ON THE INTERPOLATION PROBLEM:
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ABSTRACT. These are my notes from Joe Harris’s talk at the Joint Meetings of the AMS and MAA. Any mistakes are my own.

We begin with the problem of finding a polynomial of degree \( d \), call it \( f \), such that \( f(z_i) = c_i \) for \( i = 1, \cdots, d + 1 \). This has a unique solution. Where do we go from there? One direction would be polynomials in many variables. If one makes this step than the clean statement here instantly becomes horribly false.

We introduce a notion of rank. In particular, we might fix a set of points and consider the map from polynomials of degree \( d \) to \( \mathbb{C}^e \), where \( e \) is the number of points we have fixed. How large or small could this evaluation map be? In particular, is it surjective or injective?

This leads us to the answer that interpolation can be done for general configurations. However, interpolation may fail if the points lie on a variety of small degree.

We turn now to some discussion about algebraic curves. To begin we ask: how do we classify algebraic curves? Now-a-days we know to fix the genus of the curve and then to consider the moduli space. The geometry of the moduli space carries a lot of the information that we care about. However, 150 years ago we tried to classify them by their genus, their degree, and which \( \mathbb{P}^r \) that they live in (\( g, d, \) and \( r \)). Given the triple \((g, d, r)\) does there exist a curve? The answer to this question is still not known today.

To prove that interpolation may fail and explain why we need to prove that polynomials have a positive dimension common zero locus. For instance, stating a slightly more general problem: consider 2 points in \( \mathbb{C}^2 \) and look for a degree 2 polynomial in two variables with assigned values and directional derivative at these points. So we have

\[
\rho : V_2 \rightarrow \mathbb{C}^6
\]

where \( \dim(V_2) = 6 \), since it is spanned by \( \{x^2, x, 1, x^2y, xy, y, x^2y^2, xy^2, y^2\} \). Then finding such a polynomial cannot be done in general! The map has a kernel. It is found by requiring the vanishing at each point and double vanishing at another. This configuration contains a line. This is a theorem of Alexander, Hirshwitz, Harbourne-Hirschowitz?.

In light of this condition about algebraic varieties, we may ask if it does not lie on an algebraic variety how close to having surjectivity are we? In the case of polynomials in one variable we have the constraint that it has at most \( d \) zeros. In two variables, we have Bezout’s theorem, stating that two polynomials will meet in \( N \) points where \( N \) is the product of their degrees.

**Theorem 0.1** (Lazerfield). If \( Q_1, \cdots, Q_{r+1} \) are linearly independent quadratics in \( \mathbb{P}^r \) with finite common locus, then

\[
\deg(\Gamma) \leq \frac{3}{4} \cdot 2^r
\]
Remark. Harris had a funny quote. He said the ratio of conjectures and hypotheses to theorems in this area is like a twinky.

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