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Abstract. These are the notes from the talk given by Andrew Booker on November 4, 2005 at the University of Wisconsin. His talk was part of the annual Midwest Number Theory Day. The notes are incomplete.

1. Introduction

Given \( f(x) \in \mathbb{Z}[x] \) we can ask how does \( f \) factor modulo \( p \). Let \( K \) be the splitting field of \( f \). WE study \( \text{Gal}(K/\mathbb{Q}) \) and the frobenius conjugacy classes for each prime \( p \).

Take a representation of \( \rho : \text{Gal}(K/\mathbb{Q}) \to GL_n(\mathbb{C}) \) irreducible and nontrivial. This is called the Galois representation. Then

\[
L(s, \rho) = \prod_p \frac{1}{\det(I - p^{-s} \rho(frobenius))}
\]

the \( \rho(frobenius) \) only depends on the conjugacy class.

This is an Artin \( L \)-function.

Conjecture 1.1 (1923-Artin). \( L(s, \rho) \) has analytic continuation to \( \mathbb{C} \) and satisfies a functional equation.

We have the following results which break down by dimension.

1. \( \dim(\rho) = 1 \): The expectation here is that \( L(s, \rho) = L(s, \chi) \), \( \chi \) a Dirichlet character or \( L(s, \rho) = L(s, \lambda) \), \( \lambda \) a Hecke character over a number field. This result was proved by Artin and uses Class Field Theory.

2. \( \dim(\rho) = 2 \) then \( L(s, \rho) = L(s, f) \) with \( f \) a modular form or a Maass form. This is know if \( \text{Gal}(K/\mathbb{Q}) \) is solvable. Covers most of the cases. The result is due to Langlands and Tunnell.

3. \( \dim(\rho) = n > 2 \): then the expectation is that \( L(s, \rho) = L(s, \pi) \) where \( \pi \) is an automorphic representation of \( GL_n \). The reason for suspecting this is because of Langlands functoriality/reciprocity. Some spurious cases are known. The conjecture is due to Langlands.

Theorem 1.2 (Bouwer, 1947). \( L(s, \rho) = \prod_{j=1}^n L(s, \lambda_j)^{e_j} \) where the \( \lambda_j \) are Hecke characters over fields between \( K \) and \( \mathbb{Q} \) and \( e_j \) are integers.

Date: November 4, 2005.
Remark. This is not factoring! Some of the $e_j$ can be less than 0.

Corollary 1.3. WE get the functional equation for the $L(s, \rho)$ inherited from the $L(s, \lambda_j)$.

This problem in general is hard and not much is known. So we ask a simpler question

2. A Simpler Question

Question. Given a Galois representation $\rho$ Can we tell whether $L(s, \rho)$ is holomorphic/modular?

Modular would mean $L(s, \rho) = L(s, f)$ or $L(s, \pi)$. Certainly modular would imply holomorphic. But is there something in the other direction? If it is holomorphic can we say anything about its modularity?

Converse Theorem. Analytic properties of $L(s, \rho)$ and “twists” imply modularity.

We know this for $GL_n$ by the amount of twists is too much, in some sense. As a precise example we have the following theorem.

Theorem 2.1 (Booker- 2003). For two dimensional $\rho$ if $L(s, \rho)$ is holomorphic then it is modular.

Can we say anything about higher dimensions? Let us turn our discussion back to Bouwer’s theorem 1.2.

3. Bouwer’s Theorem

We have $L(s, \rho) = \frac{N(s)}{D(s)}$. In general, the denominator will have zeros in nontrivial cases. How do we check that every zero in the bottom is a zero in the top. Write $N_f(T) = \#$ zeros in the box which goes from $-1$ to $2$ in the real direction and $0$ to $iT$ in the imaginary direction. Then if we call the box $B(T)$, then we have

$$N_f(T) = \frac{1}{2\pi i} \int_{B(T)} \frac{f'}{f}(s)ds.$$ 

Now the graph of $N_f(T)$ will be a step function but we cannot actually exactly compute $N_f(T)$ so near each jump in the function there is a fuzzy region that is some measure of tolerance. This uncertainty causes problems.

We know that $N_L(T) = N_N(T) - N_D(T)$. So the moral here is that we can only compute the number of net zeros with some precision.

4. Comments

Sarnak mentions this result in his first lecture at the fields institute in 2003.

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