = 1 \) we do not get convergence. By Stirling’s approximation each term is \( \ll (1 + |t_j|)^{2s-2} \) and so we need \( \text{Re} (s) \) negative enough to get convergence. Note that \( \langle f g \text{Im} (\cdot)^k, u_j \rangle \sim t_j^k \) because

\[ \left( \langle f g \text{Im} (\cdot)^k, u_j \rangle \right)^2 \sim L \left( \frac{1}{2}, f \times g \times u_j \right) \]

and the \( \Gamma \) factors introduce the power of \( k \) (details of this are probably a bit off). For some reasons we do not go into here, this series converges for \( \text{Re} (s) < \frac{1}{2} - \frac{k}{2} \). The result is a spectral expansion of the series

\[ \sum_{m} \frac{A(m) B(m + h)(s)}{m^s} \]

which converges for \( \text{Re} (s) > 1 \). So there is no region where both of these series converge.

The fix is to keep \( \delta > 0 \) and you get an expansion on each side that are equal everywhere. So you get equality that is an equality of analytically continued functions. There are some consequences of
this result. First we have
\[
\frac{1}{2\pi i} \int_{(2)} D(s,h)X^s g(s)ds = \sum_{m \sim X} A(m)B(m+h)G\left(\frac{m}{X}\right) = \sum \text{res}(D(s,h))X^{\frac{1}{2}+it_j}
\]
for \(s = \frac{1}{2} \pm it_j - r\) with \(r \in \mathbb{N}\). The residues are \(\frac{\lambda_j(h)}{h \pm it_j}G_{r,j}\) where the \(G_{r,j}\) is the residue of the Gamma factors. In the result we need to bound the shifted \(D(s,h)\) to the line \((-1)\), say. This yields
\[
\sum c_{r,j} h^{\frac{1}{2}} g\left(\frac{1}{2} \pm it_j - r\right) \left(\frac{h}{X}\right)^{r-\frac{1}{2} \pm it_j} + \cdots
\]
So there is a fundamental change in behavior when \(h > X\) and \(h < X\). You must push the integral to at least \(\frac{1}{2} - \frac{k}{2}\) to use this. This gives a non-trivial bound (estimate) when \(h < X^2\).

Remark. Blomer took a different approach and got to the same obstruction.

Suppose we sum on \(h\) to produce \(\sum \frac{D(s,h)}{h^{w}}\). Then the spectral side turns into
\[
(*) \sum_j L(s+w-\frac{1}{2},u_j) \frac{\Gamma(\cdot)\Gamma(\cdot)}{\Gamma(\cdot)\Gamma(\cdot)}
\]
Replacing the sum on integers by a sum on eigenvalues. Observe that the ratio of Gamma factors is \(\sim \frac{1}{|1+it_j|^s-2s}\). Hence, as \(s \to -\infty\) looks itself like a Dirichlet series.

Say we have \(\sum_{h \equiv a (\text{mod } q)} \frac{D(s,h)}{h^{w}}\). This introduces hyper-Kloosterman sums and so there is a charge of improving these bounds. You may also try to attack
\[
\frac{1}{\phi(Q)} \sum_{\chi (Q)} L\left(\frac{1}{2},f,\chi\right)L\left(\frac{1}{2},g,\chi\right)
\]
If \(f \neq g\) should be \(C_{f,g} + O(Q^{-1/2})\) and \(C_{f,g} \sim L(f \times g, 1)\). This is, however, out of reach in the case \(f = g\). The asymptotic is known for many values of \(Q, Q\) prime for instance. This can be boiled down to a certain double Dirichlet series \(\sum_{h \equiv 0 (\text{mod } Q)} \frac{A(m)B(m+h)}{m^\alpha h^w} =: Z_Q(s,w)\). There are lines of poles that obstruct the analytic continuation of this series. Subtracting these poles to continue the series and then we can add them back in. There is a critical point at \((1 - \frac{k}{2}, \frac{1}{2}) = (s,w)\). One the other hand, we have
\[
Z_Q\left(\frac{1}{2} - \frac{k}{2} - \epsilon, 1 + \epsilon\right) \ll Q^{\theta+\frac{1}{2}}
\]
\[
Z_Q\left(1 - \frac{k}{2}, \frac{1}{2}\right) \ll Q^{\theta}\frac{1}{2}^2
\]
If we can do anything to keep \(\epsilon < 0\) then we win by convexity. One the other hand we can use these methods to establish
\[
\sum_{|Q-X|} \frac{1}{\phi(Q)} \sum_{\chi} L(f,\chi)L(g,\chi) = CY + O(X^\epsilon).
\]