Selberg proved that there are a positive proportion of zeros of $\zeta$ on the critical line. What percentage do we get from Selberg’s method if we do it right?

Levinson and Montgomery proved that
\[ \left| \{ \rho : 0 \leq \gamma \leq T, 0 < \beta < \frac{1}{2} \} \right| = \left| \{ \rho' : \zeta'(\rho') = 0, 0 < \gamma' < T, 0 < \beta' < \frac{1}{2} \} \right| + \log(T). \]

Write $\zeta(s) = \chi(s)\zeta(1-s)$ then
\[ \frac{\chi'(s)}{\chi(s)^{1/2}} \zeta(1-s) = \chi(s)^{-1/2} \zeta'(s) + \chi(1-s)^{-1/2} \zeta'(1-s). \]

When $\text{Re}(s) = 1/2$ this is the sum of the conjugates. So zeros of $\zeta$ come to imaginary values of $\chi(s)^{-1/2} \zeta'(s)$. Levinson and Montgomery do something a little different, but this is easier. At this point we set out to count zeros of $\zeta'(1-s)$ to the right of 1/2.

The Littlewood Lemma leads to the consideration of the integral
\[ \int_0^T |\zeta'(a + it)|^2 dt \]
after applying the AM-GM inequality. But applying AM-GM yields a big loss. So we apply a mollifier to do better.

We will mollify $\zeta'$ with $1/\zeta$. Why?

$\zeta' = \zeta' \zeta$ and $\zeta'/\zeta$ is normal and does go up linearly and so in some sense it does not “need” a mollifier.

The same arguments we use here would work for any particular Dirichlet $L$-function and yield 40% of the zeros on the critical line. We will look at $q \leq Q$, $\chi$ primitive of modulus $q$ and all zeros up to height $\log(Q)$. There are $cQ^2$ zeros since
\[ \phi^*(q) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \approx q. \]

They prove that at least 55% zeros are on the critical line. Note, for $\zeta$ the length of mollifier is $T^\theta$ and with $\theta = 1$ the percentage is 55%.
An element of the proof is a sort of asymptotic large sieve. Montgomery and Vaughn proved that
\[\int_0^T \left| \sum_{n \leq N} a_n n^it \right|^2 dt = (T + O(N)) \sum_{n \leq N} |a_n|^2\]

On the other hand, the large sieve in the family of interest is
\[\sum_{q \leq Q} \frac{\phi(q)}{q} \sum_{\chi} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \leq (Q^2 + N) \sum_{n \leq N} |a_n|^2\]

The asymptotic large sieve gives a formula for
\[S := \sum_q W\left(\frac{q}{Q}\right) \sum_{\chi} \left| L\left(\frac{1}{2}, \chi\right) M\left(\frac{1}{2} \chi\right) \right|^2\]

The result looks similar to the following asymptotic
\[
\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| \sum_{n \leq N} \frac{a_n}{n^{1/2+it}} \right|^2 dt
= T \left( \sum_{m,n \leq N} \frac{a_m a_n (m,n)}{mn} \left( \log\left( \frac{T(m,n)^2}{2\pi mn} \right) + 2\gamma \right) + o(1) \right)\]

for \(N \leq T^{1/2-\epsilon}\). It is conjectured that this asymptotic should hold for \(N \leq T^{1-\epsilon}\). If this is the case then it would imply the Lindelöf Hypothesis.

The first step in proving such a theorem is the approximate functional equation. With that in hand \(S\) becomes
\[
\sum_q W\left(\frac{q}{Q}\right) \sum_{h,k} \sum_{m,n} V\left(\frac{\pi mn}{q}\right) \frac{1}{\sqrt{mn}} \sum_{\chi} \chi(mh) \overline{\chi(nk)}.
\]

At this point one can see that we really need to only consider primitive \(\chi\). If we didn’t then the non-primitive would count too much and you lose the appropriate calculation. The number of primitive is
\[
\sum_{\mu(c)\phi(d)} \mu(c)\phi(d)
\]

Putting this in we obtain
\[
\sum_{hm \equiv nk \pmod{d}} W\left(\frac{cd}{Q}\right) \frac{a_n a_k}{\sqrt{hk}} \sum_{m,n} V\left(\frac{\pi mn}{cd}\right) \frac{\phi(d)\mu(c)}{\sqrt{mn}}.
\]

We split this on \(c < C\) and \(c > C\).
\[
\mathcal{U} + \mathcal{L} := \sum_{c < C} (\cdots) + \sum_{c > C} (\cdots)
\]
Notice that in $Ud$ is necessarily big while in $c > C$ $d$ is necessarily small. To handle $L$ they use all characters modulo $d$ to rewrite the summation obtaining something like the following

$$\sum_{c > C, d} \mu(c) \sum_{m,n} V\left(\frac{\pi mn}{cd}\right) \psi(mh) \overline{\psi}(nk)$$

The idea is to open the function $V$ and introduce $L$-functions. Then one can read off the main term and the original large sieve can be applied to handle the error term.

One surprising step is that the $\psi = \psi_0$ term cancels with the $\sum_{c < C}$ summation!

Along the way one applies the estimate

$$\sum_{\chi} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 \ll \varphi^*(q) \log(q)^4$$

It is perhaps surprising that one does not need the additional summation on $q$ in this estimate.